CERTAIN PROPERTIES OF ESSENTIAL NUMERICAL RANGES OF BOUNDED OPERATORS ON BANACH SPACES

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

To my Mother, Miriam Angulu Wegulo.

Abstract

The numerical range of an operator on a Hilbert space has been extensively researched on. The concept of numerical range of an operator goes back as early as 1918 when Toeplitz defined it as the field of values of a matrix for bounded linear operators on a Hilbert space. Major results like convexity, that is the Toeplitz-Hausdorff theorem, the relationship of the spectrum and the numerical range, the essential spectra and the essential numerical range, have given a lot of insights. Most of these results have been on Hilbert spaces. As for Banach spaces there is still work to be done. There is scanty literature on the properties of the essential spectra and the essential numerical ranges on Banach spaces. The objectives of this study were to determine the properties of the essential spectrum and the properties of the essential numerical range, and to investigate the relationship between the essential spectrum and the essential numerical range for operators on Banach spaces. To study the properties of the essential spectra, we defined various parts of the spectra and using known theorems, we established the duality properties of these parts. For the essential numerical range, we applied the approach of Barraa and Müller which considers a measure of noncompactness instead of the usual essential norm on the Calkin algebra. We finally extended the existing relations between the spectra and the numerical range to the setting of the essential spectrum and essential numerical range. We hope that the results of this study will be significant to both Applied Mathematicians and theoretical physicists for further research.

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CHAPTER 1

INTRODUCTION

1.1 Background information

The concept of the numerical range of an operator on a Hilbert space H was presented by O. Toeplitz [11] in 1918. The Toeplitz-Hausdorff Theorem establishes the convexity of the numerical range for any operator on a Hilbert space. The concept of the numerical range on a Banach space X was extended by Bauer and Lumer [15] who showed that the numerical range implemented by an operator on a Banach space is not necessarily convex. There were certain properties established by Toeplitz on the numerical range in the Hilbert space, which were also true for the numerical range in the Banach space. For example the spectrum of an operator is contained in the closure of its numerical range. This property makes the numerical range to serve as a tool used to localise the spectrum [11]. In the last fifty years, several substantial attempts have been made to relate some of the important features of the spectral theory of normal operators from the realm of Hilbert spaces to the more general setting of Banach spaces. Probably the most prominent and ambitious step in the early development of abstract spectral theory was the systematic investigation of spectral operators on Banach spaces that was initiated by Dunford [18]. For a comprehensive treatment of these operators, we refer the reader to the monographs of Dunford and Schwartz [19]. Moreover, a wide field of applications in analysis shows that the geometry of Banach spaces can be quite different from that of Hilbert spaces. For example, the unit ball of a Banach space may have corners, and closed convex sets need not possess a unique vector of smallest norm [12, 13]. The most important geometric property absent in general Banach spaces is a notion of perpendicularity (or orthogonality) which is determined by considering the inner product. However, a linear space equipped with an inner product can be made into a Banach space if it is made to be complete in the metric defined by the inner product. Such spaces possess many of the more pleasant properties of the Hilbert spaces which makes it possible to study the numerical range on quotient spaces using compact operators [12, 13]. In 1968, Stampfli, Fillmore and Williams [49] defined an essential numerical range on bounded linear operators on a Hilbert space. It was shown that the essential numerical range is the set of the intersection of the closure of the compact cosets and the convex hull of the essential spectrum is contained in the essential numerical range. According to a result of Stampfli, Fillmore and Williams [48, 49, 50], if T is a normal operator then the convex hull of the essential spectrum of T is equal to the essential numerical range of T. The notion of an essential spectrum is important in the spectral theory. Essential spectra are subsets of the spectrum which are invariant under compact perturbation of the given operator. These spectra are obtained, for example, by the strengthening of the invertibility of the operator using the Fredholm environment. In this study, we consider two numerical ranges on Banach spaces, namely the essential algebraic numerical range and the essential spatial numerical range. For Banach space operators, the properties of the essential algebraic numerical range have been remarkably studied in literature [3, 11, 13]. Surprisingly, the first reasonable attempt for the corresponding study of the essential spatial numerical range was by Baraa and Müller, in 2005 [9]. The reasons behind this strange observation isn't apparent but it is noted that for a successful study of the properties of this numerical range, it is important to consider another norm which is a "measure of non-compactness" instead of the usual essential norm $\|.\|_e$. The literature on the study of essential spatial numerical range for Banach space operators still remains very scanty. In fact, one visible study is the work by Barraa and Müller [9].

1.2 Basic Concepts and Preliminaries

The following basic concepts are necessary in the study of the essential spectrum and the essential numerical range of operators on a Banach space. These can be found in [8, 42, 43, 45].

Definition 1.2.1

Let X, Y be linear spaces over the same scalar field $\mathbb{K}$, which can be $\mathbb{R}$, the real field or $\mathbb{C}$, the complex field. A mapping $T : X \to Y$ is said to be *linear*, if

$$T(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) = \alpha(T\mathbf{x}_1) + \beta(T\mathbf{x}_2)$$

for all $\alpha, \beta \in \mathbb{K}$ and $\mathbf{x}_1, \mathbf{x}_2 \in X$. The *domain* of T denoted by $D(T)$ is defined by $D(T) = \{\mathbf{x} \in X : T\mathbf{x} \in Y\}$, while the *range* of T denoted by $R(T)$ is defined by $R(T) = \{\mathbf{y} = T\mathbf{x} : \mathbf{x} \in X\}$. The *null space* of T denoted by $N(T)$ is the set $N(T) = \{\mathbf{x} \in X : T\mathbf{x} = 0\}$. The set $N(T)$ is also called the *kernel*of T.

T is called a *linear operator* if it is a linear mapping from X into itself i.e., $T : X \to X$.

If the range $R(T)$ is contained in the scalar field $\mathbb{K}$ then T is called a *linear functional* on X.

A linear operator $T : X \to Y$ is said to be *bounded* if for all $\mathbf{x} \in X$ there exists $M > 0$ such that $\|T\mathbf{x}\| \leq M\|\mathbf{x}\|$. We shall denote the set of all bounded linear operators of X into Y by $\mathcal{L}(X, Y)$. Also $\mathcal{L}(X, X) = \mathcal{L}(X)$. A linear transformation $T : X \to Y$ is compact precisely when, for each bounded sequence $\{x_i\}_{i\in\mathbb{N}}$ in X the sequence $\{Tx_i\}_{i\in\mathbb{N}}$ has a sub-sequence that converges in Y. We denote the set of all compact operators as $\mathcal{K}(X)$. If a linear operator T gives a one-to-one map of $D(T)$ onto $R(T)$, then the inverse map T^{-1} gives a linear operator on $R(T)$ onto $D(T)$, such that $T^{-1}T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in D(T)$ and $TT^{-1}\mathbf{y} = \mathbf{y}$ for all $\mathbf{y} \in R(T)$.

Definition 1.2.2

For each vector $\mathbf{x}$ in a linear space X there corresponds a real number denoted $||\mathbf{x}||$ called the *norm* of $\mathbf{x}$, satisfying the following properties:

(i) $||\mathbf{x}|| > 0$ for any $\mathbf{x} \in X$, and $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = 0$ (Strict positivity),

(ii) $||\lambda \mathbf{x}|| = |\lambda| ||\mathbf{x}||$ for all $\mathbf{x} \in X$ and scalars λ (Homogeneity), and

(iii) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ for all $\mathbf{x}, \mathbf{y} \in X$ (Triangle inequality).

The non-negative real number $||\mathbf{x}||$ can be seen as the length of the vector $\mathbf{x}$. A *normed linear space* $(X, ||.||)$ is a linear space on which a norm has been defined. If the norm defines a metric on the linear space as $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ for $\mathbf{x}, \mathbf{y} \in X$, then the normed space becomes a metric space. A *Banach space*X is a complete normed linear space.

Definition 1.2.3

A non-negative function $||.|| : X \to \mathbb{R}$ is called a *semi-norm* (or a pseudo-norm) if $||.||$ satisfies properties (ii) and (iii) of the norm.

Definition 1.2.4

Let X and Y be Banach spaces on which a norm $||.||$ has been defined. The operators $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$ are said to be *adjoint* with respect to the norm $||.||$ if the scalar product $\langle S\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T\mathbf{y} \rangle$, $(\mathbf{x} \in X, \mathbf{y} \in Y)$. T can also be denoted as S^*.

1.2.1 Dual spaces and Annihilators

Definition 1.2.5

The set of all linear functionals on a vector space X denoted by X^* is called the dual space of X. If $f \in X^*$ then dim Ker (f)+ dim Im (f) = dim X, where Imf is the range of f. Since Im$(f) \subset \mathbb{K}$, we have either Im$(f) = 0$ the zero linear functional or Im$(f) = \mathbb{K}$, i.e. f is surjective. Moreover for finite dimensional vector spaces we have dim$X$ = dim(X^*) [7].

For an arbitrary subset M of a Banach space X, the annihilator of M in X^* is denoted by $M^\perp$ and is defined as

$$M^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in M\}.$$

For an arbitrary subset N of the Banach space X^*, the annihilator of N in X is denoted by $N_\perp$ and is defined as

$$N_\perp = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in N\}.$$

We refer to [33, 43].

Definition 1.2.6

The dual space X^* of a normed linear space X which is the set of all bounded linear functional also has what is known as the second dual space X^{**}. If X^{**} is isomorphic to X then X is called *reflexive*.

Let X be a Banach space. The following are some important classes of bounded operators on X. An operator $T \in (\mathcal{L}(X)$ is

(i) Self-adjoint or Hermitian if $T^* = T$,

(ii) Normal if $T^*T = TT^*$,

(iii) Positive if $T^* = T$ and $\langle T(x), x\rangle \geq 0$ for all $\mathbf{x} inX$,

(iv) Unitary if $T^*T = TT^* = l$,

(v) Projection if $T^2 = T = T^*$.

One has the implications that Projection $\Rightarrow$ Positive $\Rightarrow$ Self-adjoint $\Rightarrow$ Normal.

1.2.2 Quotient spaces and Quotient mapping

Definition 1.2.7

Let M be a subspace of a linear space X and let the coset of an element $x \in X$ be defined by $x + M = \{x + m : m \in M\}$. Then the distinct cosets form a partition of X. If addition and scalar multiplication are defined by

$$(x + M) + (y + M) = (x + y) + M$$

and

$$\alpha(x + M) = \alpha x + M,$$

then these cosets constitute a linear space denoted by X/M called the *quotient space* of X with respect to M. The origin in X/M is the coset $0 + M = M$, and the negative of $x + M$ is $(-x) + M$.

Definition 1.2.8

For any vector space X equipped with a semi-norm $\|.\|$, and any closed subspace $M \subseteq X$, the norm of $x + M \in X/M$, defined by $\|x + M\|_{X/M} = \inf_{y \in x+M} \|y\| = \inf_{v \in M} \|x + v\|$ for $x \in X$ is called the *quotient norm*.

Definition 1.2.9

The *quotient mapping* is defined as a map $q : V \to V/S$ where V is a vector space and S is an ideal in V. Here $q(v) = v + S$ for all $v \in V$. This map is called the canonical projection of V onto S (Projection modulo S). It is linear since $q(\alpha u + \beta v) = (\alpha u + \beta v) + S = \alpha(u + S) + \beta(v + S) = \alpha q(u) + \beta q(v)$. The canonical projection $q : V \to V/S$ is a surjective linear transformation with $\text{Ker}(q) = S$, see [8, 20]

Any linear transformation $T : V \to W$, can be factored through the projection map $q : V \to V/S$ if $S \subset \text{Ker}(T)$. Let $T \in \mathcal{L}(V, W)$ and $S \subset \text{Ker}(T)$ be a subspace of V then there is a unique linear transformation $\tilde{T} : V/S \to W$ with the property that $\tilde{T} \circ q = T$. Moreover, $\text{Ker}(\tilde{T}) = \{v + S : v \in \text{Ker}(T)\}$ and $\text{Im}(\tilde{T}) = \text{Im}(T)$. Thus the image of any linear transformation with domain V is isomorphic to a quotient space of V. Conversely, any quotient space V/S is the image of the surjective canonical projection map $q : V \to V/S$. Thus up to isomorphism, images of linear transformations on a vector space V are the same as the quotient spaces of V. The natural map $q : V \to V/S$ is a contraction and is an open map.

1.2.3 Compact operators and Fredholm operators

Definition 1.2.10

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the vector space of all bounded linear maps from X to Y. This is a Banach space when endowed with the operator norm $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$. Let $\mathbb{K}$ be a field. The dual space $\mathcal{L}(X, \mathbb{K})$ of X is denoted by X^*, and its elements are continuous linear functionals. A linear transformation $T : X \to Y$ is said to be *compact* if for each bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in X the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ in Y has a subsequence that converges in Y. The set of all compact operators from X to Y is denoted $\mathcal{K}(X,Y)$. We denote as $\mathcal{K}(X)$, the set $\mathcal{K}(X,X)$

Definition 1.2.11

Let X denote an infinite dimensional Banach space. Suppose $T \in \mathcal{L}(X)$ is a bounded operator with closed range. There are two natural Banach spaces associated with T. These are the kernel of T, $\ker(T) = \{x \in X : Tx = 0\} = N(T)$ and the co-kernel of T, $\text{coker}(T)$ or $\text{codim}(T) = X/R(T)$. The dim $\text{coker}(T) = \dim \ker(T^*)$ for every operator $T \in \mathcal{L}(X)$ whose range is closed and of finite dimension in X. An operator $T \in \mathcal{L}(X)$ is said to be *Fredholm* if both the dimensions of the kernel $N(T)$ and the co-kernel (codimension) are finite as complex vector spaces, (i.e. $\dim N(T) < \infty$, the codim $R(T) < \infty$ and its range is closed). The number $\dim \ker T$ measures the degree of failure of existence of solutions. We define the nullity of T to be the dimension of $\ker(T)$. The defect of T is the codimension of $T(X)$ in X denoted $\text{codim}(T)$

REMARK 1.2.12

$\text{codim}\, R(T) = \dim\, (X/R(T))$ and the index of T, denoted by $\text{ind}(T)$ is given by $\text{ind}(T) = \dim N(T) - \text{codim} R(T)$.

We shall denote the set of all Fredholm operators from X to Y by $F(X,Y)$ and $F(X) = F(X,X)$. Evidently, every invertible operator is a Fredholm operator of index zero. In fact, the dimension of the null space of either T or T^* will equal the codimension of the range of the other, so that $\text{ind}(T^*) = -\text{ind}(T)$. The classical index theorem asserts that the index acts as a homomorphism, in the sense that $\text{ind}(TS) = \text{ind}(T) + \text{ind}(S)$. Thus the product of two Fredholm operators is again a Fredholm operator. Since invertible operators are Fredholm operators of index zero, it follows that the Fredholm property as well as the index are preserved under similarity [20].

Definition 1.2.13

Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$.

(i) T is *upper semi-Fredholm* denoted $\Phi_+(X,Y)$ if $R(T)$ is closed and dim ker(T)$< \infty$.

(ii) T is *lower semi-Fredholm* denoted $\Phi_-(X,Y)$ if codim $R(T) < \infty$.

(iii) T is Fredholm if dim ker$T < \infty$ and codim$R(T) < \infty$ i.e., $\Phi_+(X,Y) \cap \Phi_-(X,Y)$.

(iv) T is *semi Fredholm* if it is either upper semi-Fredholm or lower semi-Fredholm, i.e., $T \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$

Definition 1.2.14

For every $T \in \mathcal{L}(X)$, the *Fredholm region* $\rho_e(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is a Fredholm operator$\}$, is an open subset of $\mathbb{C}$. Evidently, $\rho_e(T)$ contains the *resolvent spectrum* of T, $\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is invertible$\}$. Its complement $\sigma_e(T) = \mathbb{C} \setminus \rho_e(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a Fredholm operator$\}$ is the *essential spectrum* of T, which is a closed subset of the spectrum $\sigma(T)$ of T. The *spectrum* of T, $\sigma(T)$ is defined as the set $\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$. The Fredholm alternative ensures that $\sigma_e(T) \subseteq \{0\}$ for every compact operator $T \in \mathcal{L}(X)$. [33, 43]

1.2.4 Algebra and ideals

Definition 1.2.15

An *algebra* A is a vector space over a field $\mathbb{K}$ together with a bilinear map (product) xy such that $(xy)z = x(yz)$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$, and $(\alpha x)(\beta y) = \alpha\beta(xy)$, for all $x, y, z \in A$ and $\alpha, \beta \in \mathbb{K}$.

A *subalgebra* on A is a vector subspace B such that $b, b' \in B$ and $\alpha \in \mathbb{K}$ implies that bb', $b + b'$ and αb are in B (sometimes called a linear manifold). The algebra is real (or complex) if the scalar field is real (or complex). If the algebra A contains an element e such that $ex = xe$ for every $x \in A$, then e is called the *unit* of the algebra. It is unique if it exists. An algebra A is commutative if $xy = yx$ for all $x, y \in A$.

An *algebra norm* (*algebra-semi-norm*) $\|.\|$ on an algebra A is a norm (semi-norm) such that $\|xy\| \leq \|x\|\|y\|$, which shows that the norm is sub-multiplicative. $(A, \|.\|)$ forms *a normed algebra*. It is unital if it admits a unit 1. Moreover, $\|ab - a'b'\| \leq \|a\|\|b - b'\| + \|a - a'\|\|b'\|$ and so is jointly continuous [12].

Example 1.2.16

Let X be a linear space over $\mathbb{K}$. $\mathcal{L}(X)$ with the product defined by composition $(S \circ T)x = S(Tx)(x \in X)$, is an algebra

Definition 1.2.17

A complete normed algebra is called a *Banach algebra*. Given a normed algebra A with two Banach spaces X and Y such that $Y = X^*$ (the dual space of X), with $\langle , \rangle$ denoting the natural bilinear form, (i.e. $\langle x, y\rangle = y(x)$, $x \in X$, and $y \in Y$), then $(X, Y, \langle , \rangle)$ is a *Banach pairing*, and Y is a Banach A-module (the dual module of X). Multiplication is defined by $(ya)(x) = y(ax)(a \in A, x \in X, y \in Y)$. We then define $(X, Y, \langle , \rangle)$ as a *Banach A-module pairing*. If $(X, Y, \langle , \rangle)$ is a Banach pairing, then $(\mathcal{L}(X, Y), \langle , \rangle)$ with the norm of an operator $T \in \mathcal{L}(X)$ defined as $\|T\| = \max(\|T\|, \|T^*\|)$, is a Banach algebra. It is clear that $\mathcal{L}(X, Y, \langle , \rangle)$ is a sub-algebra of $\mathcal{L}(X)$ and that $\|.\|$ is an algebra norm. Let (S_n) be a Cauchy sequence in $\mathcal{L}(X, Y, \langle , \rangle)$ with respect to the norm $\|.\|$, then (S_n) and (S_n^*) are Cauchy sequences in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively. Therefore there exists $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$ with $\lim_{n\to\infty}\|S_n - S\| = 0$

and $\lim_{n\to\infty}\|S_n^* - T\| = 0$. For all $x \in X, y \in Y$, $\langle Sx, y\rangle = \lim_{n\to\infty}\langle S_n x, y\rangle = \lim_{n\to\infty}\langle x, S_n^* y\rangle = \langle x, Ty\rangle$. Thus $S \in \mathcal{L}(X, Y, \langle,\rangle)$, $S^* = T$ and $\lim_{n\to\infty}\|S_n - S\| = 0$. So the algebra $\mathcal{L}(X)$ is complete hence it is a Banach algebra.

A *Banach subalgebra* of a Banach algebra A is a closed subalgebra of A which contains 1. Banach algebras can be classified in general into function algebras, operator algebras or group algebras according as multiplication is defined pointwise, by composition or by convolution.
If we consider a non trivial Hilbert space H then $\mathcal{L}(H)$, the set of all bounded linear operators on H is a Banach algebra . This special Banach space has the adjoint operation $T \to T^*$. A subalgebra of $\mathcal{L}(H)$ which are self-adjoint are called C^*-algebras [12].

Definition 1.2.18

An *ideal* I in an algebra A is a vector space such that for all $a \in A$ and $b \in I$, we have $ab \in I$, or $ba \in I$ that is simultaneously a left and right ideal in A. There are two *trivial ideals*. These are $I = \{0\}$ and $I = A$ itself. An ideal is a linear subspace $I \subseteq A$ that is invariant under both the left and right multiplications, i.e. $AI + IA \subseteq I$. An algebra A is called *simple* if the trivial ideals are the only ideals. An ideal I in an algebra A is *proper* if $I \subset A$. A *maximal ideal* is a proper ideal that is not contained in any other proper ideal in A. An ideal I in an algebra A is *modular* if there exists an element $u \in I$ and $a \in A$ such that $a(1-u)$ and $(1-u)a$ are both in I.

Definition 1.2.19

A *homomorphism* from an algebra A to an algebra B is a linear map $\varphi : A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. A *monomorphism* is an injective homomorphism and an *isomorphism* is a bijective homomorphism. The kernel of φ, $\ker(\varphi)$ is an ideal I in A while the range or image of φ is a subalgebra of B and $\varphi(I) = 1$. Let I be an ideal in A, then the quotient map $q : A \to A/I$, is a homomorphism [11].

Let X denote an infinite dimensional complex Banach space and $\mathcal{L}(X)$ the set of bounded linear operators on X. The vector space $\mathcal{K}(X)$ of all compact operators on X, is a closed algebraic ideal on $\mathcal{L}(X)$. The quotient vector space $\mathcal{L}(X)/\mathcal{K}(X)$, whose elements are denoted as $[T]$ under the algebraic operations; $[S+T] = [S]+[T], [\lambda T] = |\lambda|[T], [S.T] = [S].[T]$ is a unital algebra with the identity $[I]$ as its unit element. We shall write $[I]$ simply as I. The quotient vector space under the quotient norm $\|T\|$ $=\inf\{\|S\| : S \in [T]\}$ $=\inf$ $\{\|S\| : S-T \in \mathcal{K}(X)\}$ is a Banach space. Since the quotient norm also satisfies the properties $\|[S][T]\| \leq \|[S]\|\|[T]\|$ and $\|[I]\| = 1$, $\mathcal{L}(X)/\mathcal{K}(X)$ is a unital Banach algebra. This algebra is called the Calkin algebra of X and is denoted $C(X)$. The quotient map (also called the canonical projection) of $\mathcal{L}(X)$ onto $C(X)$ will be denoted by q. That is $q : \mathcal{L}(X) \to C(X)$ is defined by $q(T) = [T] = T + \mathcal{K}(X)$. Since $\|q(T)\| \leq \|T\|$ it follows that q is a contraction. If A is a normed algebra then the closure of an ideal $\overline{I}$ is an ideal.

Suppose I is a proper ideal of an algebra A forming the quotient vector space A/I, then we have a natural linear map $x \in A \to \hat{x} = x + I \in A/I$ of A onto A/I. Since I is a two sided ideal one can define multiplication in A/I by:

$$(x+I)(y+I) = xy + I(x, y \in A).$$

This multiplication turns A/I into an algebra and the natural map $x \to \hat{x}$ becomes a surjective homomorphism of algebras having the given ideal I as its kernel. If I is modular, then A/I is unital and $(u+I)$ is the unit, where $u \in A$. Conversely, if A/I is unital then I is modular.

If I is a closed ideal in a normed algebra A then A/I is a normed algebra with the quotient norm $\|x+I\| = \inf_{y\in I}\|x+y\|$. If I is a proper closed ideal in a Banach algebra A with normalized unit 1 then the unit of A/I satisfies $\|\hat{1}\|$ $=\inf_{z\in A}\|1+z\| = 1$, hence the unit of A/I is also normalized. More generally, if I is a closed ideal in an arbitrary Banach algebra A (with or without unit) then A/I is a Banach space [33].

1.2.5 Convex sets

A set S is *convex* if it contains the line segment $\overrightarrow{x_1x_2}$, whenever $x_1, x_2 \in S$. Thus $\alpha_1 x_1 + \alpha_2 x_2 \in S$ whenever $\alpha_1 + \alpha_2 = 1$, and $\alpha_1, \alpha_2 > 0$. A subset S of a linear space X over a field $\mathbb{K}$ is said to be *absolutely convex* if $x, y \in S$, and α, $\beta \in \mathbb{K}$ are such that $|\alpha| + |\beta| \leq 1$ implies $\alpha x + \beta y \in S$

The *convex hull* of S is the intersection of all convex sets which contain S, denoted $\text{conv}(S)$. The convex hull of S consist of all points which are expressible in the form $\alpha_1 x_1 + \cdots + \alpha_n x_n$, where $x_1, \ldots, x_n \in S$, $\alpha_k > 0$ for each $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} \alpha_k = 1$.

Definition 1.2.20

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if its domain is a convex set and for all x, y in its domain, and all $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq f(x) + (1 - \lambda)f(y)$.

1.2.6 Spectra of linear operators

Let X be a Banach space and $T \in \mathcal{L}(X)$. Then *the spectrum* of T, $\sigma(T)$ is defined as the set $\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. We denote $T - \lambda I$ by T_λ. The *resolvent* set of T, $\rho(T) = \mathbb{C} \setminus \sigma(T)$. The *Spectral radius* of T, $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. It is clear that $0 \leq r(T) \leq \|T\|$.

The *Point spectrum* (the set of eigenvalues) of T, is defined as $\sigma_p(T) = \{\lambda \in \mathbb{C} : T_\lambda \text{ is not injective}\}$.

The *approximate point spectrum* of T is defined as, $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \text{there exists a sequence } (x_n), \|x_n\| = 1, \lim_{n\to\infty}(T - \lambda I)x_n = 0\}$. The *Surjectivity spectrum* of T is defined as $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective }\}$, while the *compression spectrum* of T by $\sigma_{com}(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I) \text{ is not dense in } X\}$.

1.2.7 The essential spectrum of bounded operators

Recall that an operator is Fredholm if its kernel and co-kernel are finite-dimensional. The *essential spectrum* of an operator T, usually denoted $\sigma_e(T)$, is the set of all complex numbers λ such that $\lambda I - T$ is not a Fredholm operator. The *essential spectral radius* is denoted by $r_e(T)$and is defined as $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. Some of the main properties of the essential spectrum known from the Hilbert space case studies include;

(a) The essential spectrum is always a closed set, and is a subset of the spectrum.

(b) If T is self-adjoint, the spectrum is contained on the real axis.

(c) The essential spectrum is invariant under compact perturbations. That is, if K is a compact, self-adjoint operator on a Hilbert space X, then the essential spectra of T and that of $T + K$ coincide.

1.2.8 Numerical Ranges

The numerical range of any linear operator T on a Hilbert space H is the set of complex numbers $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$.
Some important properties of the numerical range on a Hilbert space are given in the proposition below:

Proposition 1.2.21
Let T be a linear operator on a Hilbert space H. Its numerical range $W(T)$ satisfies the following properties:

(a) *$W(T)$ contains every eigenvalue of T,*

(b) *$W(T)$ lies in the disc $\{|w| \leq \|T\|\}$,*

(c) *The closure $\overline{W(T)}$ of $W(T)$ contains the spectrum of T, i.e., $\sigma(T) \subset \overline{W(T)}$,*

(d) *$W(T)$ is convex, and*

(e) *$W(T)$ is invariant under unitary equivalence of operators. Thus $W(T) = W(U^*TU)$ where U is unitary.*

For details, see [1, 5, 33, 34]. There are various ways to generalize the numerical range in the Banach algebras of bounded linear operators on both Hilbert and Banach spaces. In our study, we focus on the spatial numerical range and the algebraic numerical range as defined on the Banach space. Let X denote a complex Banach space, $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators acting on X with the operator norm $\|.\|$. The norm of an operator $T \in \mathcal{L}(X)$ is given by $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$. Its unit sphere is $S(X) = \{x \in X : \|x\| = 1\}$, and X^* its dual space. Let I denote the identity operator on X. For $T \in \mathcal{L}(X)$, the *spatial numerical range* is defined by $W(T) = \{\langle Tx, x^*\rangle : x \in X,\ x^* \in X^*,\ \|x\| = 1 = \|x^*\| = \langle x, x^*\rangle\}$, which can also be written as $W(T) = \{x^*(Tx) : x \in S(X), x^* \in S(X^*), x^*x = 1\}$, see [49, 53]. In the case where X is a Hilbert space the definition reduces to $W(T) = \{\langle Tx, x\rangle\ : x \in X, \|x\| = 1\}$ which is the well known definition of the Hilbert space numerical range.

For an operator T on a Banach space X with dual X^*, the *algebraic numerical range* is defined as the set $V(T) = \{\phi(Tx) : x \in X, \phi \in X^*, \|\phi\| = \|x\| = \phi(x) = 1\} \subseteq \mathbb{C}$, which can also be expressed as $V(T) = \{\phi(T) : \phi \in \mathcal{L}(X)^*, \|\phi\| = \phi(I) = 1\}$.

The algebraic numerical range is considered to be the closure of the spatial numerical range, i.e., $V(T) = \overline{W(T)}$. A thorough treatment of the spatial numerical range and the algebraic numerical range can be found in [9, 12, 13].

Definition 1.2.22

Let X be an infinite-dimensional Banach space and $T \in \mathcal{L}(X)$. Denote the *essential spatial numerical range* by $W_e(T)$ which is the set of all complex numbers λ with the property that there are nets $(u_\alpha) \subset X, (u_\alpha^*) \subset X^*$ such that $\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^*\rangle = 1$ for all α, $u_\alpha \to 0$ weakly and $\langle Tu_\alpha, u_\alpha^*\rangle \to \lambda$ [9].

Definition 1.2.23

Let X be an infinite-dimensional Banach space and $T \in \mathcal{L}(X)$. Denote by $\mathcal{K}(X)$ the ideal of all compact operators acting on a complex Banach space

X, and let q be the canonical projection from $\mathcal{L}(X)$ onto the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Denote further by $\|T\|_e$ *the essential norm* of T where $\|T\|_e = \inf\{\|T+K\| : K \in \mathcal{K}(X)\}$. The *essential algebraic numerical range* $V_e(T)$ of T is denoted by $V_e(T) = \{x^*((T+K)x)) : x \in \mathcal{L}(X), x^* \in \mathcal{L}(X)^*, \|x\| = \|x^*\| = x^*(x) = 1, K \in \mathcal{K}(X)\}$ The essential algebraic numerical range $V_e(T)$ of $T \in \mathcal{L}(X)$ can be characterized by $V_e(T) = V(q(T)) = V(T+K)$ where $K \in \mathcal{K}(X)$. That is $V_e(T) = V\left(q(T), \mathcal{L}(X)/\mathcal{K}(X), \|\|_e\right)$ [9].

Definition 1.2.24

Let X be a reflexive Banach space and $T \in \mathcal{L}(X)$. The essential spacial numerical range of T is defined by $W_e(T) = \text{conv}(V_e(T))$, where $\text{conv}(V_e(T))$ is the convex hull of $V_e(T)$, see[9].

1.3 Statement of the Problem

Let X be an infinite dimensional Banach space and $\mathcal{L}(X)$ the set of bounded linear operators on X. The algebraic and structural properties of the spectra and the numerical range for the Hilbert space operators have remarkably been studied over the past few decades. The corresponding study of the essential spectra and the essential numerical range for $T \in \mathcal{L}(X)$ which are operators on Banach spaces, does not have as much literature. In this study, we have determined the properties of the essential spectra and the essential numerical range for operators in Banach spaces. We have further investigated the relationship between the essential numerical range and the essential spectra.

1.4 Objective of the Study

The main objective is to investigate the properties of the essential numerical range and the essential spectra for operators on Banach spaces as a corresponding study to what has been done in Hilbert spaces. The specific objectives are:

(i) To determine the properties of the essential spectrum on Banach spaces.

(ii) To determine the properties of the essential numerical range on Banach spaces.

(iii) To investigate the relationship between the essential spectrum and the essential numerical range on Banach spaces.

1.5 Significance of the Study

The study of the properties of the essential numerical range of operators both on the Hilbert and Banach spaces provides a clear picture of the characteristics of the operators. The spectrum of an operator defines the algebraic properties of an operator regardless of the norm used. Since the spectrum is contained in the closure of the numerical range one can identify appropriate operators by studying their numerical ranges. The Banach space has more of these properties than the Hilbert space because of the diverse norms one can apply. The study of the essential spectra and the essential numerical ranges will be key in identifying and classifying operators. This makes their applications in different fields of study much easier. The study will therefore be important to physicists and applied mathematicians.

1.6 Methodology

To achieve our first objective we defined the various parts of the spectrum using the various studies done on Banach spaces. We used the closed range theorem to determine the duality properties of these parts of the spectrum and the corresponding parts of the adjoints. For the second objective, we used the methods applied by Barraa and Müller in their paper [9]. To determine the properties of the essential numerical range we considered another norm on the Calkin algebra instead of the usual essential norm on the Calkin algebra. This norm is a measure of non compactness. This had been introduced by Barraa and Müller [9]. We used the existing relations in literature on both Hilbert and Banach spaces between the spectra and numerical ranges and established the relationship between the essential spectrum and the essential numerical range especially for compact operators. Overall, we used existing methods, used previously by other mathematicians in this area of research. Discussions with experts in this area gave us more insights where necessary.

1.7 Organization of the thesis study

In Chapter 1, we give definitions and basic concepts to enable us give the problem statement of the study with ease. In Chapter 2, we review related literature that outlines the development of the study of the essential spectra and essential numerical ranges. It is in Chapter 3 where we establish some properties of the essential spectra while in Chapter 4 we study the properties of the essential numerical ranges. These properties are established in the general setting of Banach spaces. Finally we give a summary of our study as well as recommendations derived from our results in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

In this chapter we review related literature that will enable us expose the knowledge gap. The essential spectra and its properties have been considered in a number of studies over the past few decades. For instance, in 1981, Fialkow [21] described the Fredholm essential spectrum and index function for a class of operators on the Hilbert spaces. He also described the essential spectra and index functions of the restrictions of these operators to norm ideals. These results thus complemented the spectral analysis of multiplications initiated by Lumer [37].

Zima [58] established a theorem on the spectral radius of the sum of linear operators. The application of this theorem to a functional differential equation of neutral type was also given. In his study, Laursen [33] used the well established local spectral theory in investigating the semi-Fredholm spectrum of a continuous linear operator. He also examined the retention of the semi-Fredholm spectrum under weak intertwining relations where it is shown, inter alias, that if two decomposable operators are intertwined asymptotically by a quasiaffinity then they have identical semi-Fredholm spectra. The results are applied to multipliers on commutative semi-simple Banach algebras. Later, Alekho [5] in the year 2000 strongly developed the perturbation theory in many directions and found plural applications to wide classes of linear operators on Banach spaces (See, [31, 20]). The notion of an essential spectrum is important in the perturbation theory. So, essential spectra are subsets of the spectrum which are invariant under a perturbation of the given operator by operators of the concrete form. These spectra are obtained, for example, at the expense of the strengthening of the non-invertibility definition. On the other hand, the spectral theory of positive operators occupies a major place in the general concept of operators on Banach lattices (See [3] which has the main stages of the development and the achievements in this direction). However, in spite of numerous attempts, the general operator theory is united and connected with the theory of positive operators. In [1], the authors established some results concerning a certain class of semi-Fredholm and Fredholm operators via the concept of measures of non-compactness. Moreover, they established a fine description of the Schechter essential spectrum of closed densely defined operators. These results were then exploited to investigate the Schechter essential spectrum of a multidimensional neutron transport operator. Salvador and Slaviša [44] considered the conditions for continuity of spectrum which are given for the approximative point spectrum and defect spectrum on the set $T + \mathcal{K}(X)$, where $T \in \mathcal{L}(X)$ and $\mathcal{K}(X)$ is the set of compact operators. They assumed continuity of the spectrum at $T \in \mathcal{L}(X)$ and gave sufficient conditions for continuity of spectrum at $T + K$, where $K \in \mathcal{K}(X)$. Recently, Dehici [17] studied the diverse properties satisfied by the Wolf and Weyl essential spectra of bounded linear operators and their links with the structures of Banach spaces. He divided the structures of Banach spaces into two categories; those which have subspaces that have an unconditional basis and those which contain hereditarily indecomposable subspaces. He answered questions within the scope of bounded linear operators theory and Fredholm, semi-Fredholm perturbations by exploiting the two directions of the geometry of Banach spaces. Later the same year, the authors in [14] proved a variant of Hildebrandt theorem which asserts that the convex hull of the essential spectrum of an operator T on a complex Hilbert space is equal to the intersection of the essential numerical ranges of operators which are similar to T. As a consequence, it gives a necessary and sufficient condition for zero not being in the convex hull of the essential spectrum of A. Most recently in 2017, Breuer and

Latif [15] established that the essential spectrum of a Schrodinger operator T on $\mathcal{L}^2$ is equal to the union of the spectra of right limits of T. The natural generalization of this relation to $\mathbb{Z}^n$ is known to hold as well. In the study the possibility of generalizing this characterization of $\sigma_e(T)$ to Schrodinger operators was done. The essential spectrum σ_e of a bounded self-adjoint operator T is shown to be the complement in the spectrum of the discrete spectrum, i.e. $\sigma_e(T) = \sigma(T) \setminus \sigma_{dis}(T)$, where $\sigma_{dis}(T)$ is the set of isolated eigenvalues of T of finite multiplicity. They considered the possibility of extending a characterization that holds in the 1-dimensional case to describes the essential spectrum using the concept of right limits.
Apparently, most of the studies mentioned above on the properties of the essential spectra are based on the Hilbert space. The corresponding study on Banach spaces is much less complete. This study therefore focuses on the properties of the essential spectra on Banach spaces. Specifically, the various parts of the essential spectra will be defined and their duality relations established.

Similarly, just as is the case for the essential spectra, the study of the essential numerical range is several decades old now. Initiated by Lumer [37] in 1961, the concept of the numerical range was then well presented by Stampfli, Fillmore and Williams [49] in 1968. In their work they considered the numerical range in an arbitrary Banach algebra with identity, and studied its relation to various growth conditions on the resolvent. An extension of the spatial numerical range for a class of operators on locally convex spaces was then latter outlined by Moore in 1969 [49].
In [22], the authors remarked that the theory of the numerical range for linear operators on normed linear spaces and for elements of normed algebras is now firmly established and the main results of the study were conveniently presented by Bonsall and Duncan in 1971 and 1973 [11, 12]. An extension of the algebra numerical range for elements of locally m-convex algebras was presented by Giles and Koehler also in 1973 [22]. In their paper, the aforementioned authors contributed by extending the concept of spatial numeri-

cal range to a wider class of operators on locally convex spaces.

Lancaster [32] in 1975 gave two results which indicated a set theoretic relationship between the boundary of the numerical range and the essential numerical range. Several applications were derived including the first extensive study on the essential numerical range by Williams [54] in 1977, where a simple proof was given of Lancaster's theorem that the convex hull of the numerical and essential numerical ranges of a Hilbert space operator is the closure of the numerical range. In 1981 Legg and Townsend [36], gave attention to a general problem of the residue class $T + \mathcal{K}(X)$ in the Calkin algebra. They wanted to find out that if the residue class possesses a certain property, does there exist a representative $T + K$ possessing the same property? For example, if X is a separable Hilbert space, then for each $T \in \mathcal{L}(X)$ there exists $K \in \mathcal{K}(X)$ such that the spectrum of $T + K$ equals the Weyl spectrum of T, See [51]. For each $T \in \mathcal{L}(X)$, does there exist $K \in \mathcal{K}(X)$ such that the numerical range of $T + K$ equals the essential numerical range of T? This question is answered in the affirmative for $X = l_p, 1 < p < \infty$ and for $X = l_1$ in the case where the essential numerical range of A has no interior points. Specifically, given an operator T, there exists a compact perturbation $T + C$ such that the numerical range of $T + K$ equals the essential numerical range of A. This result has also been established for essentially Hermitian operators on l_1 [51]. In the same year [52], Stout, showed that any operator T is in the kernel (hull (compact operators)) in some $\mathcal{L}(X)$ if and only if 0 is in the essential numerical range of T. In 1988, Puttmadaiah and Gowda [39] characterized the spatial numerical range of a normal operator on a smooth reflexive Banach space to be closed and convex. This generalized the theorem for normal operators on Hilbert spaces. A few more results concerning the spatial numerical ranges of convexoid and iso-abelian operators are also obtained. In 2005, Baraa and Müller [9] introduced and studied the properties of the essential numerical range for Banach space operators. This generalized the corresponding well-known concept for Hilbert space operators. In [40], it is shown that the elliptical shape of the numerical range of quadratic

operators holds also for the essential numerical range. The latter is described quantitatively, and based on that, sufficient conditions are established under which the c-numerical range is also an ellipse. Several examples were considered, including singular integral operators with the Cauchy kernel and composition operators. Mecheri [38] then offered a simple proof that convexoid operators on Hilbert space are normaloid, and gave an example to show that the converse fails. Later, Abdolaziz and Mohammad [2] studied the spatial numerical range of operators on weighted Hardy spaces and conditions for closedness of numerical range of compact operators. After giving some background material, a useful formula for the spatial numerical range of operators on weighted Hardy space was given. They proved that the spatial numerical range of finite rank operators on weighted Hardy spaces is star shaped; though, in general, it does not need to be convex.
The study of the numerical range and numerical radius has an extensive history, and there is a great deal of current research on these concepts and their generalizations. In particular, the subject has connections and applications to various areas including C^* algebras, iterations methods, operator theory, dilation theory, Krein space operators, factorizations of matrix polynomials, unitary similarity and many others. (see [11, 12, 13, 28], and their reference) All these constitute a very active field of research in operator theory. The numerical range of an operator, like the spectrum, is a subset of the complex plane whose geometrical properties should say something about that operator. Our major concern was to compare the properties and utility of the essential numerical range and the essential spectrum. In [56] Williams showed that an operator $T \in \mathcal{L}(H)$ is normaloid if and only if it is convexoid. It is known that the part "if" in Williams result is not true [56]. An example which contradicts the part if in Williams result and a simple proof of the part only if of this result is given in this thesis. A necessary and sufficient condition for an operator $T \in \mathcal{L}(H)$ to be convexoid is also given. Again, based on the preceding literature, the properties of the essential numerical range are well established in the setting of Hilbert spaces while the

corresponding study in Banach spaces is wanting. In fact the only study we came about on the general Banach space is the work of Baraa and Müller [9]. There's therefore a need to study the properties of the essential numerical range on Banach spaces.

In relating the essential spectrum and the essential numerical range several results can be found in [3, 11, 12, 13, 25, 26, 27, 28, 33, 34, 40] and their references.

In 1967, Williams [55] presented an extension of the relation that the spectrum of an operator on a Hilbert space H is contained in the closure of the numerical range of the operator, to bounded linear operators on a Banach space, and certain nonlinear transformations on a real or complex Hilbert space. The extension was mild, as he showed that if zero is not in the closure of the numerical range of an operator A, then the spectrum of $(A^{-1}B)$ is contained in $\overline{W(B)}/\overline{W(A)}$ for any operator B on H. Here the set on the right is by definition the set of quotients b/a with b in $\overline{W(B)}$ and a in $\overline{W(A)}$. The extension had interesting consequences. For example it implied that if A is strictly positive and B is greater than 0, then the product AB has a nonnegative spectrum. Also, if A is positive and B is self-adjoint then the product AB has real spectrum. In 1968, Stampfli, Fillmore and Williams [49] defined an essential numerical range for linear bounded operators in a Hilbert space. It was shown that the essential numerical range is the intersection of the closed compact cosets and the convex hull of the essential spectrum of an operator T was contained in the essential numerical range of T. According to a result of Stampfli, Fillmore and Williams [49], if T is normal then the convex hull of the essential spectrum of an operator T equals the essential numerical range of T, but equality does not hold in general. In their work the convex hull of the essential spectrum of a bounded linear operator defined on a separable Hilbert space is obtained in terms of intersections of appropriate bi-operative numerical ranges. After establishing that the essential numerical range on a Hilbert space is invariant under norm-preserving isomorphisms, Stampfli, Fillmore and Williams [49] considered the numerical range in an arbitrary

Banach algebra with identity, and studied its relation to various growth conditions on the resolvent. They have shown that, if q is an algebra homomorphism of norm 1, from a complex Banach algebra A with unit into another such algebra A', then the essential numerical range is contained in the essential numerical range of (x) for each x in A. It has been established that the convexity of the numerical range leads to effective numerical algorithms for its graphical representation, at least for finite dimensional operators [25, section 5.6], and this has implications for locating the spectrum. It has been shown that the entire spectrum of an operator T, and hence its convex hull, belongs to the closure of the numerical range of T [25, section 1.2]. Hildebrand [29] has shown that, upon intersecting the closures of the numerical ranges of all the bounded operators on a Hilbert space H similar to T, one obtains precisely the convex hull of the spectrum. He has also shown that the numerical range lacks similarity invariance unlike the spectrum. In this way, the numerical range plays a role in spectral location similar to that of the Gershgorin set of matrix theory (see, for example, [25, section 5.2]. For maps ϕ assuming some kind of normal form in a Banach space the numerical range of the compositions of ϕ, C_ϕ is easy to determine. The challenge arises in trying to show that composition operators induced by maps conformally conjugate to such normal forms have similarly shaped numerical ranges. Each elliptic automorphism of the unit sphere is conformally conjugate to one that fixes the origin, that is to a rotation. The normal form for such an elliptic automorphism is therefore a map. The notion of essential numerical range appears naturally here, and this set is shown to be characterized in the way one would expect by analogy with the essential spectrum. In 1973 Amelin, [6] defined a numerical range for two closed, linear operators for the purpose of obtaining some new results on the stability of index of a Fredholm operator perturbed by a bounded or relatively bounded operator.

Zarantonello [57] introduced the concept of numerical range of nonlinear Hilbert space operator, and proved that the numerical range contains the spectrum. He applied this connection for solving the nonlinear functional equations. In 1991, Verma [53], gave a generalization of Zarantonellos result on the numerical range of a nonlinear Hilbert space valued operator to the case of the Banach space valued operator. The results obtained provide a constructive method for solving the nonlinear functional equations similar to that of Zarantonello [57]. Agnes [4], in 2014, showed that the numerical of positive operators on Banach lattices have many properties that resemble the spectral properties of positive operators. In particular, they have shown that the numerical radius is always contained in the closure of the numerical range. Moreover, the numerical range is symmetric with respect to the real axis. For irreducible operators on suitable L_p-spaces Agnes proved a rotational symmetry for the numerical range. In addition, they determined the numerical range and radius for some concrete operators. More research on the relationship of the essential numerical range and the essential spectra in Banach spaces will make a strong and better understanding of the many new and important operator classes coming out of computational linear algebra and applications. Almost all of the available literature have their results well established on the Hilbert spaces. But for Banach spaces each class of operators carries particular structure properties reflecting those of the class of applications and those of the norm used. This makes their study more tedious. In this way, the numerical range will remain a vital and growing part of operator theory whose literature is still scarce.

The following fundamental theorems of functional analysis will be useful in this study:

Theorem 2.0.1 (Open Mapping Theorem)

Let X and Y be Banach spaces. Then every surjective continuous linear mapping $T : X \to Y$ is open.

Theorem 2.0.2 (Closed Graph Theorem)

Let X and Y be Banach spaces. Then every closed mapping $T : X \to Y$ is continuous.

Theorem 2.0.3 (Ascoli's theorem)

If x_n is a bounded sequence from $C[a,b]$ such that $x_n's$ form an equi-continuous sequence (i.e., to each $\varepsilon > 0$ corresponds a $\delta > 0$ such that $|x(t_1) - x(t_2)| < \varepsilon$ whenever $x \in \mathbb{C}$ and $t_1, t_2 \in [a,b]$ such that $|t_1 - t_2| < \delta$), then x_n contains a convergent subsequence. (Convergent in the topology of $C[a,b]$)

Theorem 2.0.4 (Krein-Milman)

A non-void compact convex subset K of a locally convex linear topological space X has at least one extremal point.

Theorem 2.0.5 (Closed Range Theorem)

Let X and Y be Banach spaces. Let $D(T)$ be a dense linear subspace of X and let $T : D(T) \to Y$ be a closed linear operator with null space $N(T) = \{x \in D(T) : Tx = 0\}$ and the range space $R(T) = \{Tx : x \in D(T)\}$. Then the following are equivalent:

(a) $R(T)$ *is closed in* Y,

(b) $R(T^*)$ *is closed in* X^*,

(c) $R(T^*) = N(T)^\perp$,

(d) $T : D(T) \to R(T)$ *is open,*

(e) $T^* : D(T^*) \to R(T^*)$ *is open, and*

(f) $R(T) = N(T^*)^\perp$.

Theorem 2.0.6 (Characterization of compact operators)

Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$, then the following conditions are equivalent:

(a) *T is compact.*

(b) *For each bounded set $S \in X$ the set $T(S)$ is relatively compact in Y.*

(c) *For each bounded sequence $\{x_n\}$ in X the sequence $\{Tx_n\}$ admits a subsequence that converges in Y.*

The following theorem is a characterization of the approximate point spectrum which is well known in literature, (see for instance [8]).

Theorem 2.0.7 (Characterization of the approximate point spectrum)
For $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, the following are equivalent:

(a) $\lambda I - T$ *is bounded below (i.e. there exists* $c > 0$ *such that* $\|(\lambda I - T)x\| \geq c\|x\|$*).*

(b) $N(\lambda I - T) = \{0\}$ *and* $R(\lambda I - T)$ *closed.*

(c) $\lambda \notin \sigma_{ap}(T)$.

The following characterization of the spectra will be useful in the next chapter.

Proposition 2.0.8
Let X be a non-zero Banach space and $T \in \mathcal{L}(X)$ be a non-invertible isometry. Then:

(i) $\sigma(T) = \nabla(0,1)$ *and*

(ii) $\sigma_{ap}(T) = \partial\nabla(0,1)$

where $\nabla(0,1)$ *denotes a closed unit ball and* $\partial\nabla(0,1)$ *its boundary.*

For proof, see [8].

Theorem 2.0.9 (Annihilator Theorem)
Let M be a closed linear subspace of a normed vector space X. Let $q : X \to X/M$ denote the canonical quotient mapping. Let $T \in \mathcal{L}(X)$ be an operator for which $T(M) \subseteq M$. Then the following canonical identifications hold:

(i) $(X/M)^* \cong M^\perp$ *and*

(ii) $X^*/M^\perp \cong M^*$.

Theorem 2.0.10 (Hahn-Banach Extension theorem)

Let E be a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $F \subseteq E$ be a linear subspace. Then for every $\psi \in F^*$ there exists some $\phi \in E^*$ such that $\phi = \psi$ on F and $\|\phi\| = \|\psi\|$.

Theorem 2.0.11 (Characterization of compact sets)

Let (X, d) be a metric space and $A \subseteq X$, then the following characterization of compact sets holds:

(i) *A is precompact if and only if every sequence in A contains a sub-sequence that is Cauchy,*

(ii) *A is compact if and only if every sequence in A contains a convergent sub-sequence,*

(iii) *A is compact if and only if A is precompact and complete,*

(iv) *A is precompact if and only if its closure $\overline{A}$ is precompact and*

(v) *A is relatively compact if and only if A is precompact and $\overline{A}$ is complete.*

For details and proofs of the above results, we refer to [8, 20, 24, 33, 43, 45, 47].

CHAPTER 3

PROPERTIES OF ESSENTIAL SPECTRA

In this chapter, X denotes an infinite dimensional complex Banach space and $\mathcal{L}(X)$, the set of all bounded linear operators on X. In Sections 3.1 and 3.2, we review some properties of compact operators and Fredholm operators which are vital in this study. In Section 3.3, we establish some algebraic properties of the essential spectra, while it is in Section 3.4, where we present our major results of this chapter which correspond to the first objective of this study. Specifically, we define various parts of the essential spectra and establish their duality relations.

3.1 Compact operators

In the next proposition, we give some basic properties of compact operators. Even though known in literature, we give our alternative shorter proofs.

Proposition 3.1.1

Let E and F be normed spaces. Then the following hold:

(a) *A linear transformation $T : E \to F$ is compact precisely when, for each bounded sequence $\{x_i\}_{i\in\mathbb{N}}$ in E, the sequence $\{Tx_i\}_{i\in\mathbb{N}}$ has a subsequence that converges in F.*

(b) *Every compact linear transformation $T : E \to F$ is continuous.*

(c) *Every continuous linear transformation $T : E \to F$ with dim $R(T) < \infty$ is compact.*

(d) *The identity $I : E \to E$ is compact precisely when dim $E < \infty$.*

(e) *For $T \in \mathcal{K}(E)$ and $0 \neq \lambda \in \mathbb{F}$, we have dim $(N(\lambda I - T)) < \infty$.*

PROOF. **(a)** Follows from general characterization of relative compactness given in Theorem 2.0.11.

(b) This follows from the fact that relative compact sets in normed spaces are bounded.

(c) By continuity of T, there exists $c > 0$ such that $T($ Ball $(E) \subseteq c\,$Ball$(R(T))$. The latter set is compact since dim $R(T) < \infty$ and so $\overline{T(Ball(E))}$ is compact. This means that T is compact.

(d) From assertion (c), it is clear that $I : E \to E$ with dim $(R(I)) < \infty$ is compact. The result now follows immediately from the fact that $R(I) = E$. Conversely, if I is compact, then Ball(E) is relatively compact hence dim $(E) < \infty$.

(e) Let $\mathcal{N} = N(\lambda I - T)$ denote the null space of $\lambda I - T$. This means $(\lambda I - T)x = 0$ for all $x \neq 0$ on $\mathcal{N}$. Hence $\lambda I \mid_{\mathcal{N}} = T \mid_{\mathcal{N}}$ are compact, where $T \mid_{\mathcal{N}}$ is a restriction of T on $\mathcal{N}$. It then follows that dim $(\mathcal{N}) < \infty$, by assertion (d).

□

Example 3.1.2

Let $\alpha_{ij} \in \mathbb{C}^n$ for all $i, j \in \mathbb{N}$ such that $\alpha = \sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 < \infty$. Then the definition

$$Tx = \left(\sum_{k=1}^{\infty} \alpha_{ik} x_k \right)_{i \in \mathbb{N}}$$

yields a compact linear map $T : l^2 \to l^2$.

$$\text{Note: } Tx = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & . & . & . \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & . & . & . \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \end{pmatrix}.$$

PROOF. For each $x \in l^2$, we have by Cauchy-Schwartz inequality,

$\|Tx\|_2^2 = \sum_{i=1}^{\infty} |Tx_i|^2 = \sum_{i=1}^{\infty} | \sum_{k=1}^{\infty} \alpha_{ik} x_k|^2 \leq \sum_{i=1}^{\infty} (\sum_{k=1}^{\infty} |\alpha_{ik}||x_k|)^2 \leq \sum_{i=1}^{\infty} (\sum_{k=1}^{\infty} |\alpha_{ik}|^2)(\sum_{k=1}^{\infty} |$

So $\|Tx\|_2 \leq \sqrt{\alpha}\|x\|_2$ for all $x \in l^2$. $(*)$ Thus $T : l^2 \to l^2$ is bounded and linear with $\|T\| \leq \sqrt{\alpha}$.

It remains to show that $T(\text{Ball}(l^2))$ is relatively compact in l^2. It suffices to show that $T(\text{Ball}(l^2))$ is bounded and uniformly convergent. l^2-convergence holds for the sequences in $T(\text{Ball}(l^2))$.

Now for all $x \in \text{Ball}(l^2)$, we have $\|Tx\|_2 \leq \sqrt{\alpha}$ by $(*)$. Also for all $n \in \mathbb{N}$, we have $\sum_{i=n}^{\infty} |Tx_i|^2 \leq \sum_{i=n}^{\infty} (\sum_{k=1}^{\infty} |\alpha_{ik}|^2 |x_k|^2) \leq \sum_{i=n}^{\infty} \sum_{k=1}^{\infty} |\alpha_{ik}|^2 < \infty$, independent of x. Hence the sequence is bounded and uniformly convergent. $\square$

3.2 Fredholm Operators

Let X, Y and Z be Banach spaces over $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$. Then the following proposition gives some characterization of Fredholm operators. (See [46, 47, 33, 34])

Proposition 3.2.1

Let $F(X, Y)$ be the set of all Fredholm operators from X into Y. Then:

(a) *$T \in F(X, Y)$ if and only if $T^* \in F(X^*, Y^*)$ and $ind(T^*) = -ind(T)$.*

(b) *$T \in F(X, Y)$ if there exist linear maps $P, Q : Y \to X$ with $PT \in F(X)$ and $TQ \in F(Y)$. Then $T \in F(X, Y)$.*

(c) *$T \in F(X,Y)$ and $S \in F(X,Y)$, implies $ST \in F(X,Y)$ and $ind(ST) = ind(S) + ind(T)$.*

Having stated the proposition above, we give an alternative prove of the next result which gives the necessary and sufficient conditions for an operator to be Fredholm. We refer to [33, 46] for details.

Theorem 3.2.2

For $T \in \mathcal{L}(X,Y)$, the following are equivalent:

(a) *T is Fredholm,*

(b) *There exists $S \in F(Y,X), P \in \mathcal{L}(X)$ with $P^2 = P$ and $\dim R(P) < \infty, Q \in \mathcal{L}(Y)$ with $Q^2 = Q$ and $\dim R(Q) < \infty$ such that $ST = I_X + P$ and $TS = I_Y + Q$, where I_X and I_Y are identities in X and Y respectively.*

PROOF. (b) implies (a): Since P and Q are compact, we know that $I_X + P$ and $I_Y + Q$ are Fredholm. So ST and TS are Fredholm. Now part (b) of proposition above applies.

(a) implies (b): Since $\dim N(T) < \infty$, there exists some $P \in \mathcal{L}(X)$ with $P^2 = P$ and $R(P) = N(T)$. For $X_1 = N(P)$, we obtain from $P^2 = P$ that $X = R(P) \oplus N(P) = N(T) \oplus X_1$.

Moreover we have $Y = R(T) \oplus Y_1$ for some finite dimensional space $Y_1 \subseteq Y$. So again there exists some $Q \in \mathcal{L}(Y)$ with $Q^2 = Q$ and $R(Q) = Y_1$ and $R(I - Q) = R(T)$. Now, $T|_{X_1} : X_1 \to R(T)$ is a bijective linear operator between Banach spaces, so $(T|_{X_1})^{-1} : R(T) \to X_1$ is continuous by the open mapping theorem.

Define $S = i \circ (T|_{X_1})^{-1} \circ (I_Y + Q)$ where $i : X_1 \to X$ is the canonical inclusion mapping. So $S \in \mathcal{L}(X,Y)$ with $TS = I_Y + Q$. Also $ST = 0 = I_{X_1} + P$ on X. But $N(T) = R(P)$ so $ST = I_{X_1} = I_{X_1} + P$ and $X_1 = N(P)$. Since $X = N(T) \oplus X_1$ it follows that $ST = I_X + P$ on X as desired. □

Next, we provide with details an example of a Fredholm operator.

Example 3.2.3

Consider the unilateral right shift $T : l^1 \to l^1$ given by $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$ where $\mathbf{x} = (x_k)_{k \in \mathbb{N}} \in l^1$. We would like to find the eigenvalues of T, the spectra and the approximate point spectra of T as well as all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is a Fredholm operator and we compute its index.

We begin by proving that T is a non-invertible isometry. For $\mathbf{x} = (x_1, x_2, x_3, \cdots)$ and $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$, we find that $\|T\mathbf{x}\|_1 = 0 + \sum_{k=1}^{\infty} |x_k| = \|\mathbf{x}\|_1$. Hence T is an isometry. By taking supremum over all $\mathbf{x} \in \text{Ball}(l^1)$ on both sides, we get that $\|T\| = 1$.

All the elements in $R(T)$ are of the form $(0, x_1, x_2, x_3, \ldots)$. Therefore the basis element $(1, 0, 0, \cdots) \notin R(T)$, so $R(T) \neq l^1$. Hence T is not surjective and therefore not invertible. This makes T a non-invertible isometry. Using Proposition 2.0.8, we then conclude that $\sigma(T) = \nabla(0, 1)$ (a closed unit disc) and $\sigma_{ap}(T) = \partial\nabla(0, 1)$ from Proposition 2.0.8.

Next, we now show that $\sigma_p(T) = \emptyset$. Let $\lambda \in \sigma_p(T)$. This is equivalent to $\lambda\mathbf{x} = T\mathbf{x}$ for some $\mathbf{x} \in l^1$, $\mathbf{x} \neq 0$. Since for any $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, we have $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$; it follows that $\lambda x_1 = 0, \lambda x_2 = x_1, \lambda x_3 = x_2, \ldots$. Now if $\lambda = 0$, we have $x_1 = x_2 = x_3 = \cdots = 0$ implying that $\mathbf{x} = 0$. Hence $\lambda = 0$ is not an eigenvalue of T.

On the other hand, if $\lambda \neq 0$, then $x_1 = x_2 = x_3 = \cdots = 0$. Again $\mathbf{x} = 0$. So $\lambda \neq 0$ is not an eigenvalue of T and therefore $\sigma_p(T) = \emptyset$.

Now we would like to find all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is Fredholm. The following three cases arise:

Case1 : $|\lambda| > 1$.

If $|\lambda| > 1$, then $\lambda \notin \sigma(T)$ and this means that $\lambda I - T$ is invertible i.e. bijective. So $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T) = l^1$. Hence $\dim(N(\lambda I - T)) = 0$ and codim $(R(\lambda I - T)) = 0$. Therefore ind $(\lambda I - T) = 0 + 0 = 0$ and so $\lambda I - T$ is Fredholm.

Case2 : $|\lambda| = 1$.

If $|\lambda| = 1$, then $\lambda \in \sigma_{ap}(T)$. Recall from Theorem 2.0.7, $\lambda \notin \sigma_{ap}(T)$

implies $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T)$ closed. Since $\sigma_p(T) = \emptyset$, it follows that $N(\lambda I - T) = \{0\}$. So $\lambda \in \sigma_{ap}(T)$ is equivalent to $R(\lambda I - T)$ fails to be closed. Therefore codim $R(\lambda I - T) = \infty$. So $\lambda I - T$ is not Fredholm.

Case3 : $|\lambda| < 1$.

If $|\lambda| < 1$, then it follows that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{ap}(T)$

But $\lambda \notin \sigma_{ap}(T)$ if and only if $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T)$ is closed (Theorem 2.0.7). Hence dim $(N(\lambda I - T)) = 0$. Now we want to find codim $R(\lambda I - T) =$ dim $l^1/R(\lambda I - T)$. Recall from Theorem 2.0.9 that $(l^1/R(\lambda I - T))^* \cong R(\lambda I - T)^\perp = N(\lambda I - T^*)$. We know that: $T^*(\mathbf{x}) = (x_2, x_3, x_4, \cdots)$, the unilateral left shift.

Therefore $\mathbf{x} \in N(\lambda I - T^*)$ implies that $(\lambda I - T^*)\mathbf{x} = 0$ and hence $\lambda \mathbf{x} = T^*\mathbf{x}$. Comparing the elements of $\lambda \mathbf{x}$ and $T^*\mathbf{x}$, we find that $\lambda x_1 = x_2, \lambda x_2 = x_3, \lambda x_3 = x_4$, etc. Taking $x_1 = x_1$, and $x_2 = \lambda x_1$, then $x_3 = \lambda x_2 = \lambda^2 x_1$, while $x_4 = \lambda x_3 = \lambda^3 x_1$, and so on. This gives a general pattern of $x_k = \lambda^{k-1} x_1$. Hence $\mathbf{x} = x_1(1, \lambda, \lambda^2, \cdots)$ and so the dim $N(\lambda I - T^*) = 1$. From Theorem 2.0.5, it therefore follows that codim $(R(\lambda I - T)) = 1$. But the dim $(N(\lambda I - T)) = 0$. Therefore ind $(\lambda I - T) =$ dim $(N(\lambda I - T))-$ codim $(R(\lambda I - T)) = 0 - 1 = -1$ and hence $\lambda I - T$ is Fredholm.

3.3 Algebraic properties of the essential spectrum

In this section, we give some algebraic properties of the essential spectrum of bouded operators acting on Banach spaces. We begin by proving the following characterization;

Theorem 3.3.1

Let X be a complex Banach space. Let the quotient space $C(X) = \mathcal{L}(X)/\mathcal{K}(X)$ *be endowed with the usual vector space operations and the canonical quotient norm. Let* $q : \mathcal{L}(X) \to C(X)$ *denote the corresponding quotient map-*

ping so that $q(T) = T + \mathcal{K}(X)$ *for all* $T \in \mathcal{L}(X)$*. For* $T^* \in \mathcal{L}(X^*)$*, define an equivalent map* $\tilde{q}$ *by* $\tilde{q}(T^*) = T^* + \mathcal{K}(X^*)$*. Then:*

(a) $C(X)$ *is an algebra with respect to the multiplication:*

$$(T + \mathcal{K}(X))(S + \mathcal{K}(X)) = TS + \mathcal{K}(X)$$

for all $T, S \in \mathcal{L}(X)$. $C(X)$ *is known as the Calkin algebra.*

(b) *Given an arbitrary* $T \in \mathcal{L}(X)$*, then* T *is a Fredholm operator on* X *if and only if* $q(T)$ *is invertible in the Calkin algebra* $C(X)$

(c) $\sigma_e(T) = \sigma(q(T))$

(d) $\sigma_e(T)$ *is compact and nonempty for each* $T \in \mathcal{L}(X)$ *provided that* X *is of infinite dimension.*

(e) *For* $T^* \in \mathcal{L}(X^*)$*, we have that* $\tilde{q}(T^*) = q(T)^*$*. In particular,* $C(X)^* = C(X^*)$.

PROOF. To prove (a), it suffices to prove that the multiplication $(T + \mathcal{K}(X))(S + \mathcal{K}(X)) = TS + \mathcal{K}(X)$ is well defined and that the quotient norm is sub-multiplicative. Indeed for the well definition, let $T_1 + \mathcal{K}(X) = T_2 + \mathcal{K}(X)$ and $S_1 + \mathcal{K}(X) = S_2 + \mathcal{K}(X)$. Then $T_1 - T_2 \in \mathcal{K}(X)$ and $S_1 - S_2 \in \mathcal{K}(X)$. Assume $T_1 = T_2 + T_3$ and $S_1 = S_2 + S_3$ for some $T_3, S_3 \in \mathcal{K}(X)$. Then $T_1S_1 = (T_2 + T_3)(S_2 + S_3) = T_2S_2 + T_2S_3 + T_3S_2 + T_3S_3$. Thus $T_1S_1 = T_2S_2 + \mathcal{K}(X)$ (since $T_2S_3 + T_3S_2 + T_3S_3 \in \mathcal{K}(X)$) which further implies that $T_1S_1 - T_2S_2 \in \mathcal{K}(X)$. Hence $T_1S_1 + \mathcal{K}(X) = T_2S_2 + \mathcal{K}(X)$ as desired.
Next, we prove that the quotient norm is sub-multiplicative. To do this we need to show that $\|q(T)q(S)\| \leq \|q(T)\|\|q(S)\|$ for all $T, S \in \mathcal{L}(X)$.
Now suppose $\varepsilon > 0$ is arbitrarily chosen. Then by definition of the quotient norm, there exists $T_1, S_1 \in K(X)$ such that $\|q(T)\| + \varepsilon \geq \|T + T_1\|$ and $\|q(S)\| + \varepsilon \geq \|S + S_1\|$. Then $\|q(T)q(S)\| = \|q(T + T_1)q(S + S_1)\| \leq \|(T + T_1)\|\|(S + S_1)\| \leq (\|q(T)\| + \varepsilon)(\|q(S)\| + \varepsilon)$. Letting $\varepsilon \to 0$, we obtain the desired result.

For (b), we need to prove that if $T \in \mathcal{L}(X)$, then T is Fredholm on X if and only if $q(T)$ is invertible in $C(X)$. Now, since T is Fredholm on X, then by Theorem 3.2.2 choose S, P and $Q \in \mathcal{L}(X)$, with $P^2 = P, Q^2 = Q$, dim $(R(P)) < \infty$ and dim $(R(Q)) < \infty$ such that $TS = I + P$ and $ST = I + Q$. Since dim $(R(P)) < \infty$ and dim $(R(Q)) < \infty$, it follows that P, Q are compact, that is, $P, Q \in \mathcal{K}(X)$. But $(T + \mathcal{K}(X))(S + \mathcal{K}(X)) = I + \mathcal{K}(X)$ and $(S + \mathcal{K}(X))(T + \mathcal{K}(X)) = I + \mathcal{K}(X)$, implies that $(T + \mathcal{K}(X))^{-1} = (S + \mathcal{K}(X))$. Hence $q(T)$ is invertible in $C(X)$.

Conversely, suppose $q(T)$ is invertible in $C(X)$ and let $S \in \mathcal{L}(X)$ be such that $q(S) = q(T)^{-1}$. Then $q(S)q(T) = q(ST)$ and $q(T)q(S) = q(TS)$. But $TS = I + K(X)$ and $ST = I + K(X)$. So both TS, ST are Fredholm. From this result it is clear that $N(T) \subseteq N(ST)$ which is finite dimensional, and $R(T) \supseteq R(ST)$ which is finite co-dimensional. Therefore $N(T)$ is finite dimensional while $R(T)$ is finite co-dimensional and hence T is Fredholm, as required.

For (c), recall that $\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not Fredholm in } X\}$. We wish to show that $\sigma_e(T) = \sigma(q(T))$, that is, for $\lambda \in \mathbb{C}$, we have $\lambda \notin \sigma_e(T)$ if and only if $\lambda \notin \sigma(q(T))$ or for $\lambda \in \mathbb{C}, \lambda I - T$ is Fredholm on X if and only if $\lambda I - q(T)$ is invertible in $C(X)$. It thus suffices to prove the fact that T is Fredholm on X if and only if $q(T)$ is invertible in $C(X)$. For the proof of this fact simply look at (b).

For (d), we need to prove that $\sigma_e(T)$ is compact and nonempty for each $T \in \mathcal{L}(X)$ provided X is infinite dimensional. Indeed, if dim $X < \infty$, then $\mathcal{L}(X) = \mathcal{K}(X)$ and so $C(X) = \mathcal{K}(X)/\mathcal{K}(X)$ which yields $\sigma(q(T)) = \emptyset$. This contradicts the fact that the spectrum of a nonzero operator is nonempty and compact, so dim $X = \infty$. Hence $\sigma_e(T) = \sigma(q(T))$ is compact and nonempty whenever dim $X = \infty$.

The assertion (e) is clear from the fact that an operator is compact if and only if its adjoint is compact. Indeed, $q(T)^* = (T + \mathcal{K}(X))^* = T^* + \mathcal{K}(X)^* = T^* + \mathcal{K}(X) = \tilde{q}(T^*)$. □ In the next result, we relate the essential spectrum of an operator and that of its adjoint. Even though known, we apply the closed range theorem to simplify the proof.

Theorem 3.3.2

Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then $\sigma_e(T) = \sigma_e(T^*)$

PROOF. It suffices to prove that for $\lambda \in \mathbb{C}$, $\lambda \notin \sigma_e(T)$ if and only if $\lambda \notin \sigma_e(T^*)$. Equivalently, for $\lambda \in \mathbb{C}$, $\lambda I - T$ is Fredholm if and only if $\lambda I - T^*$ is Fredholm. Thus it suffices to prove that $T : X \to Y$ is Fredholm if and only if $T^* : Y^* \to X^*$ is Fredholm.

Recall that T is Fredholm if and only if dim $(N(T)) < \infty$, and dim $(Y/R(T)) < \infty$. It therefore suffices to show that dim $(N(T^*)) =$ dim $(Y/R(T))$ and dim $(X^*/R(T^*)) =$ dim $(N(T))$.

Now $N(T^*) = R(T)^\perp \cong (Y/R(T))^*$. Since $\dim(Y/R(T)) < \infty$, it follows that dim $(Y/R(T))^* < \infty$. So dim $(N(T^*)) =$ dim $(Y/R(T))^* = \dim(Y/R(T)) < \infty$. By the closed range theorem, $R(T^*)$ is closed in X^* and equals $N(T)^\perp$.

Now $X^*/R(T^*) = X^*/N(T)^\perp \cong N(T)^*$.

Since dim $(N(T)) < \infty$, we have dim $(N(T)^*) =$ dim $N(T)$ and so,

dim $(X^*/R(T^*)) =$ dim $(N(T)) < \infty$. Therefore the dimension of $N(T)$ and $Y/R(T)$ are finite and hence T is Fredholm.

Conversely, if T^* is Fredholm, then dim $(N(T^*)) < \infty$ and dim $(X^*/R(T^*)) < \infty$. It therefore suffices to show that dim $(N(T)) =$ dim $(X^*/R(T^*))$ and dim $(Y/R(T)) =$ dim $(N(T^*))$. Now $X^*/R(T^*) = X^*/N(T))^\perp \cong N(T)^*$. Since $\dim(X^*/R(T^*)) < \infty$, it follows that dim $(N(T)^*) =$ dim $(N(T)) < \infty$. By the closed range theorem, $N(T^*) = R(T)^\perp \cong (Y/R(T))^*$.

But dim $(N(T^*)) < \infty$, so it implies that dim $(Y/R(T))^*) =$ dim $(Y/R(T)) < \infty$, as desired. $\square$ An immediate consequence of the above theorem is the following corollary:

Corollary 3.3.3

For $S, T \in \mathcal{L}(X), \sigma_e(T+S) = \sigma_e((T+S)^*)$.

PROOF. Since S,$T \in \mathcal{L}(X)$, $S + T \in \mathcal{L}(X)$, and therefore from Theorem 3.3.2, it follows that $\sigma_e(T+S) = \sigma_e((T+S)^*)$ as desired. $\square$

Another consequence that relates the essential spectral radius of an operator T, $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$ and that of its adjoint is the following:

Corollary 3.3.4

For S, $T \in \mathcal{L}(X)$, the following hold;

(i) $r_e(T^*) = r_e(T)$.

(ii) $r_e(T+S) = r_e((T+S)^*)$.

PROOF. Assertion (i) follows immediately from Theorem 3.3.2 above as well as the definition of the essential spectral radius, while (ii) is a consequence of Corollary 3.3.3. □ The next theorem relates the essential spectrum of an operator and that of its scalar multiple.

Theorem 3.3.5

For $T \in \mathcal{L}(X), \alpha \in \mathbb{R}$ with $\alpha \neq 0$, we have $\sigma_e(\alpha T) = \alpha\sigma_e(T)$. Moreover, $r_e(\alpha T) = |\alpha| r_e(T)$.

PROOF. Recall that $\lambda \in \sigma_e(\alpha T)$ if and only if $\lambda I - \alpha T$ is not Fredholm. Factoring out the scalar we have $\alpha(\frac{\lambda}{\alpha} I - T)$ is not Fredholm, and hence $\frac{\lambda}{\alpha} I - T$ is not Fredholm.

The latter statement is equivalent to $\frac{\lambda}{\alpha} \in \sigma_e(T)$, and hence $\lambda \in \alpha\sigma_e(T)$.

So $\lambda \in \sigma_e(\alpha T)$ if and only if $\lambda \in \alpha\sigma_e(T)$.

Therefore $\sigma_e(\alpha T) = \alpha\sigma_e(T)$ as claimed. By the definition of the essential spectral radius, it immediately follows that $r_e(\alpha T) = |\alpha| r_e(T)$. □

REMARK 3.3.6

Relating the spectra of two operators and that of their sum on Banach spaces is not obvious. In fact if A and B are two operators on a Banach space, then in general $\sigma(A), \sigma(B)$ and $\sigma(A+B)$ are not related. The question has been to find the conditions necessary for them to be related. This question together with related ones were considered in [41, 58]. For the essential spectra analogue, Shapiro and Snow [46] considered such questions and obtained among others the following result:

Theorem 3.3.7

Let $A, B \in \mathcal{L}(X)$ and suppose that A and B commute, then we have:

(i) $\sigma_e(A+B) \subseteq \sigma_e(A) + \sigma_e(B)$.

Moreover, if B is Fredholm, then

(ii) $\sigma_e(AB) \subseteq \sigma_e(A)\sigma_e(B)$.

Now if we scale the operators A and B using scalars α and β, we obtain the following result,

Corollary 3.3.8

Let A and B be defined as in Theorem 3.3.7 above. Then for $\alpha, \beta \in \mathbb{R}$, we have

(i) $\sigma_e(\alpha A + \beta B) \subseteq \alpha\sigma_e(A) + \beta\sigma_e(B)$.

Moreover, if B is Fredholm, then

(ii) $\sigma_e((\alpha A)(\beta B)) \subseteq \alpha\beta\sigma_e(A)\sigma_e(B)$.

PROOF. Using Theorems 3.3.5 and 3.3.7 above, we have $\sigma_e(\alpha A + \beta B) \subseteq \sigma_e(\alpha A) + \sigma_e(\beta B) = \alpha\sigma_e(A) + \beta\sigma_e(B)$, which proves (i). The proof of (ii) is similar. Indeed, $\sigma_e((\alpha A)(\beta B)) \subseteq \sigma_e(\alpha A)\sigma_e(\beta B) = \alpha\beta\sigma_e(A)\sigma_e(B)$. □

Another consequence giving the essential spectral radii relations is the following,

Corollary 3.3.9

Let A and B be defined as in Theorem 3.3.7. Then for $\alpha, \beta \in \mathbb{R}$, we have

(i) $r_e(\alpha A + \beta B) \leq |\alpha| r_e(A) + |\beta| r_e(B)$.

(ii) $r_e((\alpha A)(\beta B)) \leq |\alpha\beta| r_e(A) r_e(B)$.

PROOF. Follows immediately from the definition of the essential spectral radius and the Corollary 3.3.8 above. □

3.4 Parts of the essential spectrum and Duality

Recall from Theorem 3.3.1 part (c), that the essential spectrum $\sigma_e(T)$ is (by definition) the spectrum of the coset $T + \mathcal{K}(X)$ in the Calkin algebra $C(X) = \mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the ideal of all compact operators on X. More precisely, $\sigma_e(T) = \sigma(q(T))$. Using this fact, we introduce the following parts of the essential spectrum on $\mathcal{L}(X)$:

Definition 3.4.1

(i) Essential approximate point spectrum, $\sigma_{ap}^{ess}(T)$

$\sigma_{ap}^{ess}(T) = \{\lambda \in \mathbb{C}$: there exists $x_n \subset X$ such that$(q(T) - \lambda I)x_n \to 0$ as $n \to \infty\}$.

(ii) Essential surjectivity spectrum, $\sigma_{su}^{ess}(T)$

$\sigma_{su}^{ess}(T) = \{\lambda \in \mathbb{C} : q(T) - \lambda I$ is not surjective $\}$.

(iii) Essential point spectrum, $\sigma_{p}^{ess}(T)$

$\sigma_{p}^{ess}(T) = \{\lambda \in \mathbb{C} : (q(T) - \lambda I)x = 0$ for some $x \neq 0, x \in X\}$. In other words $\sigma_{p}^{ess}(T)$ is the set of eigenvalues of $q(T)$.

(iv) Essential compression spectrum, $\sigma_{com}^{ess}(T)$

$\sigma_{com}^{ess}(T) = \{\lambda \in \mathbb{C} : R(q(T) - \lambda I)$ is not dense in $C(X)\}$.

From the above definition, it's apparent that $\sigma_{ap}^{ess}(T) = \sigma_{ap}(q(T))$, $\sigma_{su}^{ess}(T) = \sigma_{su}(q(T))$, $\sigma_{p}^{ess}(T) = \sigma_p(q(T))$ and $\sigma_{com}^{ess}(T) = \sigma_{com}(q(T))$, where σ_{ap}, σ_{su}, σ_p and σ_{com} are the usual approximate point spectrum, surjectivity spectrum, point spectrum and compression spectrum respectively. For a comprehensive theory on these parts of the spectrum, we refer the reader to [33, 41].
There are some obvious relations between the various parts of the essential spectrum defined above. In particular, we give the following result,

Theorem 3.4.2

Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then

(i) $\sigma_p^{ess}(T) \subseteq \sigma_{ap}^{ess}(T)$

(ii) $\sigma_{com}^{ess}(T) \subseteq \sigma_{su}^{ess}(T)$

(iii) $\sigma_e(T) = \sigma_p^{ess}(T) \cup \sigma_{su}^{ess}(T)$.

In particular, $\sigma_e(T) = \sigma_{ap}^{ess}(T) \cup \sigma_{com}^{ess}$.

PROOF. Assertion (i) Follows from the definition. For (ii), let $\lambda \in \sigma_{com}^{ess}(T)$. Then $\overline{R(q(T) - \lambda I)} \neq C(X)$, which implies that $q(T) - \lambda I$ is not surjective, as desired. Moreover,

$$\begin{aligned}
\sigma_e(T) &= \sigma(q(T)) = \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not invertible}\} \\
&= \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not bijective }\} \\
&= \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not surjective or} \\
&\quad\ q(T) - \lambda I \text{ is not injective}\} \\
&= \sigma_p^{ess}(T) \cup \sigma_{su}^{ess}(T),
\end{aligned}$$

which proves (iii). Finally, by the characterization of the approximate point spectrum given by Theorem 2.0.7, we have that $\lambda \in \sigma_{ap}(q(T))$ if and only if either $q(T) - \lambda I$ is not injective or $R(q(T) - \lambda I)$ is not closed. This completes the proof. □

In the next results, we shall use duality theory to provide connections between these various parts of the essential spectrum of a bounded operator on X and the corresponding parts of the spectrum of the adjoint operator on the dual space X^*. We also give a relationship between the essential surjectivity spectrum of an operator and the essential approximate point spectrum of its adjoint on a Banach space, and vice versa.

Theorem 3.4.3

Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$, then $\sigma_{ap}^{ess}(T) = \sigma_{su}^{ess}(T^*)$, where $\sigma_{su}^{ess}(T^*) = \sigma_{su}(\tilde{q}(T^*))$ and $\sigma_{ap}^{ess}(T) = \sigma_{ap}(q(T))$.

PROOF. To prove this theorem we shall show that $\mathbb{C} \setminus \sigma_{ap}^{ess}(T) = \mathbb{C} \setminus \sigma_{su}^{ess}(T^*)$. Using the characterization of the approximate point spectrum given by Theorem 2.0.7, it suffices to prove that $q(T)$ is bounded below if and only $\tilde{q}(T^*)$ is surjective. Since $q(T)$ is bounded below, it follows that $R(q(T))$ closed and $N(q(T)) = \{0\}$. By Closed Range theorem (Theorem 2.0.5) $R(\tilde{q}(T^*)) = R(q(T)^*) = N(q(T))^{\perp} = \{0\}^{\perp} = C(X)$, and so $\tilde{q}(T^*)$ is surjective as desired.
Conversely, let $R(\tilde{q}(T^*)) = C(X^*)$ and so $R(\tilde{q}(T^*))$ is closed. By Closed Range theorem, $R(q(T))$ is closed and $N(q(T)) = R(\tilde{q}(T^*))^{\perp} = (C(X^*))^{\perp} = \{0\}$. Therefore $q(T)$ is bounded below, and this completes the proof. □

Theorem 3.4.4
Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$, then $\sigma_{su}^{ess}(T) = \sigma_{ap}^{ess}(T^*)$, where $\sigma_{su}^{ess}(T) = \sigma_{su}(q(T))$ and $\sigma_{ap}(T^*) = \sigma_{ap}^{ess}(\tilde{q}(T^*))$.

PROOF. We prove that $\mathbb{C} \setminus \sigma_{su}^{ess}(T) = \mathbb{C} \setminus \sigma_{ap}^{ess}(T^*)$. That is, for $\lambda \in \mathbb{C}$ we have; $\lambda \notin \sigma_{su}^{ess}(T)$ if and only if $\lambda \notin \sigma_{ap}^{ess}(T^*)$. Thus $\lambda I - q(T)$ surjective is equivalent to $\lambda I - \tilde{q}(T^*)$ bounded below. It therefore suffices to prove that $q(T)$ is surjective if and only if $q(T^*)$ bounded below. Now let $R(q(T)) = C(X)$, then by the Closed range theorem, $R(\tilde{q}(T^*))$ is closed and $N(\tilde{q}(T^*)) = R(q(T))^{\perp} = \{0\}$. Hence $\tilde{q}(T^*)$ is bounded below.
Conversely, if $\tilde{q}(T^*)$ is bounded below, then $R(\tilde{q}(T^*))$ is closed and $N(\tilde{q}(T^*)) = R(q(T))^{\perp} = \{0\}$. By the Closed range theorem $R(q(T))$ is closed and $R(q(T)) = N(\tilde{q}(T^*))^{\perp} = \{0\}^{\perp} = C(X)$. Therefore $q(T)$ is surjective, and this completes the proof. □
The following theorem now gives the relationship of the compression essential spectrum of an operator with the essential point spectrum of its adjoint on a Banach space and vice versa.
Theorem 3.4.5
Let X be an infinite dimensional Banach space and let $T \in \mathcal{L}(X)$. Then we have, $\sigma_{com}^{ess}(T) = \sigma_{p}^{ess}(T^*)$, where $\sigma_{p}^{ess}(T^*) = \sigma_p(\tilde{q}(T^*))$ and $\sigma_{com}^{ess}(T) = \sigma_{com}(q(T))$.

PROOF. We wish to prove that $\mathbb{C} \setminus \sigma_{com}^{ess}(T) = \mathbb{C} \setminus \sigma_p^{ess}(T^*)$. That is, for $\lambda \in \mathbb{C}$, we have $\lambda \notin \sigma_{com}^{ess}(T)$ if and only if $\lambda \notin \sigma_p^{ess}(T^*)$. This means that $\lambda I - q(T)$ has a dense range, and this is equivalent to $\lambda I - \tilde{q}(T^*)$ being injective. So $\overline{R(\lambda I - q(T))} = C(X)$ if and only if $N(\lambda I - q(T^*)) = \{0\}$. It suffices to prove that $\overline{R(q(T))} = C(X)$ if and only if $N(\tilde{q}(T^*)) = \{0\}$. Since $N(\tilde{q}(T^*)) = R(q(T))^{\perp}$ and from the Hahn-Banach extension theorem (Theorem 2.0.10), it follows that for each $M \subseteq Y$,

$$\overline{M} = Y \Leftrightarrow M^{\perp} = \{0\}. \tag{3.1}$$

Now from Equation (3.1), $N(q(T^*)) = R(q(T))^{\perp} = \{0\}$ which completes the proof. □

Theorem 3.4.6

Let X be an infinite dimensional Banach space and let $T \in \mathcal{L}(X)$. Then, $\sigma_p^{ess}(T) \subseteq \sigma_{com}^{ess}(T^*)$, where $\sigma_p^{ess}(T) = \sigma_p(q(T))$ and $\sigma_{com}^{ess}(T^*) = \sigma_{com}(\tilde{q}(T^*))$.

PROOF. By replacing T with T^* in Theorem 3.4.5, we get $\sigma_{com}^{ess}(T^*) = \sigma_p^{ess}(T^{**})$. Therefore, we obtain $\sigma_p^{ess}(T) \subseteq \sigma_p^{ess}(T^{**}) = \sigma_{com}^{ess}(T^*)$ as desired. □

REMARK 3.4.7

If X is reflexive, that is, $X \cong X^{**}$; then $\sigma_p^{ess}(T) = \sigma_{com}^{ess}(T^*)$.

CHAPTER 4

PROPERTIES OF ESSENTIAL NUMERICAL RANGES

In this chapter as well, X denotes an infinite dimensional complex Banach space and $\mathcal{L}(X)$ the space of bounded linear operators on X. In section 4.1 we give the algebraic properties of the algebraic essential numerical range while section 4.2 is devoted to the study of the properties of the essential spatial numerical range. The relationship between the essential spectra and the essential numerical range in the setting of a Banach space is considered in the last section 4.3.

4.1 Essential Algebraic Numerical Range

The Calkin algebra over an arbitrary Banach space where the essential version of the numerical range is considered, has been an area of considerable study. The essential algebraic numerical range is given as $V_e(T) = V(q(T)) = V(T+K)$, where $T \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$. This is the numerical range of $q(T)$ in the quotient space $\mathcal{L}(X)/\mathcal{K}(X)$. We define the essential algebraic numerical range as the set $V_e(T) = \{f((T+K)x) : x \in S(X), f \in S(X^*), f(x) = 1, f(K) = 0\}$.

Some of the known properties of the essential algebraic numerical range $V_e(T)$ on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ can be found in [9], and we summarize them in the following theorem:

Theorem 4.1.1

For $T \in \mathcal{L}(X)$, the following properties hold.

(i) $V_e(T)$ *is a nonempty compact convex set and* $\sigma_e(T) \subseteq V_e(T)$ *where* $\sigma_e(T)$ *denotes the essential spectrum of* T,

(ii) $V_e(T) = \{0\}$ *if and only if* $T \in \mathcal{K}(X)$,

(iii) $V_e(T) = \bigcap\{V(T+K,) : K \in \mathcal{K}(X)\}$,

(iv) $V_e(T) = \{f(T) : f \in \mathcal{L}(X)^*, f(I) = 1 = \|f\|, f(\mathcal{K}(X)) = 0\}$ *and*

(v) $exp(-1)\|T\|_e \leq max\{|\lambda| : \lambda \in V_e(T) \leq \|T\|_e\}$.

The details on the properties above can be found in [9].

As an extension of the properties above, we give more algebraic properties of $V_e(T)$.

Theorem 4.1.2

For $T, S \in \mathcal{L}(X)$ and α, $\beta \in \mathbb{C}$, we have:

(i) $V_e(\alpha T) = \alpha V_e(T)$.

(ii) $V_e(T+S) \subseteq V_e(T) + V_e(S)$.

(iii) $V_e(\alpha I + T) = \alpha + V_e(T)$.

(iv) $V_e(\alpha T + \beta S) \subseteq \alpha V_e(T) + \beta V_e(S)$.

PROOF. To prove (i), let p be a complex number. Then $p \in V_e(T)$ if and only if $|p - \lambda| \leq \|T + K - \lambda\|$ for each complex number λ and each compact operator K. So $p \in V_e(\alpha T)$ if and only if $|p - \lambda| \leq \|\alpha((T+K) - \lambda)\| = \alpha\|T + K - \lambda\|$ for each pair of complex numbers α and λ and each compact operator K. Hence $V_e(\alpha T) = V(q(\alpha T)) = V(\alpha q(T)) = \alpha V(q(T)) = \alpha V_e(T)$. For (ii), we have $V_e(T+S) = V(q(T+S)) = V((T+S)+K) = V((T+K)+(S+K)) \subseteq V(T+K) + V(S+K) = V(q(T)) + V(q(S)) = V_e(T) + V_e(S)$. The assertion

(iii) follows from the fact that I and T commute.

The proof of (iv) follows from (i) and (ii) above. From(ii), we have that $V_e(\alpha T+\beta S) \subseteq V_e(\alpha T)+V_e(\beta S)$. Now using (i) we obtain $V_e(\alpha T+\beta S) \subseteq V_e(\alpha T)+V_e(\beta S) = \alpha V_e(T)+\beta V_e(S)$. □

Now let $\nu_e(T) = \sup\{|\lambda| : \lambda \in V_e(T)\}$, be the *essential algebraic numerical radius*, that is, the numerical radius associated with $V_e(T)$. Then we obtain the following consequence of the above theorem;

Corollary 4.1.3

For $T, S \in \mathcal{L}(X)$ and $\alpha,\beta \in \mathbb{C}$, we have:

(i) $\nu_e(\alpha T) = |\alpha|\nu_e(T)$,

(ii) $\nu_e(T+S) \leq \nu_e(T)+\nu_e(S)$,

(iii) $\nu_e(\alpha I+T) = |\alpha|+\nu_e(T)$, *and*

(iv) $\nu_e(\alpha T+\beta S) \leq |\alpha|\nu_e(T)+|\beta|\nu_e(S)$.

PROOF. Follows immediately from Theorem 4.1.2 and the definition of $\nu_e(T)$, that is, $\nu_e(T) = \sup\{|\lambda| : \lambda \in V_e(T)\}$. Indeed, let $\lambda \in V_e(\alpha T) = \alpha V_e(T)$. Then there exists $\mu \in V_e(T)$, such that $\lambda = \alpha\mu$, implying that $|\lambda| = |\alpha||\mu|$. Now taking supremum over all $\lambda \in V_e(\alpha T)$, we get $\sup_{\lambda \in V_e(\alpha T)}|\lambda| = |\alpha||\mu|$. Again taking supremum over all $\mu \in V_e(T)$, we obtain $\sup_{\lambda \in V_e(\alpha T}|\lambda| = |\alpha| \sup_{\mu \in V_e(T)}|\mu|$. Therefore $\nu_e(\alpha T) = |\alpha|\nu_e(T)$.

For (ii) we have from Theorem 4.1.2 assertion (ii) that $V_e(T+S) \subseteq V_e(T)+V_e(S)$. Let $\lambda \in V_e(T+S)$, then there exist $\lambda_1 \in V_e(T)$ and $\lambda_2 \in V_e(S)$ such that $|\lambda| \leq |\lambda_1|+|\lambda_2|$. Now taking supremum over all $\lambda \in V_e(T+S)$, then over all $\lambda_1 \in V_e(T)$ and finally over all $\lambda_2 \in V_e(S)$, we obtain $\sup_{\lambda \in V_e(T+S)}|\lambda| \leq \sup_{\lambda_1 \in V_e(T)}|\lambda_1| + \sup_{\lambda_2 \in V_e(S)}|\lambda_2|$. Hence $\nu_e(T+S) \leq \nu_e(T)+\nu_e(S)$, as desired.

For (iii) and (iv), we first note that $\nu_e(T+S) \leq \nu_e(T)+\nu_e(S)$. Using (i) we obtain $\nu_e(\alpha I+T) = |\alpha|+\nu_e(T)$, and $\nu_e(\alpha T+\beta S) \leq |\alpha|\nu_e(T)+|\beta|\nu_e(S)$. □

4.2 Essential Spatial Numerical Range

There is scarse literature on the essential spatial numerical range for Banach space operators. Part of the study that we have come across is the work by Barraa and Müller [9]. In their attempt to study some properties of the essential numerical range $W_e(T)$ on Banach spaces, the authors considered a measure of non-compactness instead of the essential norm in the Calkin algebra. They remarked that this is a probable reason why the essential numerical range $W_e(T)$ has not been extensively studied for Banach space operators. For $T \in \mathcal{L}(X)$, where X is an infinite dimensional Banach space, we define a seminorm $\|.\|_\mu$ on $\mathcal{L}(X)$ by $\|T\|_\mu = \inf\{\|T|_M\| : M \subset X$ a subspace of finite codimension $\}$. Following [9], $\|.\|_\mu$ is a measure of non-compactness, that is, $\|T\|_\mu = 0$ if and only if T is compact. Moreover, $\|.\|_\mu$ is an algebra norm on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. As a result, another type of essential numerical range $V_\mu(T)$ defined by $V_\mu(T) = V(T, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu)$ was introduced. In particular, $V_\mu(T)$ is the set of all $\lambda \in \mathbb{C}$ such that there is a functional $\tilde{\Phi} \in (\mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu)^*$ satisfying $\|\tilde{\Phi}\| = 1 = \tilde{\Phi}(I + \mathcal{K}(X))$ and $\tilde{\Phi}(T + \mathcal{K}(X)) = \lambda$. Equivalently, there is a functional $\Phi \in \mathcal{L}(X)^*$ such that $\Phi(\mathcal{K}(X) = 0, \Phi(I) = 1, \Phi(T) = \lambda$ and $|\Phi(S)| \leq \|S\|_\mu$ for all $S \in \mathcal{L}(X)$, where X is a general Banach space.

Before looking at the properties of the essential spatial numerical range $W_e(T)$, we summarize some properties of $V_\mu(T)$ in the following theorem:

Theorem 4.2.1

Let $T \in \mathcal{L}(X)$. Then

(i) $V_\mu(T) = conv(W_e(T))$.

(ii) $V_\mu(T) \subset V_e(T)$.

(iii) $exp(1)\|T\|_\mu \leq max\{|\lambda| : \lambda \in V_{\mu(T)}\} \leq \|T\|_\mu$.

Theorem 4.2.2

Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then

(i) *$V_\mu(T)$ is a closed, convex and compact subset of $\mathbb{C}$,*

(ii) *$V_\mu(T) = \{0\}$ if and only if T is compact,*

(iii) *$V_\mu(T+K) = V_\mu(T)$ for $K \in \mathcal{K}(X)$, and*

(iv) *$V_\mu(T+S) \subseteq V_\mu(T) + V_\mu(S)$, where $S \in \mathcal{L}(X)$.*

PROOF. To prove that $V_\mu(T)$ is closed, let $\lambda_n \in V_\mu(T)$ be such that $\lambda_n \to \lambda$ as $n \to \infty$. Then for each $n \in \mathbb{N}$, there exists $(\Phi_n)_n \subseteq \mathcal{L}(X)^*$ such that $\Phi_n(\mathcal{K}(X)) = 0$, $\Phi_n(I) = 1$, $\Phi_n(T) = \lambda_n$ and $\|\Phi_n(S)\| \le \|S\|_\mu$ for all $S \in \mathcal{L}(X)$. Now, since $(\Phi_n)_n$ is bounded, let $\Phi_n \to \Phi$ as $n \to \infty$. Then $\Phi(I) = \lim_{n\to\infty}\Phi_n(I) = \lim_{n\to\infty} 1 = 1$; $\Phi(T) = \lim_{n\to\infty}\Phi_n(T) = \lim_{n\to\infty}\lambda_n = \lambda$; $\Phi(\mathcal{K}(X)) = \lim_{n\to\infty}\Phi_n(\mathcal{K}(X)) = 0$, and for all $S \in \mathcal{L}(X)$, $\|\Phi(S)\| = \|\lim_n \|\Phi_n(S)\| = \lim_n \|\phi_n(S)\| \le \lim_n \|S\| = \|S\|$. It therefore follows that $\lambda \in V_\mu(T)$ and this proves that $V_\mu(T)$ is closed as claimed.
Following [9], $V_\mu(T) = conv(W_e(T))$ which clearly indicates that $V_\mu(T)$ is convex since it is a convex hull of $W_e(T)$. In general, we know that $V_\mu(T) \subset V_e(T)$. But $V_e(T)$ is compact from Theorem 4.1.1 and $V_\mu(T)$ is closed. The result then follows immediately from the fact that a closed subset of a compact set is compact. This proves (i).
Now, for any $T \in \mathcal{L}(X)$, we have that $V_\mu(T) = \{0\}$ if and only if $V_e(T) = \{0\}$ which is true if and only if T is compact. This proves (ii).
The proof of (iii) follows from the definition of $V_\mu(T)$ and from the fact that $\|.\|_\mu$ is a measure of non-compactness.
For (iv), from the sum property of the algebraic numerical range, we have for $S \in \mathcal{L}(X)$,

$$\begin{aligned} V_\mu(T+S) &= V(T+S, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) \\ &\subseteq V(T, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) + V(S, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) \\ &= V_\mu(T) + V_\mu(S). \end{aligned}$$

□

If $\nu_\mu(T) = \sup\{|\lambda| : \lambda \in V_\mu(T)\}$, then ν_μ is the numerical radius corresponding to the numerical range $V_\mu(T)$ and we can deduce the following corollary.

Corollary 4.2.3

Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then

(i) $\nu_\mu(T) = 0$ *if and only if T is compact,*

(ii) $\nu_\mu(T+K) = \nu_\mu(T)$, *for $K \in \mathcal{K}(X)$,*

(iii) $\nu_\mu(T+S) \leq \nu_\mu(T) + \nu_\mu(S)$ *where $S \in \mathcal{L}(X)$.*

PROOF. Follows from the definition of $\nu_\mu(T)$ and Theorem 4.2.1 above. For (i), we use assertion (ii) of Theorem 4.2.1, where $V_\mu(T) = \{0\}$ if and only if T is compact. This is equivalent to the fact that $\lambda \in V_\mu(T)$, $\lambda = 0$, if and only if T is compact. Equivalently $\lambda \in V_\mu(T)$, $|\lambda| = 0$ if and only if T is compact. Hence sup $_{\lambda \in V_\nu(T)}|\lambda| = 0$, if and only if T is compact. For (ii), it is clear from the definition of $\nu_\mu(T)$ and assertion (iii) of Theorem 4.2.1. For (iii), $V_\mu(T+S) \subseteq V_\mu(T) + V_\mu(S)$. This means that for $\lambda \in V_\mu(T+S)$ there exist $\lambda_1 \in V_\mu(T)$ and $\lambda_2 \in V_\mu(S)$ such that $\lambda = \lambda_1 + \lambda_2$. So $|\lambda| \leq |\lambda_1| + |\lambda_2|$. This implies that sup $_{\lambda \in V_\mu(T+S)}|\lambda| \leq \sup_{\lambda_1 \in V_\mu(T)}|\lambda_1| + \sup_{\lambda_2 \in V_\mu(S)}|\lambda_2|$. Hence $\nu_\mu(T+S) \leq \nu_\mu(T) + \nu_\mu(S)$. $\square$

It's important to take note that in general $V_\mu(T) \subset V_e(T)$, but if X is a Hilbert space we obtain equality, that is, $V_\mu(T) = V_e(T)$, see [9].

The next result gives some properties of the essential spatial numerical range of a bounded operator T, $W_e(T)$.

Theorem 4.2.4

Let X be an infinite dimensional Banach space, and $T \in \mathcal{L}(X)$, Then the following properties hold.

(i) $W_e(T)$ *is nonempty, closed, non-convex and compact subset of the complex plane $\mathbb{C}$,*

(ii) $W_e(T) = \{0\}$ *if and only if T is compact,*

(iii) $W_e(\beta T) = \beta W_e(T)$ *for some* $\beta \in \mathbb{C}$,

(iv) $W_e(T+S) \subseteq W_e(T) + W_e(S)$, *where* $S \in \mathcal{L}(X)$, *and*

(v) $W_e(\alpha T + \beta S) \subseteq \alpha W_e(T) + \beta W_e(S)$, *where* $S \in \mathcal{L}(X)$ *and* β, $\alpha \in \mathbb{C}$.

PROOF. Following [9], $\sigma_e(T) \subset W_e(T)$. Since $\sigma_e(T)$ is nonempty, it follows that $W_e(T)$ is nonempty as well. The non-convexity of $W_e(T)$ is immediate from the relation: $V_\mu(T) = \text{conv}(W_e(T))$ where $\text{conv}(W_e(T))$ is the convex hull of $W_e(T)$. To prove that $W_e(T)$ is closed, let $\lambda_n \in W_e(T)$ be such that $\lambda_n \to \lambda$ as $n \to \infty$. We want to show that $\lambda \in W_e(T)$. Since $\lambda_n \in W_e(T)$, choose nets which are partially ordered on subsets of X and X^* by the relation $\leq$ as $(u_\alpha) \subset X$, $(u_\alpha^*) \subset X^*$ such that $\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^* \rangle = 1$ for each α and $u_\alpha \to 0$ weakly. Fix n such that $|\langle Tu_\alpha, u_\alpha^* \rangle - \lambda_n| < \frac{1}{n}$. Then

$$|\langle Tu_\alpha, u_\alpha^* \rangle - \lambda| \leq |\langle Tu_\alpha, u_\alpha^* \rangle - \lambda_n| + |\lambda_n - \lambda| < \frac{1}{n} + |\lambda_n - \lambda| \to 0$$

as $n \to \infty$ and therefore $\lambda \in W_e(T)$. The compactness of $W_e(T)$ follows from the compactness of $V_\mu(T)$ since $W_e(T)$ is a closed subset of $V_\mu(T)$. This completes the proof of (i).

To prove (ii), take note that $W_e(T) \subset \text{conv}\ (W_e(T)) = V_\mu(T) = \{0\}$ if and only if T is compact. Since $W_e(T)$ is nonempty, the latter statement is equivalent to $W_e(T) = \{0\}$ if and only if T is compact, as desired.

For (iii), let $\lambda \in W_e(T)$. This is equivalent to having $(u_\alpha) \subset X$, $(u_\alpha^*) \subset X^*$ such that $\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^* \rangle = 1$ for all $\alpha, u_\alpha \to 0$ weakly and $\langle Tu_\alpha, u_\alpha^* \rangle \to \lambda$. Then $\beta W_e(T)$ is equivalent to $\beta \langle Tu_\alpha, u_\alpha^* \rangle \to \beta\lambda$. This in turn is equivalent to $\langle \beta Tu_\alpha, u_\alpha^* \rangle \to \beta\lambda$. Since $\beta\lambda \in \mathbb{C}$, it follows that $\beta W_e(T) = W_e(\beta T)$.

To prove (iv), let $\lambda \in W_e(T+S)$. Then there exists $(u_\alpha) \subset X$, $(u_\alpha^*) \subset X^*$ such that $\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^* \rangle = 1$ for all $\alpha, u_\alpha \to 0$ weakly and $\langle (T+S)u_\alpha, u_\alpha^* \rangle \to \lambda$. Then $\langle Tu_\alpha + Su_\alpha, u_\alpha^* \rangle \to \lambda$ is equivalent to $\langle Tu_\alpha, u_\alpha^* \rangle + \langle Su_\alpha, u_\alpha^* \rangle \to \lambda$. This implies that $\langle Tu_\alpha, u_\alpha^* \rangle \to \lambda_1$ and $\langle Su_\alpha, u_\alpha^* \rangle \to \lambda_2$ where $\lambda = \lambda_1 + \lambda_2$. Thus, $\lambda_1 \in W_e(T)$ and $\lambda_2 \in W_e(S)$ with $\lambda = \lambda_1 + \lambda_2$. Hence

$\lambda \in W_e(T) + W_e(S)$.

Assertion (v) follows immediately from assertions (iii) and (iv) above. From (iii), we have that $W_e(\beta T) = \beta W_e(T)$ for some $\beta \in \mathbb{C}$. Now using (iv) we obtain $W_e(\alpha T + \beta S) \subseteq \alpha W_e(T) + \beta W_e(S)$, where $S \in \mathcal{L}(X)$ and β, $\alpha \in \mathbb{C}$. $\square$

Define the essential spatial numerical radius $\omega_e(T)$ by

$$\omega_e(T) = \sup\{|\lambda| : \lambda \in W_e(T)\}.$$

Then the following is an immediate consequence of Theorem 4.2.3 above.

Corollary 4.2.5

For $T, S \in \mathcal{L}(X)$ and α, $\beta \in \mathbb{C}$, we have

(i) $\omega_e(T) = 0$ *if and only if* $T \in \mathcal{K}(X)$,

(ii) $\omega_e(\beta T) = |\beta|\omega_e(T)$,

(iii) $\omega_e(T + S) \leq \omega_e(T) + \omega_e(S)$, *and*

(iv) $\omega_e(\alpha T + \beta S) \leq |\alpha|\omega_e(T) + |\beta|\omega_e(S)$.

PROOF. Follows from the definition of $\omega_e(T)$ and Theorem 4.2.3 above. For (i), we use assertion (ii) of Theorem 4.2.3, where $W_e(T) = \{0\}$ if and only if T is compact. This is equivalent to $\lambda \in W_e(T)$, $\lambda = 0$, if and only if T is compact, which is further equivalent to $\lambda \in W_e(T)$, $|\lambda| = 0$ if and only if T is compact. Hence $\sup_{\lambda \in W_e(T)} |\lambda| = 0$, if and only if T is compact. For (ii), it is clear from the definition of $\omega_e(T)$ and assertion (iii) of Theorem 4.2.3.

For (iii), since $W_e(T + S) \subseteq W_e(T) + W_e(S)$, let $\lambda \in W_e(T + S)$. Then there exist $\lambda_1 \in W_e(T)$ and $\lambda_2 \in W_e(S)$ such that $|\lambda| \leq |\lambda_1| + |\lambda_2|$. This implies that $\sup_{\lambda \in W_e(T+S)} |\lambda| \leq \sup_{\lambda_1 \in W_e(T)} |\lambda_1| + \sup_{\lambda_2 \in W_e(S)} |\lambda_2|$. Hence $\omega_e(T + S) \leq \omega_e(T) + \omega_e(S)$.

For (iv), Since $W_e(\alpha T + \beta S) \subseteq \alpha W_e(T) + \beta W_e(S)$, we have

$$\begin{aligned}
\omega_e(\alpha T + \beta S) &= \sup\{|\lambda| : \lambda \in W_e(\alpha T + \beta S)\} \\
&\leq \sup\{|\alpha||\lambda_1| + |\beta||\lambda_2| : \lambda_1 \in W_e(T), \lambda_2 \in W_e(S)\} \\
&\leq \sup\{|\alpha||\lambda_1| : \lambda_1 \in W_e(T)\} + \sup\{|\beta||\lambda_2| : \lambda_2 \in W_e(S)\} \\
&= |\alpha|\omega_e(T) + |\beta|\omega_e(S).
\end{aligned}$$

□

4.3 Relationship between Essential spectra and Essential numerical range

In this section, we attempt to relate the essential spectra and the essential numerical range for Banach space operators.

Theorem 4.3.1

If $T \in \mathcal{L}(X)$ is normal, then conv($\sigma_e(T)) = V_e(T)$.

PROOF. Since $\sigma_e(T) \subset V_e(T)$ and T is normal, conv$\sigma_e(T) \subset \bigcap\{V_p(T) : p \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of equivalent norms to the essential norm. Since $\sigma_e(T)$ is compact, conv$(\sigma_e(T))$ is a compact convex set and is therefore the intersection of the open circular discs containing $\sigma_e(T)$. Suppose then that $|\lambda - \alpha| < r$, $(\lambda \in \sigma_e(T))$, then $p(T - \alpha I) < r$ and so there is a $p \in N$ with $p(T - \alpha I) < r$. But then it is obvious that $|\lambda - \alpha| < r$, $(\lambda \in V_e(T))$, and so $\bigcap\{V_p(T) : p \in N\}$ is contained in every open circular disc that contains $\sigma_e(T)$. □

For compact operators, the following theorem details the relation between the essential spectrum and the essential algebraic numerical range;

Theorem 4.3.2

If $T, S \in \mathcal{K}(X)$ and $\alpha \in \mathbb{C}$, then

(i) $\sigma_e(T) = V_e(T) = \{0\}$,

(ii) $\sigma_e(T+S) = V_e(T+S) = V_e(T) + V_e(S)$,

(iii) $\sigma_e(T^*) = V_e(T^*)$, *and*

(iv) $\sigma_e(\alpha T) = V_e(\alpha T)$.

PROOF. For $T \in \mathcal{K}(X)$, $V_e(T) = \{0\}$. But $\sigma_e(T) \subseteq V_e(T)$, and since the spectrum $\sigma_e(T)$ is nonempty, the result follows, and this proves (i).

To prove (ii), since $T, S \in \mathcal{K}(X)$, $T+S \in \mathcal{K}(X)$ and $\{0\} = \sigma_e(T+S) \subseteq V_e(T+S) = V_e(T) + V_e(S) = \{0\}$. So the equality holds.

Assertion (iii) follows from the fact that $\sigma_e(T) = \sigma_e(T^*)$, while assertion (iv) is obvious. □

Consequently, if $r_e(T)$ is the essential spectral radius in the sense that $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$, then we have the following result;

Corollary 4.3.3

If $T \in \mathcal{K}(X)$, *then* $r_e(T) = \nu_e(T) = 0$.

PROOF. This follows from the fact that $\sigma_e(T) = V_e(T) = \{0\}$, given by Theorem 4.3.2 above. □

REMARK 4.3.4

We have determined the relationship between the essential spectra and the essential algebraic numerical range for Banach space operators specifically when the operator is compact. The relation between the essential spectra and the essential spatial numerical range has not been investigated in this thesis and remains open for further research.

CHAPTER 5

SUMMARY AND RECOMMENDATIONS

5.1 Summary

In this thesis we considered the properties of the essential spectra and the essential numerical ranges of bounded linear operators in the general setting of Banach spaces. We determined some algebraic properties of the essential spectrum and went further to define the various components of the essential spectrum. In Theorems 3.4.2, 3.4.3 and 3.4.4 we have established the relation between the various parts of the essential spectrum of a bounded operator and the corresponding adjoint operator. For the essential numerical ranges on Banach spaces, we have extended the known algebraic properties that can be found in Theorems 4.1.2, 4.2.1 and 4.2.3. Specifically we have shown that the spatial numerical range for Banach space operators is a closed non-convex and compact subset of the complex plane. On the relationship between the essential spectra and essential numerical ranges, we have shown that for compact operators, the essential spectrum and the essential algebraic numerical range coincide and is a singleton set with element 0, see Theorem 4.3.2. Moreover, for normal operators, we have proved that the convex hull of the essential spectrum is the essential algebraic numerical range for Banach space operators.

5.2 Recommendation

From the results of this study, we recommend the following for further research:

(i) The properties of the essential spatial numerical range have not been exhaustively studied on Banach spaces. Further studies can still be done to establish more properties, for example, on the joint operators, commuting operators, composition operators, adjoint operators, among others.

(ii) The study of the essential spectra can be extended to unbounded operators on Banach spaces.

(iii) The spatial numerical range has not been widely studied compared to other types of numerical ranges. We remarked that the lack of equality in the relation $W_e(T) \neq W(q(T))$ might be a probable reason. It would be interesting to study and find the properties that would enable this equality to be achieved. In this study we have established that for compact operators, $W_e(T) = \{0\}$ which is a trivial case for this equality. The non trivial cases remain open.

(iv) Not much has been done on the relation between the essential spectrum and the essential spatial numerical range and since not much is known about $W_e(T)$, it would be interesting to study this relation.

(v) The consideration for the spectrum being equal to essential spectrum for a given operator (i.e. $\sigma(T) = \sigma_e(T)$) has been widely studied in Hilbert spaces along classes of operators starting with normal operators. This can be correspondingly taken up in Banach spaces.

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