# INTEREST RATE RISK MODELING USING SEMI-HEAVY TAIL DISTRIBUTIONS OF NORMAL VARIANCE-MEAN MIXTURES: CENTRAL BANK OF KENYA INTEREST RATES.

BY

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# DECLARATION

This research project is my own work and has not been presented for a degree award in any other institution.

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This project has been submitted for examination with my approval as the university supervisor.

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# Dedication

I dedicate this work to Jane Oguta and Roseline Auma.

# Abstract

Derivative prices such as options and bond prices as well as swaps depend on the distributional assumptions of the underlying economic variables, normally, interest rates. The risk associated with changes in interest rates may worsen the value of the contract that depend on it since the values of these assets (derivative contracts) are affected directly by the fluctuations in interest rates. The distribution of interest rates, therefore, needs to be well understood to reduce the risks of losses associated with it. The Binomial Option pricing model assumes that interest rates are constant, with no returns, throughout the life of the option. Another common assumption of the underlying economic variables is that their returns are normally distributed with constant volatility. These assumptions have been used in pricing derivatives and currencies and has led to over-pricing and in some cases under-pricing. These assumptions have been considered inaccurate and misleading. This research uses mixture models exhibiting properties that appropriately capture the peakedness and skewness of interest rates as fundamental variables in pricing. The models of the Normal Variance-Mean Mixtures shows better performance than the normal distribution. The GARCH model is used under the assumption that 91-day Treasury Bills interest rates follow a Generalized Hyperbolic distribution while the Commercial Bank interest rates follows a Normal Inverse Gaussian distribution. A 99pc Value at Risk is then computed for the two models and calculates the minimum expected returns in the subsequent months. This research forms a foundation for the development of advanced pricing models that incorporates the fluctuations of interest rate in the pricing industry.

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# CHAPTER 1

# GENERAL INTRODUCTION

## 1.1 Preliminary Information

Interest rates are one of the most common underlying variables in the pricing of financial derivatives such as options, equity, and bond valuation. They can simply be put as determinant of the prices of other derivatives. The most common rates according to the central bank of Kenya are the treasury bills' interest rates and the commercial bank's weighted average rates. A treasury bill is a financial instrument which is issued to lenders by the authority through the central bank (which is a fiscal agent) in order to raise project finances on a short-term basis. They mature after a period of 91, 182, and 364 days from the beginning of the contract and are sold at discounted prices to reflect the investor's return and redeemed at face (par) value. On the other hand, a weighted average rate is a group of data that analyze the calculated weighted average interest rate of the flow of transaction of the previous working day of the relevant month.

The risk, as measured by the volatility of interest rates and their Kurtosis, exhibit high frequency fluctuation over very short time intervals (seconds or minutes). The returns from interest rates exhibit a behavior of fat tails and peakedness with a skewed distribution curve. This clearly invalidates the assumption that they follow a normal distribution with constant volatility. They also exhibit common statistical properties such volatility clustering and non-normality [1],[19] Moreover, the data, as in, [19] are fat tailed, and non-linear, with leptokurtic empirical distribution, meaning that they are more peaked with heavier tails than the normal distribution. By the property of semi heavy tails, a risk manager and derivative pricing manager need to use models relative to the characteristics of the data to bring out the right context and interpretation of the data in question. These models can capture the semi-heavy tails of the data and other statistical properties. Some studies have suggested the use of Gaussian assumption in modeling, pricing and forecasting financial derivatives which to date has been rendered obsolete. The use of the Black-Scholes Option pricing model [18], [41] and [4], the Random Walk model [46], and the Geometric Brownian Motion [49] have based their assumption on the fact that the distribution of interest rates is normal [52], while others have assumed a log-normal distribution. This assumption has been rendered obsolete and invalidated by [30] which confirms that these returns do not follow the normality assumption.

Despite the vast research on the financial derivative market and distribution of asset returns in the stock market and equity prices, very little is known about the risk associated with interest rates, as a financial variable, and its effect on pricing derivatives and currency options using the Black-Scholes options pricing model, GBM, and RW financial time series data forecasting models, as underlying variables. As much as the use of models with normality assumptions has been rampant and considered appropriate, the use of new more flexible, and analytically tractable models has been developed but has since not been applied in modeling interest rates risk. Therefore, modeling the CBK interest rates using semi-heavy tail distributions developed by [48] and used by [6], [25], [10], [29] and [13] in risk management has so far not been used in modeling interest rate risk.

### 1.2 Statement of the Problem

In derivative pricing market, good investment strategies are fundamental factors in financial success and independence. This comes from the proper awareness of the economic variables and distribution of risks. Investment in financial instruments such as government bonds and options requires an investor to have detailed knowledge in the distribution of interest rates. The assumption that returns of these variables follow a normal distribution has become obsolete and invalid [30] and has contributed to under-pricing, and in some cases, over-pricing derivative contracts and currencies that has attracted losses to the investors. In addition, during hedging and pricing of risk dependent derivative claims (claims with volatility as underlying variables) such as Variance and Volatility Swaps (pays difference between future price and stock price based on the variance and the volatility respectively), incorporating constant variance and volatility may lead to losses. The use of Brownian Motion (Geometric Brownian Motion) [49], n-period Binomial Model and the Cox-Ross-Rubinstein model and the famous Black-Scholes Option pricing model [18] assume that the underlying variables such as interest rates are normally distributed with a constant volatility which in practice is not true. Besides, the normal model allows for a chance that the interest rates can be negative which is also highly unlikely.

## 1.3 Justification of the Study

This study seeks to provide a better interest rates model that can be used in place of a normal distribution to model the underlying interest rates as used in derivative pricing models such as the C.R.R and Black-Scholes models as well as hedging and pricing volatility dependent claims such as Variance and Volatility Swaps. This will improve accuracy of these models therefore reducing risks of mis-pricing.

## 1.4 Objectives of the Study.

### 1.4.1 General Objectives

This project aims at modeling interest rates using "Semi-Heavy" tail distributions and forecast the returns using special cases of GARCH models.

### 1.4.2 Specific Objectives

The general objective will be achieved through considering the following specific objectives;

1. To construct the Normal Variance-Mean Mixtures and their properties using direct Integration approach.

2. To fit the four distributions to the 91-day Treasury Bills interest rates and the Commercial Banks Weighted Average rates and provide the best model in each case.

3. To forecast the interest rates returns using the special cases of "GARCH-Hyperbolic" models.

### 1.5 Significance of the Study

The financial market investment has been at a risk of huge loss due to incorrect assumptions about underlying economic variables such as interest rates. The underlying interest rates, especially are important in pricing of common interest rate derivatives in the stock market like the government bonds and the options, interest-bearing assets and volatility dependent Swaps. If these rates are erroneously assumed, then the overall problem is that there will be inaccurate (under or over-pricing) pricing of these contracts. This work therefore will attempt to apply flexible distributions to these underlying variables and show that this family of distributions provides a better capture of the statistical properties of the interest rates data than the assumed normality. This will therefore ensure that the right pricing of derivative contracts and other financial market variables are correctly computed based on correct assumptions. Through estimation and forecast of the time-varying returns using special cases of GARCH model, investors and managers can be able to manage financial risks and reduce losses. In the long-run, this will attract increase return and reduce risks to the investors therefore improve their wealth.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Introduction

In this chapter, we will review the development of interest rate modeling right from the time of the Brownian Motion, Ho-Lee model, and models with normal assumptions up to the time the normal mixtures with semi-heavy tail distributions were developed. We look at the studies done so far on these distributions, their relevance, shortcomings, and how they help us answer our research questions.

## 2.2 Models Under Normality Assumption

### 2.2.1 Brownian Motion

This model was first developed by Robert Brown to observe the irregular pollen grains from plants as they move on the surface of a liquid. He admitted that he had no scientific explanation for his observed phenomena [17]. This was later studied by Albert Einstein in 1905 [3] who proposed the existence of Brownian Motion which was later proven by Norbert Wiener in 1923 [24]. The standard Wiener process can be used to model asset returns but its main weakness is that its mean is zero, meaning that the growth rate of return is zero which, because of inflation [45], is not true in practice. With the modern application of Brownian Motion to model financial markets, the GBM was applied by Black and Scholes [27], Merton [50], and Paul Samuelson among others. Later in 1973, Black and Scholes derived an option valuation model [27] which was extended by Merton [50]. They however assumed that stock returns follow a Brownian Motion process.

### 2.2.2 Arbitrage-Free and Equilibrium Interest Rate Models

Different scholars developed these models with their analysis beginning with the observed market price of financial instruments or a benchmark instrument, which is fairly priced. These models assume a random process with a drift term (mean) and volatility (variance) of interest rates. The Ho-Lee model was in 1986 by scientists Ho and Lee [34] to derive a model for movement of the arbitrage free interest rate and apply this model to interest rates. However, the model assumes that the market is friction-less, has discrete points in time, and is complete (a market with negligible transaction costs and perfect information). In this model, there is no mean reversion (an assumption that interest rates eventually move back to their average), and volatility is independent of the level of the short rate. In short, the Ho-Lee model is normal. The Hull-White model was later introduced by John Hull and Allan White in 1990 [35] to price interest rate derivative securities by allowing for mean reversion. This model (HW) [35] is a generalization of the Ho-Lee model [34] and its purpose was to overcome the weakness of [35] but still, it was a normal model. Kalotay-Williams-Fabozzi model (KWF thereafter) was later introduced by Andrew Kalotay, George Williams, and Frank J. Fabozzi in 1993 [38] to model bond rates and embedded options, it did not allow for mean reversion but allowed for changes in short rates to be modeled by natural logarithms of returns to attain stationarity. The Black-Karasinski model (BK thereafter) was introduced by Fisher Black and Piotr Karasinski in 1991 [11] to price bonds and options when short rates are lognormal. The BK was a logarithmic extension of the KWF. Black-Derman-Toy was earlier introduced by Fischer Black, Emmanuel Derman, and William Toy in 1990 [28] as a one-factor model for interest rates and Treasury bond options. Later, Heath-Jarrow-Morton (HJM herein) model was introduced in 1997 [22] and derived by Jeffrey Andrew [5] as a general continuous-time multi-factor model to price bonds and the term structure of interest rates. In these models, term structure dynamics are described using fundamental economic variables assumed to affect interest rates. The Vasicek was introduced in 1977 [54], the Cox, Ingersoll, and Ross in 1985 [20], Brennan and Schwartz [16], and Longstaff and Schwartz model [39] used economic variables affecting interest rates and used to price the term structure of interest rates.

In summary, these models have common limitations. They make an assumption that the interest rates follow a normal distribution which in practice is not true. The Vasicek model and the Hull and White models allows for a negative interest rates which in reality is not very likely. They also assume that the volatility as shown by the variance of interest rates is constant, which is also not true in practice. In derivative pricing, the most commonly used methods of options pricing are the famous continuous-time Black-Scholes Option pricing model [12], [51] and [19], the discrete time Binomial model [21], the Cox-Ross-Rubinstein model [21], [44] and the Stochastic models such as the Heston, SABR and ARCH models. These models bear a common assumption that the volatility and the interest rates are known and constant, which in practice is nearly impossible. The true implied volatility (volatility measured using the historical information) is not constant over time. Since the volatility is a key component in option pricing using these models, ignoring its variability nature by assuming that it is constant and known poses high risks in pricing process. There is need to incorporate models that have the capabilities to model the volatility skew and the tails in to the pricing models. The development of financial models has since taken a turn when other models were introduced in the literature. Berndorff-Nielsen in 1977 introduced hyperbolic distributions [48] as a model for wind blown sand and sizes of grains. The distribution used was derived from the fact that its geometric returns form a hyperbola and hence a hyperbolic distribution. The distribution was later discussed by other authors, particularly in Physics to model turbulence. A special case such as NIG was introduced by BN in 1997 [47] and found to have long tails in both directions. Because of its attractive features and analytical tractability, it has attracted attention of authors in the investment market [55]. Its tail behavior is seen as semi-heavy to mean that it is heavier than Gaussian but lighter than Pareto and non-Gaussian stable laws. Other special cases also have similar properties.

## 2.3 Organization of the Research

This research is organized as follows; Chapter three will outline the Modified Bessel function of the third kind, its properties, alternative definitions and some proofs. Construction of the mixing distributions and their properties will also be included. At the end of this chapter, we shall look at the GARCH process with hyperbolic assumptions. Chapter four will provide the construction of the Normal variance-mean mixtures and their properties, parameter estimation for each model and distribution curves the analysis of interest rates data. It will also provide parameter estimates of the GARCH model with hyperbolic assumptions. Finally, chapter five will provide the summery of the research, conclusions and recommendations for future research. An appendix for R codes used will be provided in Chapter A of the research.

# CHAPTER 3

# METHODOLOGY

## 3.1 Introduction

In this chapter various methodologies data collection and analysis which are relevant to this research such as the source, type of data and its management, and the limitation of the study. We will also show how the distributions are arrived at and the forecasting models.

## 3.2 Parameter Estimation

The distribution parameters will be obtained using Maximum Likelihood Estimation and not by hand. This is because, these distributions have modified Bessel functions which is difficult to solve by method of moments. Through method of maximum likelihood, we shall obtain the best estimates for the parameters of the distributions. The mathematics behind this method is in the next section.

## 3.3 The Modified Bessel Function of the Third Kind.

The following are the definitions and properties.

### 3.3.1 Definition 1

Represented as follows (index $\nu$ and order $\omega$ denoted by $K_\nu(\omega)$)

$$
\begin{aligned}
2K_\nu(\omega) &= \int_0^\infty x^{\nu-1} e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx \\
\omega, x > 0
\end{aligned}
\tag{3.3.1}
$$

as proposed by [48] and properties analyzed by [2]

Important properties and alternative definitions are as follows:

**Symmetry Property**

$$K_\nu(\omega) = K_{-\nu}(\omega)$$

Proof provided in [2]

### 3.3.2 Definition 2

$$K_\nu(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\omega cosht} cosh\nu t dt \tag{3.3.2}$$

### 3.3.3 Definition 3

$$K_\nu(\omega) = \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_1^\infty (t^2-1)^{\nu-\frac{1}{2}} e^{-\omega t} dt \tag{3.3.3}$$

### 3.3.4 Definition 4

$$K_\nu(\omega) = \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty (sinh\theta)^{2\nu} e^{-\omega cosh\theta} d\theta \tag{3.3.4}$$

**Proof**

From the equation;

$$K_\nu(\omega) = \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_1^\infty (t^2-1)^{\nu-\frac{1}{2}} e^{-\omega t} dt \tag{3.3.5}$$

We let $t = cosh\theta$, $dt = sinh\theta \; d\theta$ and;

$$\begin{aligned}
\therefore K_\nu(\omega) &= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty (cosh^2\theta-1)^{\nu-\frac{1}{2}} e^{-\omega cosh\theta} d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty \left[\left(\frac{e^\theta+e^{-\theta}}{2}\right)^2 - 1\right]^{\nu-\frac{1}{2}} e^{-\omega cosh\theta} sinh\theta d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty \left(\frac{e^{2\theta}+2+e^{-2\theta}-4}{4}\right)^{\nu-\frac{1}{2}} e^{-\omega cosh\theta} sinh\theta d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty \left(\frac{e^{2\theta}-2+e^{-2\theta}}{4}\right)^{\nu-\frac{1}{2}} e^{-\omega cosh\theta} sinh\theta d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^{(\nu-\frac{1}{2})} e^{-\omega cosh\theta} sinh\theta d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty (sinh\theta)^{2\nu-1} e^{-\omega cosh\theta} sinh\theta d\theta \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty (sinh\theta)^{2\nu} e^{-\omega cosh\theta} d\theta
\end{aligned}$$

### 3.3.5 Summation form of Equation 3.3.3.

$$K_\nu(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\nu-\frac{1}{2}}\frac{\Gamma(\nu+\frac{1}{2}+i)}{\Gamma(\nu+\frac{1}{2}-i)}(2\omega)^{-i} \quad (3.3.6)$$

$$= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\nu-\frac{1}{2}}\frac{\Gamma(\nu+\frac{1}{2}+i)}{\Gamma(\nu+\frac{1}{2}-i)}(2\omega)^{-i}\right] \quad (3.3.7)$$

**Proof**

From definition 3,

$$\begin{aligned}
K_\nu(\omega) &= \left(\frac{\omega}{2}\right)^{\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty (t^2-1)^{\nu-\frac{1}{2}}e^{-\omega t}dt \\
&= \left(\frac{\omega}{2}\right)^{\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}e^{-\omega+\omega}\int_1^\infty (t^2-1)^{\nu-\frac{1}{2}}e^{-\omega t}dt \\
&= \left(\frac{\omega}{2}\right)^{\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\int_1^\infty (t^2-1)^{\nu-\frac{1}{2}}e^{-\omega t+\omega}dt \\
&= \left(\frac{\omega}{2}\right)^{\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\int_1^\infty (t^2-1)^{\nu-\frac{1}{2}}e^{-\omega(t-1)}dt
\end{aligned}$$

*Let* $y=\omega(t-1) \quad t=1+\frac{y}{\omega} \quad dt=\frac{dy}{\omega}$

By substitution:

$$
\begin{aligned}
K_\nu(\omega) &= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} e^{-\omega} \int_0^\infty \left[\left(1+\frac{y}{\omega}\right)^2 - 1\right]^{\nu-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} e^{-\omega} \int_0^\infty \left(1+\frac{2y}{\omega}+\frac{y^2}{\omega^2}-1\right)^{\nu-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} e^{-\omega} \int_0^\infty \left[2\left(1+\frac{y}{2\omega}\right)\frac{y}{\omega}\right]^{\nu-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} e^{-\omega} \int_0^\infty \left[2^2\left(1+\frac{y}{2\omega}\right)\frac{y}{2\omega}\right]^{\nu-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left[2^2\left(1+\frac{y}{2\omega}\right)\frac{y}{2\omega}\right]^{\nu-\frac{1}{2}} e^{-y} dy \\
&= 2^{2(\nu-\frac{1}{2})}\left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left[\left(\frac{y}{2\omega}\right)\left(1+\frac{y}{2\omega}\right)\right]^{\nu-\frac{1}{2}} e^{-y} dy \\
&= 2^{2(\nu-\frac{1}{2})}\left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left(\frac{y}{2\omega}\right)^{\nu-\frac{1}{2}}\left(1+\frac{y}{2\omega}\right)^{\nu-\frac{1}{2}} e^{-y} dy
\end{aligned}
$$

Given that;

$$(1+\frac{y}{2\omega})^{\nu-\frac{1}{2}} = \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i}(\frac{y}{2\omega})^i$$

We have;

$$
\begin{aligned}
K_\nu(\omega) &= 2^{2(\nu-\frac{1}{2})}\left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\frac{e^{-\omega}}{\omega}\int_0^\infty \left(\frac{y}{2\omega}\right)^{\nu-\frac{1}{2}} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i}\left(\frac{y}{2\omega}\right)^i e^{-y}dy \\
&= 2^{2(\nu-\frac{1}{2})}\left(\frac{\omega}{2}\right)^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\frac{e^{-\omega}}{\omega}\int_0^\infty \left(\frac{y}{2\omega}\right)^{\nu-\frac{1}{2}+i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} e^{-y}dy \\
&= \frac{2^{2\nu}}{2\omega}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\int_0^\infty \left(\frac{y}{2\omega}\right)^{\nu-\frac{1}{2}+i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} e^{-y}dy \\
&= \frac{2^{2\nu}}{2\omega}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\int_0^\infty \left(\frac{1}{2\omega}\right)^{\nu-\frac{1}{2}+i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} y^{\nu-\frac{1}{2}+i} e^{-y}dy \\
&= \frac{2^{2\nu}}{2\omega}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\left(\frac{1}{2\omega}\right)^{\nu-\frac{1}{2}+i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} \int_0^\infty y^{\nu-\frac{1}{2}+i} e^{-y}dy \\
&= \frac{2^{2\nu}}{2\omega}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}\left(\frac{1}{2\omega}\right)^{\nu-\frac{1}{2}+i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} \;\Gamma\left(\nu-\frac{1}{2}+i+1\right) \\
&= \frac{2^{2\nu}}{2\omega(2\omega)^{\nu-\frac{1}{2}}}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}(2\omega)^{-i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} \;\Gamma\left(\nu-\frac{1}{2}+i+1\right) \\
&= \frac{2^{2\nu}}{(2\omega)^{\nu+\frac{1}{2}}}\left(\frac{\omega}{2}\right)^\nu \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}e^{-\omega}(2\omega)^{-i} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} \;\Gamma\left(\nu-\frac{1}{2}+i+1\right) \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega} \sum_{i=0}^{\nu-\frac{1}{2}} \binom{\nu-\frac{1}{2}}{i} \;\frac{\Gamma\left(\nu-\frac{1}{2}+i+1\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}(2\omega)^{-i}
\end{aligned}
$$

Since;

$$
\binom{\nu-\frac{1}{2}}{i} = \frac{\left(\nu-\frac{1}{2}\right)!}{\left(\nu-\frac{1}{2}-i\right)!i!} = \frac{\Gamma\left(\nu-\frac{1}{2}+1\right)}{\Gamma\left(\nu-\frac{1}{2}-i+1\right)} = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-i\right)}
$$

We have;

$$
\begin{aligned}
K_\nu(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\nu-\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}+i\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}*\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}-i\right)}(2\omega)^{-i} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\nu-\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}+i\right)}{\Gamma\left(\nu+\frac{1}{2}-i\right)}(2\omega)^{-i} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\sum_{i=1}^{\nu-\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}+i\right)}{i!\Gamma\left(\nu+\frac{1}{2}-i\right)}(2\omega)^{-i}\right)
\end{aligned}
$$

where i is the range of values from 0 to $(\nu - \frac{1}{2})$

$\nu$ = n + $\lambda$ and n = 0,1,2,... and $\pi$ is as a result of the solution of the modified Bessel function.

It then follows immediately from above proof that;

$$
K_{n+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{(\nu+i)!}{i!(\nu-i)!}(2\omega)^{-i}\right]
$$

**Proof**

From equation 3.3.7

$$
K_\nu(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\nu-\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}+i\right)}{i!\Gamma\left(\nu+\frac{1}{2}-i\right)}(2\omega)^{-i}\right]
$$

Making $\nu = n + \frac{1}{2}$ , we have;

$$
\begin{aligned}
K_{n+\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n+\frac{1}{2}-\frac{1}{2}}\frac{\Gamma\left(n+\frac{1}{2}+\frac{1}{2}+i\right)}{i!\Gamma\left(n+\frac{1}{2}+\frac{1}{2}-i\right)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n+\frac{1}{2}-\frac{1}{2}}\frac{\Gamma(n+1+i)}{i!\Gamma(n+1-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{(n+i)!}{i!(n-i)!}(2\omega)^{-i}\right]
\end{aligned}
$$

Alternatively,

$$
K_{\nu}(\omega)=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-1}\frac{(\nu+i-1)!}{i!(\nu-i-1)!}(2\omega)^{-i}\right]
$$

**Proof**

Making $\nu = n-\frac{1}{2}$ , we have;

$$
\begin{aligned}
K_{n-\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-\frac{1}{2}-\frac{1}{2}}\frac{\Gamma\left(n-\frac{1}{2}+\frac{1}{2}+i\right)}{i!\Gamma\left(n-\frac{1}{2}+\frac{1}{2}-i\right)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-1}\frac{\Gamma(n+i)}{i!\Gamma(n-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-1}\frac{(n+i-1)!}{i!(n-i-1)!}(2\omega)^{-i}\right]
\end{aligned}
$$

Its also seen that;

$(a)K_{\frac{1}{2}}(\omega) = K_{-\frac{1}{2}(\omega)} = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$
$(b)K_{\frac{3}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{1}{\omega}\right)$

$(c)K_{\frac{5}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{3}{\omega}+\frac{3}{\omega^2}\right)$
$(d)K_{\frac{7}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{6}{\omega}+\frac{15}{\omega^2}+\frac{15}{\omega^3}\right)$

**Proof: From,**

$$K_{n+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\nu-\frac{1}{2}}\frac{\Gamma\left(\nu+\frac{1}{2}+i\right)}{i!\Gamma\left(\nu+\frac{1}{2}-i\right)}(2\omega)^{-i}\right]$$

when $n=0$ we have;

$$\begin{aligned} K_{\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{0}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}{1!\Gamma\left(\frac{1}{2}+\frac{1}{2}-1\right)}(2\omega)^{-1}\right] \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}(1+0) \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega} \end{aligned}$$

when $n=1$ we have;

$$\begin{aligned} K_{\frac{3}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{1}\frac{\Gamma\left(\frac{3}{2}+\frac{1}{2}+1\right)}{1!\Gamma\left(\frac{3}{2}+\frac{1}{2}-1\right)}(2\omega)^{-1}\right] \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma(3)}{\Gamma(1)2\omega}\right] \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{1}{\omega}\right] \end{aligned}$$

when $n = 2$ we have;

$$\begin{aligned}
K_{\frac{5}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{2}\frac{\Gamma\left(\frac{5}{2}+\frac{1}{2}+i\right)}{i!\Gamma\left(\frac{5}{2}+\frac{1}{2}-i\right)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma\left(\frac{5}{2}+\frac{1}{2}+1\right)}{1!\Gamma\left(\frac{5}{2}+\frac{1}{2}-1\right)2\omega}+\frac{\Gamma\left(\frac{5}{2}+\frac{1}{2}+2\right)}{2!\Gamma\left(\frac{5}{2}+\frac{1}{2}-2\right)(2\omega)^2}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma(4)}{\Gamma(2)2\omega}+\frac{\Gamma(5)}{2!\Gamma(1)4\omega^2}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{6}{2\omega}+\frac{24}{8\omega^2}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{3}{\omega}+\frac{3}{\omega^2}\right]
\end{aligned}$$

Again when $n = 3$ we have;

$$\begin{aligned}
K_{\frac{7}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{3}\frac{\Gamma\left(\frac{7}{2}+\frac{1}{2}+i\right)}{i!\Gamma\left(\frac{7}{2}+\frac{1}{2}-i\right)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma\left(\frac{7}{2}+\frac{1}{2}+1\right)}{1!\Gamma\left(\frac{7}{2}+\frac{1}{2}-1\right)2\omega}+\frac{\Gamma\left(\frac{7}{2}+\frac{1}{2}+2\right)}{2!\Gamma\left(\frac{7}{2}+\frac{1}{2}-2\right)(2\omega)^2}+\frac{\Gamma\left(\frac{7}{2}+\frac{1}{2}+3\right)}{3!\Gamma\left(\frac{7}{2}+\frac{1}{2}-3\right)(2\omega)^3}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma(5)}{\Gamma(3)2\omega}+\frac{\Gamma(6)}{2!\Gamma(2)(4\omega)^2}+\frac{\Gamma(7)}{6\Gamma(1)8\omega^3}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{24}{4\omega}+\frac{120}{8\omega^2}+\frac{720}{48\omega^3}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{6}{\omega}+\frac{15}{\omega^2}+\frac{15}{\omega^3}\right]
\end{aligned}$$

For more properties and proofs, see [2]

## 3.4 The Mixing Distributions

Here, we provide distributions which will be used in the mixing process under different values of $\lambda$. The $\lambda$ values are chosen based on the required mixing distribution. We specifically consider the Barndorff-Nielsen parametization of $\omega = \delta\gamma$. This will express $\omega$ interms of other parameter as in this case, $\gamma$ and $\delta$. (see [7], [37], [6]). We proceed with construction as follows (see [7] for details)

### 3.4.1 Generalized form of the Mixing distributions and Properties

The pdf's of the Generalized form of the mixing distribution (see [31], [9] and [36]) is obtained by substituting $\delta\gamma$ into the equation 3.3.1. This is then simplified to obtain :

$$g(z) = \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{z^{\lambda-1}}{2K_{\lambda}(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \tag{3.4.1}$$

where $\delta, \gamma > 0$ with the following properties:
the n-th moment is;

$$E[Z^n] = \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_{\lambda}(\delta\gamma)}$$

and the variance:

$$Var[Z] = \left(\frac{\delta}{\gamma}\right)^2 \left[\frac{K_{\lambda+2}(\delta\gamma)}{K_{\lambda}(\delta\gamma)} - \left(\frac{K_{\lambda+1}(\delta\gamma)}{K_{\lambda}(\delta\gamma)}\right)^2\right]$$

For more properties and proofs, see [7].

From the distribution above 3.4.1, we can easily construct the distributions of interest as follows;

### 3.4.2 Construction of RIG Mixing Distribution

Suppose $\lambda = \frac{1}{2}$, and substituting in equation 3.4.1 above, we obtain;

$$\begin{aligned} g(z) = & \left(\frac{\gamma}{\delta}\right)^{\frac{1}{2}} \frac{z^{-\frac{1}{2}}}{2k_{\frac{1}{2}}(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \left(\frac{z^{-\frac{1}{2}}}{2k_{\frac{1}{2}}(\delta\gamma)}\right) e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \frac{z^{-\frac{1}{2}}}{\sqrt{\frac{2\pi}{\delta\gamma}} e^{-\delta\gamma}} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \frac{z^{-\frac{1}{2}}}{\sqrt{\frac{2\pi}{\delta\gamma}}} e^{\delta\gamma} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{\delta\gamma} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \end{aligned} \tag{3.4.2}$$

**Properties**

Here, the expectation, variance, Skewness and Kurtosis are provided. The proofs are simplified through the n-th moment generator as outlined in 3.4.1

$$
\begin{aligned}
E(Z^n) &= \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_\lambda \delta\gamma} \\
&= \frac{(\delta\gamma+1)}{\gamma^2} \\
E(Z^2) &= \left(\frac{\gamma}{\delta}\right)^2 \frac{k_{\frac{5}{2}}(\delta\gamma)}{k_{\frac{1}{2}}(\delta\gamma)} \\
&= \frac{\delta^2}{\gamma^2} * \frac{\delta^2\gamma^2+3\delta\gamma+3}{\delta^2\gamma^2} \\
&= \frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4} \\
Var(Z) &= \frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4} - \left[\frac{\delta^2\gamma^2+2\delta\gamma+1}{\gamma^4}\right] \\
&= \frac{\delta\gamma+2}{\gamma^4}
\end{aligned}
$$

The Modified Bessel function is solved as in section 3.4

Also, from;

$$SK(z) = \mu_3(Z) = E(Z^3) - 3E(Z)E(Z^2) + 2E(Z)^3$$

Recall

$E(Z) = \frac{(\delta\gamma+1)}{\gamma^2} and \quad E(Z^2) = \frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4}$ as in the properties above, we obtain;

$$E(Z^3) = \frac{\delta^3\gamma^3+6\delta^2\gamma^2+15\delta\gamma+15}{\gamma^6}$$

We therefore have, $\mu_3(Z)$ as follows;

$$
\begin{aligned}
&= \frac{\delta^3\gamma^3+6\delta^2\gamma^2+15\delta\gamma+15}{\gamma^6} - 3\Big(\frac{\delta\gamma+1}{\gamma^2}\frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4}\Big) + 2\Big(\frac{\delta\gamma+1}{\gamma^2}\Big)^3 \\
&= \frac{3\delta\gamma+8}{\gamma^6} \\
Sk(Z) &= \frac{\mu_3(Z)}{(Var(Z))^{1.5}} \\
&= \frac{3\delta\gamma+8}{\gamma^6}\Big(\frac{\gamma^4}{\delta\gamma-1}\Big)^{1.5} \\
&= \frac{3\delta\gamma+8}{(\delta\gamma-1)^{1.5}}
\end{aligned}
$$

**Suppose** $\lambda = -\frac{1}{2}$

Replacing into the equation 3.4.1, and going through the same process, we obtain the Inverse Gaussian distribution (which is a mixing distribution for the Normal Inverse Gaussian distribution) as follows;

$$
g(z) = \frac{\delta}{\sqrt{2\pi}} z^{-\frac{3}{2}} e^{(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \tag{3.4.3}
$$

**Properties**

The first, second, and third moments are studied here:

$$
\begin{aligned}
E(Z^n) &= \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_\lambda \delta\gamma} \\
&= \left(\frac{\delta}{\gamma}\right) \frac{K_{\frac{1}{2}}(\delta\gamma)}{K_{-\frac{1}{2}}\delta\gamma} \\
&= \left(\frac{\delta}{\gamma}\right) \\
E(Z^2) &= \left(\frac{\gamma}{\delta}\right)^2 \frac{k_{\frac{3}{2}}(\delta\gamma)}{k_{-\frac{1}{2}}(\delta\gamma)} \\
&= \frac{\delta^2}{\gamma^2}\frac{\delta\gamma+1}{\delta\gamma} \\
&= \frac{\delta(\delta\gamma+1)}{\gamma^3} \\
VAR(Z) &= \frac{\delta(\delta\gamma+1)}{\gamma^3} - \frac{\delta^2}{\gamma^2} \\
&= \frac{\delta}{\gamma^3}
\end{aligned}
$$

The skewness is given as;

Sk(z)=$\frac{\mu_3(z)}{(var(z))^{1.5}}$

Therefore:

$$
\mu_3(z) = \frac{3\delta}{\gamma^5}
$$

It is to be noted that, the mixing distributions can be obtained by varying the value of $\lambda$ to give different kinds of distribution which are more complex as the value of $\lambda$ increases and are difficult to solve numerically. Therefore, since we are only interested in the two studied above, we will not focus this discussion on the remaining mixing distributions.

### 3.5 The G.A.R.C.H model

Introduced by [15] to allows for conditional variance to depend on the previous lag unlike the ARCH model. We will use this model as a forecasting tool for the interest rates.

#### 3.5.1 Definition 1

$\varepsilon_t$ is called a G.A.R.C.H (x,y) process iff:

1. $E(\varepsilon_t|\varepsilon_u, u < t) = 0$

2. There exist constants $\omega$, $\alpha_i$, for $i = 1,...,y$, and $\beta_j$, for $j = 1,...,x$ such that;

$$R_t{}^2 = \omega + \sum_{i=1}^{y} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{x} \beta_j \sigma_{t-j}^2 \tag{3.5.1}$$

under a normal assumption.
where $\omega > 0$, $\alpha_i > 0$, $\beta_j > 0$

#### 3.5.2 Definition 2

Let $\eta_t$ represent an *iid* sequence $\eta$ distributed. $\varepsilon_t$ is a G.A.R.C.H(p,q) process in the sequence $\eta_t$ iff; $\varepsilon_t = \sigma_t \eta_t$ and given 3.5.1, we can sufficiently say that;

$$R_t{}^2 = \omega + \sum_{i=1}^{q} \alpha_i \sigma_{t-i}^2 \eta_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{3.5.2}$$

where $\omega$, $\alpha_i$, and $\beta_j$ are constants and positive values. $\eta$ is the distribution of the error term of the hyperbolic family.
For more properties and proof, see [26] and [15]

### 3.6 Model Selection Criteria

The models will be evaluated based on the Akaike Information Criteria because of its ability to provide the best approximating model for the true model in cases where the true model is not in the candidate set. This can be approximated as follows;

$$AIC = -2log(H) + 2n \tag{3.6.1}$$

where H is the likelihood and n the number parameters in the model.

In some cases, we shall use Goodness-of-fit tests.

# CHAPTER 4

# RESULTS AND DISCUSSION

## 4.1 Introduction

In this chapter, we will consider an application of the models discussed previously to 91-day Treasury Bills Interest rates from the Central Bank of Kenya and Commercial Bank Interest rates for the period of July 1991 to July 2022 which are carefully selected to reduce biasedness and error by having enough data. Fitting is carefully done using all the uni-variate "Semi-heavy" tail distributions with the available data to find the distribution that best captures the statistical properties of the data. We will evaluate the models using the Akaike Information Criteria (AIC) and the Log-Likelihood (Log-Lik) methods of model evaluation and rank them.

The distributions will be fitted to monthly returns data that will be obtained as follows;

Let $I(t), t > 0$, be a sequence of interest rates in a specific period of time. We investigate the monthly interest rate returns in order to allow comparison of investment in different interest rate dependent securities. The monthly continuously compounded returns, $R$, at time $t$ for $t = 1, 2, 3, 4, ...$ therefore is defined by;

$$R_t = ln\left[\frac{I(t)}{I(t-1)}\right] \tag{4.1.1}$$

where $Ri's$ are independently and identically distributed over time.

## 4.2 Data Used in the Research

Secondary data for 91-day Treasury Bills interest rates and Commercial Bank Weighted average interest rates (Base lending rates) from the Central Bank of Kenya from 1991 to 2021 were used.

The log was used to eliminate the unit root behavior intrinsic to $I_t$ and therefore achieving stationarity.

### 4.3 Mean Reversion of Interest Rates

The Rendleman and Bartter model assumes that the short-term interest rates behave like stock price only that interest rates comes back to a long term average, a phenomena that is not captured in the Rendleman and Bartter model and other models such as Ho-Lee model and CIR model which only assumes mean reversion for long term interest rates. For example, high rates causes economic slow down and less requirements for funds for the borrowers. This therefore causes interest rates to drift negatively. On the other hand, when the rates are low, the contrary happens and a positive drift is experienced. While the short-term interest rates and the yield curve tend to revert to the normal average level, the long-term rates can deviate from it. Even though the economic theory holds an assumption that the nominal interest rates experience the property of mean reversion in the long-run, the empirical studies is not conclusive. The available literature concludes that the unit rot hypothesis cannot be rejected with regard to the long-term bond yields implying that they would not be mean-reverting. Using the data available, we will test for this property of interest rates in the later section.

### 4.4 Construction of Normal Variance-Mean Mixtures and Properties

The mixing process yields distributions called the Normal Variance-Mean mixtures. This is a family of flexible distributions with special cases such as the Normal Inverse Gaussian distribution, the Variance-Gamma mixture, and Student-t distribution (the mixing distributions are Inverse Gaussian, Gamma and Inverse Gamma distributions respectively) (see [8], [7]). This family of distributions makes an assumption that the variance ($\sigma^2$X) is not fixed but is related to the mean ($\mu$+$\beta$X) [48], [53].

The pdf of the conditional distribution is of the form;

$$f(x|z) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{1}{2}\left[\frac{((x-\mu)+\beta z)^2}{z}\right]} dz \tag{4.4.1}$$

Where Z is a r.v greater than 0, $\mu$ and $\beta$ are constants. Therefore, the properties of the mixture distribution we will use are as follows;

$$E(X) = \mu + \beta E(Z) \tag{4.4.2}$$

$$Var(X) = E(Z) + \beta^2 var(Z) \tag{4.4.3}$$

$$Sk(X) = 3\beta Var(Z) + \beta^3 \mu_3(Z) \tag{4.4.4}$$

$$M_X(t) = e^{\mu t} M_Z\left(\beta t + \frac{t^2}{2}\right) \tag{4.4.5}$$

For proofs, see [8], [33]

### 4.4.1 Mixture when $\lambda = -\frac{1}{2}$

We have equation 3.4.3, as the mixing distribution and $X|Z \sim N(\mu + \beta z, z)$ as the conditional distribution, therefore, by using direct integration;

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{1}{2}\left[\frac{((x-\mu)+\beta z)^2}{z}\right]} * \frac{\delta}{\sqrt{2\pi}} z^{-\frac{3}{2}} e^{(\delta\gamma)} e^{-\frac{1}{2}\left[\frac{\delta^2}{z}+\gamma^2 z\right]} dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} \int_0^\infty z^{-2} e^{-\frac{1}{2}\left[\frac{((x-\mu)+\beta z)^2}{z}+\frac{\delta^2}{z}+\gamma^2 z\right]} dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} \int_0^\infty z^{-2} e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{z}+2(x-\mu)\beta+\beta^2 z+\frac{\delta^2}{z}+\gamma^2 z\right]} dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} e^{(x-\mu)\beta} \int_0^\infty z^{-2} e^{-\frac{(\beta^2+\gamma^2)}{2}\left[z+\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)z}\right]} dz
\end{aligned}
$$

The aim is to simplify the exponential part so we use change of variable technique by letting $z = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t \Rightarrow dz = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}dt$ Note also that; $(\beta^2+\gamma^2) = \alpha^2$, therefore

$$
\begin{aligned}
f(x) &= \int_0^\infty \left[\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t\right]^{-2} e^{-\frac{(\beta^2+\gamma^2)}{2}\left[\left(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t\right)+\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t)}\right]} \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}dt \\
&= \left[\sqrt{\frac{(x-\mu)^2+\delta^2}{(\alpha^2)}}\right]^{-1} \int_0^\infty t^{-2} e^{-\frac{\alpha^2}{2}*\sqrt{\frac{(x-\mu)^2+\delta^2}{(\alpha^2)}}(t+\frac{1}{t})} dt \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} e^{(x-\mu)\beta} \frac{\alpha}{\sqrt{(x-\mu)^2+\delta^2}} 2K_1\left[\sqrt{\alpha^2\left((x-\mu)^2+\delta^2\right)}\right]
\end{aligned}
$$

This is the pdf of a Normal Inverse Gaussian distribution.

x, $\beta$, $\mu \in \mathbb{R}$, $\delta > 0$

$K_1(*)$ is the Modified Bessel function of index of 1.

**Alpha parameter,** $\alpha$

This is the steepness parameter, hence describe the shape of the graph.

**Beta Parameter,** $\beta$

This is the asymmetry parameter, and hence determine the skewness of the graph. We can also say that this parameter captures the skewness of the distribution.

**Delta parameter, $\delta$**

This is the scale parameter. An increase in $\delta$ flattens the distribution. If $\alpha$ remains constant, an increase in $\delta$ decreases the kurtosis of the distribution.

**Mu parameter, $\mu$**

This is responsible for the location of the curve and determines where on the x axis, the curve is moved.

**Properties**

From the conditions given in 4.4.3, 4.4.4, and 4.4.5 above, we can easily obtain the properties of the mixed distribution as follows;

$$
\begin{aligned}
E(X) &= \mu + \beta E(Z) \\
&= \mu + \beta\left(\frac{\delta}{\gamma}\right) && (4.4.6) \\
Var(X) &= E(Z) + \beta^2 var(Z) \\
&= \frac{\alpha^2\delta}{\gamma^3} \\
Sk(X) &= 3\beta Var(Z) + \beta^3\mu_3(z) && (4.4.7) \\
&= \frac{3\beta\left(\frac{\delta}{\gamma^3}\right) + \beta^3\left(\frac{3\delta}{\delta^5}\right)}{\left(\frac{\delta}{\gamma^3}\right)^{1.5}} \\
&= \frac{3\beta}{\alpha(\delta\gamma)^{0.5}} && (4.4.8) \\
Kurt(X) &= 3\left(\frac{1+4\beta^2/\alpha^2}{\gamma\delta}\right) && (4.4.9)
\end{aligned}
$$

### 4.4.2 Mixture when $\lambda = \frac{1}{2}$

Through similar approach:

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{1}{2}\left(\frac{((x-\mu)+\beta z)^2}{z}\right)} * \frac{\gamma}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} dz \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} \int_0^\infty z^{-1} e^{-\frac{1}{2}\left(\frac{((x-\mu)+\beta z)^2}{z}+\frac{\delta^2}{z}+\gamma^2 z\right)} dz \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} \int_0^\infty z^{-1} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2}{z}+2(x-\mu)\beta+\beta^2 z+\frac{\delta^2}{z}+\gamma^2 z\right)} dz \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} e^{(x-\mu)\beta} \int_0^\infty z^{-1} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2+\delta^2}{z}+(\beta^2+\gamma^2)z\right)} dz \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} e^{(x-\mu)\beta} \int_0^\infty z^{-1} e^{-\frac{(\beta^2+\gamma^2)}{2}\left(z+\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)z}\right)} dz
\end{aligned}
$$

Again, we intend to simplify the exponential part hence we use the change of variable technique by letting $z = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t \Rightarrow dz = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}dt$ Note also that; $(\beta^2+\gamma^2) = \alpha^2$
Dropping the constants and replacing back later we get;

$$
\begin{aligned}
f(x) &= \int_0^\infty \left[\sqrt{\frac{(x-\mu)^2+\delta^2}{\alpha^2}t}\right]^{-1} e^{-\frac{(\beta^2+\alpha^2)}{2}\left[\left(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}t}\right)+\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)\left(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}t}\right)}\right]} \sqrt{\frac{\delta^2+(x-\mu)^2}{\alpha^2}}dt \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} e^{(x-\mu)\beta} \int_0^\infty t^{-1} e^{-\frac{\alpha^2}{2}*\sqrt{\frac{(x-\mu)^2+\delta^2}{(\alpha^2)}}\left(t+\frac{1}{t}\right)} dt \\
&= \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} 2K_0\left(\sqrt{\delta^2+\alpha(x-\mu)^2}\right) e^{\beta(x-\mu)} \qquad (4.4.10)
\end{aligned}
$$

Which is the pdf of the Reciprocal Inverse Gaussian distribution. Its properties are obtained as follows;

**Properties**

From the conditions in the sections 4.4.3, 4.4.4, and 4.4.5, we can easily obtain the properties;

$$\begin{aligned}
E(X) &= \mu+\beta E(Z) \\
&= \mu+\beta\frac{(1+\delta\gamma)}{\gamma^2} \qquad (4.4.11)\\
Var(X) &= E(Z)+\beta^2 var(Z) \\
&= \frac{(1+\delta\gamma)}{\gamma^2}+\beta^2\frac{(\delta\gamma+2)}{\gamma^4} \\
&= \frac{\alpha^2(1+\delta\gamma)\beta^2}{\gamma^4} \qquad (4.4.12)\\
Sk(X) &= 3\beta Var(X)+\beta^3\mu_3(z) \\
&= \frac{3\beta(\frac{2+\delta\gamma}{\gamma^4})+\beta^3(\frac{3\delta\gamma+8}{\gamma^6})}{(\frac{\alpha^2(1+\delta\gamma)\beta^2}{\gamma^4})^{1.5}} \\
&= \frac{3\beta\alpha^2(2+\delta\gamma)+2}{(\alpha^2(1+\delta\gamma)+\beta^2)^{1.5}} \qquad (4.4.13)
\end{aligned}$$

We also note that the latter mixed distribution (Reciprocal Inverse Gaussian Distribution) is complex as evidenced by the properties. This therefore implies that the parameter estimation using the method of moments is nearly impossible. Similarly, parameter estimation using Maximum Likelihood can prove quite complex that would require a programming language to compute.

On the other hand, a comparison of the moments of the Reciprocal Inverse Gaussian Distribution and the NIG distribution shows that, the moments of NIG are simpler and therefore estimating the parameter by maximum likelihood is even simpler, while the method of moments might prove complex.

Even though we may not go into the details of the other special cases of the GHD's, it is worth outlining them at this stage because we may use them in fitting the interest rates data in the subsequent section. These distributions are the Variance-Gamma (VG) distribution ([42] and

[40]), Student's t distribution ([56], [14]), 5-parameter Hyperbolic distribution ([43]) and the Gaussian distribution. The pdf's of the distributions are outlined below (For derivation and proofs of properties, relevant materials showing the proofs have been provided).

**The Variance-Gamma Distribution**

The mixing distribution is Gamma($\alpha$,$\beta$). For details, see [40] and McNicholas et al, 2013, [42]. The pdf of the V-G distribution is;

$$f_{V-G}(x;\alpha,\lambda,\beta,0,\mu) \quad = \quad 2\Big(\frac{\alpha^2-\beta^2}{2}\Big)^{\lambda}\Big(\frac{|x-\mu|}{\alpha}\Big)^{\lambda-\frac{1}{2}}\frac{K_{\lambda-\frac{1}{2}(\alpha|x-\mu)}e^{\beta(x-\mu)}}{\sqrt{2\pi}\Gamma(\lambda)} \qquad (4.4.14)$$

**The Student's t-distribution**

The mixing distribution is inverse gamma. For details on proof and properties, see [56], and [14]. The pdf is of the form;

$$f_{t-dist}(x;\lambda,|\beta|,\beta,\delta,\mu) \quad = \quad \frac{2^{(-\frac{n+1}{2})}\sigma^n}{\sqrt{2\pi}\Gamma(\frac{n}{2})}\left(\frac{\sqrt{\delta^2+(x-\mu)^2}}{|\beta|}\right)^{(-\frac{n+1}{2})} K_{(-\frac{n+1}{2})}\left(|\beta|\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)} \qquad (4.4.15)$$

**The Hyperbolic Distribution**

The pdf of the hyperbolic distribution is;

$$f_{hyp}(x;\alpha,\beta,\delta,\mu) \quad = \quad \frac{\sqrt{(\alpha^2-\beta^2)}}{2\alpha\beta K_1(\delta\sqrt{(\alpha^2-\beta^2)})}e^{[-\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)]} \qquad (4.4.16)$$

$\alpha > 0$, $\delta > 0$, $\mu \in \mathbb{R}$, $\beta \in (-\alpha, \alpha)$

For proof and properties, see [43]

## 4.5 Parameter Estimation

### 4.5.1 The Maximum Likelihood Estimation (MLE)

The MLE [32] is another commonly used method of parameter estimation in many statistical and Actuarial Applications apart from method of moments and the EM type algorithm [23]. In this case, the parameters are estimated by assuming that the $r_1, r_2, \ldots, r_n$ are iid. The models parameters are obtained by maximizing the following distribution (this will not be done for all the distributions of interest);

$$f_{NIG}(\alpha,\beta,\mu,\delta) = \frac{\delta}{2\pi} e^{(\delta\gamma)} e^{(x-\mu)\beta} \frac{\alpha}{\sqrt{(x-\mu)^2+\delta^2}} 2K_1 \left[\sqrt{\alpha^2\left((x-\mu)^2+\delta^2\right)}\right] \quad (4.5.1)$$

The log-likelihood is obtained by summing up all the variables at each data point as follows;

$$logL(\alpha,\beta,\mu,\delta) = \sum_{i=1}^{n} log f_{NIG}(x;\alpha,\beta,\mu,\delta) \quad (4.5.2)$$

This is done for the remaining distributions as follows;

$$logLf_{V-G}(x;\alpha,\lambda,\beta,0,\mu) = \sum_{i=1}^{n} log f_{V-G}(x;\alpha,\lambda,\beta,0,\mu) \quad (4.5.3)$$

$$logLf_{t-dist}(x;\lambda,|\beta|,\beta,\delta,\mu) = \sum_{i=1}^{n} log f_{t-dist}(x;\lambda,|\beta|,\beta,\delta,\mu) \quad (4.5.4)$$

$$logLf_{hyp}(x;\alpha,\beta,\delta,\mu) = \sum_{i=1}^{n} log f_{hyp}(x;\alpha,\beta,\delta,\mu) \quad (4.5.5)$$

The log likelihood therefore is;

$$logL(\alpha,\beta,\mu,\delta) = nlog(\alpha) - \frac{1}{2}\sum_{i=1}^{n} log[\delta^2 + (x_i-\mu)^2] + n[\delta\sqrt{\alpha^2-\beta^2} - \beta\mu] + \beta\sum_{i=1}^{n} x_i$$
$$-nlog(\pi) + nlog(\delta) + \sum_{i=1}^{n} logK_1(\alpha\sqrt{\delta^2+(x_i-\mu)^2}) \quad (4.5.6)$$

Differentiating this log likelihood with respect to each of the four parameters and equating each to zero gives a system of four equations with four parameters. However, the presence of the Bessel function ($K_1(\alpha\sqrt{\delta^2+(x_i-\mu)^2})$) in each of the four equations makes maximization problem too complex to solve for the parameters by hand. Therefore, a maximizing algorithm was introduced to solve this problem. This algorithm was introduced by Breymann and Luthi in 2013 in R to perform parameter estimation using the *ghyp* R package.
All the R codes and other materials including sample of an extract of the data will be included in the Appendix after the reference section. Data analysis begins in the following subsection.

## 4.6 Descriptive Statistics

The table below shows the moments of the 91-day Treasury Bills and the Commercial Bank interest rates including the higher moments.

The table 4.6.1 shows some summary statistics for the interest rates data.

Table 4.6.1: Summary Statistics

| Rates | Min. | 1st Qu. | Median | Mean | 3rd Qu. | Max | Skewness | Kurtos |
|---|---|---|---|---|---|---|---|---|
| T.R | -0.2896005 | -0.0188708 | -0.0005412 | -0.00110384 | 0.01578 | 0.26162 | -0.31146 | 8.5935 |
| Bank | -0.193207 | -0.0041124 | 0.0003479 | -0.00353 | 0.0059352 | 0.215995 | 0.137206 | 18.942 |

From the table above, it is sufficient to conclude that the underlying interest rates used in pricing of bonds and stocks really do not follow the normal distribution as assumed. This is manifested by Skewness not equal to zero and Kurtosis greater than 3 as in the normal distribution. When this is used in pricing derivatives, error may be committed which might lead to under or overpricing of the derivatives in question.

## 4.7 Statistical Tests for Normality

### 4.7.1 Kolmogorov-Smirnov (K-S) Test

Carrying out the K-S test on the set of data we obtain the following results.

Table 4.7.1: Kolmogorov-Smirnov test for Commercial Bank and Treasury Bills Interest Rates

| | D | P-value |
|---|---|---|
| C.B | 0.25586 | < 2.2e-16 |
| T.B | 0.16042 | 1.6042e-18 |

### 4.7.2 Shapiro-Wilk (S-W) Test

This tests how good sample data fits the normal distribution by ordering the standardazing of the sample. Performing the S-W test on the data, we obtain the following results.

Table 4.7.2: Shapiro-Wilk test for Commercial Bank and Treasury Bills Interest Rates

| | W | P-value |
|---|---|---|
| C.B | 0.64896 | < 2.2e-16 |
| T.B | 0.85533 | < 2.2e-16 |

**Conclusion:**

In the two tests we consider the following hypothesis;

H0: the data follows a normal distribution

H1: the data follows a distribution other than the normal distribution.

The criteria under this test is that if the p < $\alpha$, reject H0. From the above results, the p-values are less than the $\alpha$ of 0.05.

We therefore conclude that the data are significantly different from the normal distribution. This is because, the parameter estimates we obtained (in the following section), shows us that the data is not normally distributed.

## 4.8 Test for Mean Reversion of Interest Rates

### 4.8.1 Dickey-Fuller test for Mean Reversion

This method tests stationarity of the time series data. A stationary time series data have properties that do not depend on the time of observation of the data. This is a characteristic of a mean reversed data. The table below contains results of this test.

Table 4.8.1: Dickey-Fuller test for Commercial Bank and Treasury Bills Interest Rates

| | Dickey-Fuller | p-value |
|---|---|---|
| T.B | -8.2525 | 0.01 |
| C.B | -12.543 | 0.01 |

In this test, the hypothesis is as follows,

H0: the data is non-stationary

H1: the data is stationary.

The p-value is less that the alpha value. We therefore reject the (H0) and conclude that there is mean reversion in the data.

## 4.9 Plots of the Data

### 4.9.1 Histograms of the Interest Rates Data

The plots of the CBK rates are as shown below;

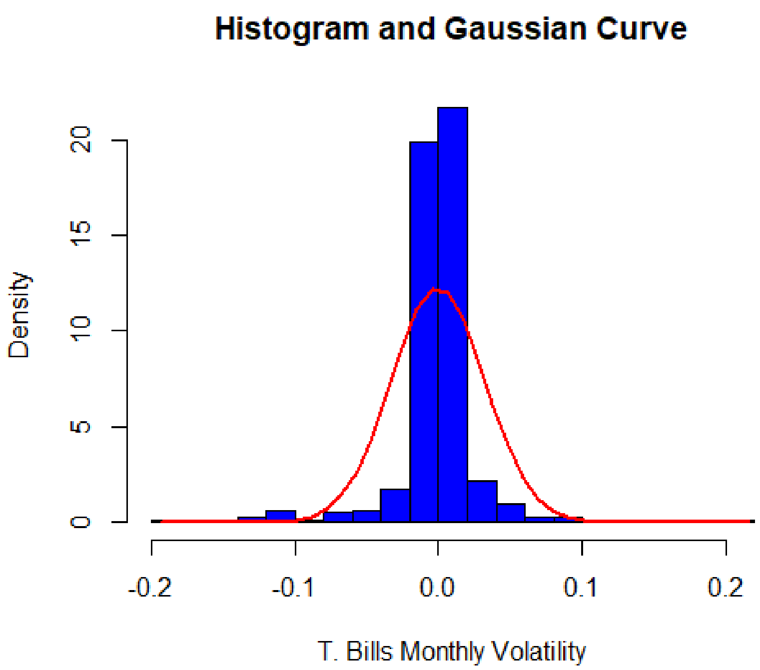


Figure 4.9.1: 91-day Treasury Bill Interest Rates

In these two figures, we have a histogram and a normal distribution curve showing the how the interest rates are distributed. From the figure, it is evident that the normal distribution does not capture the peakedness or the Kurtosis in the data. It is also seen that the data has fat tails, which the distribution is not able to capture properly.

Even though the graph shows a normal distribution with normal parameters, the higher

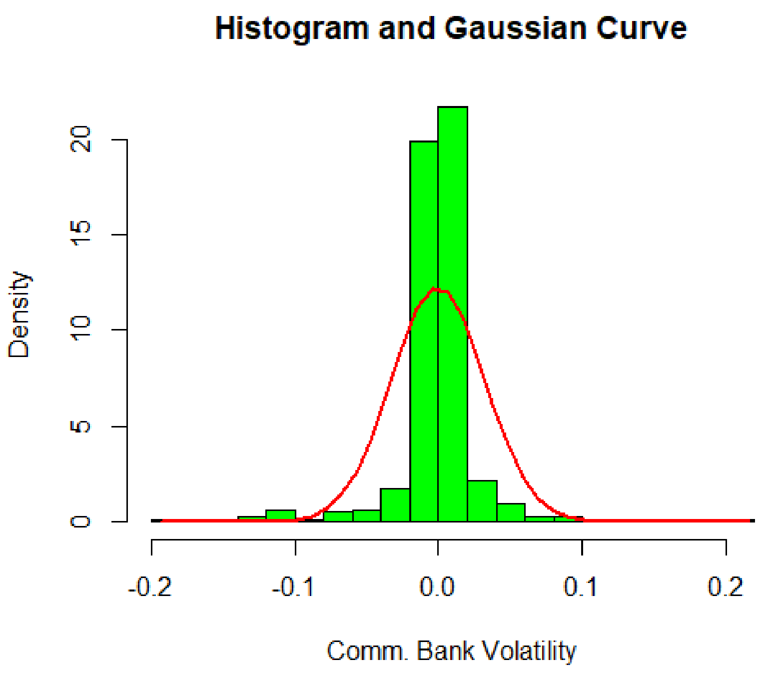


Figure 4.9.2: Commercial Bank Weighted Average Interest Rates

moments such as the Skewness and Kurtosis as well as the tail behavior, which we were interested in, are not well captured by this distribution.

### Time Series Plots

The time series plots for the data and the returns are as follows;

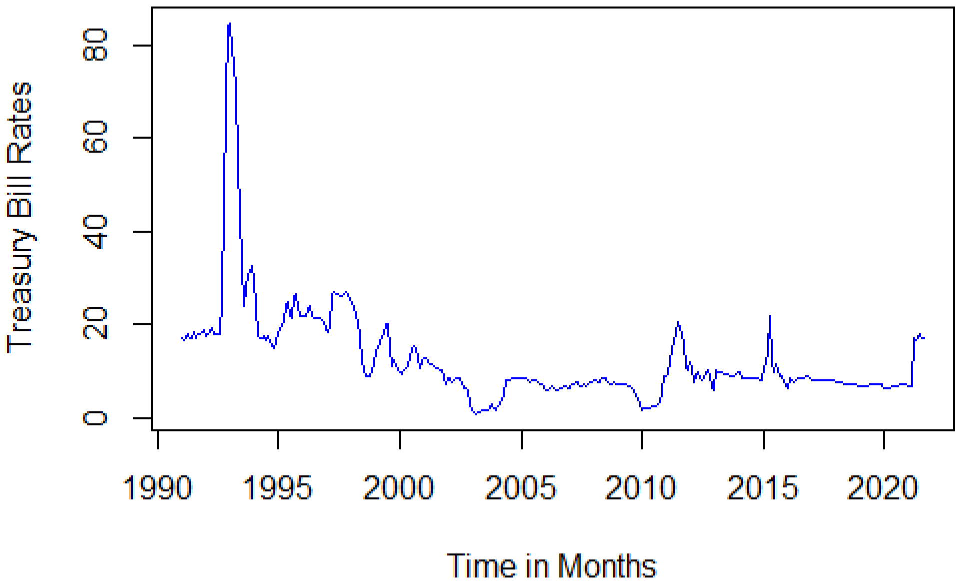


Figure 4.9.3: 91-day treasury bill interest rates time series plot

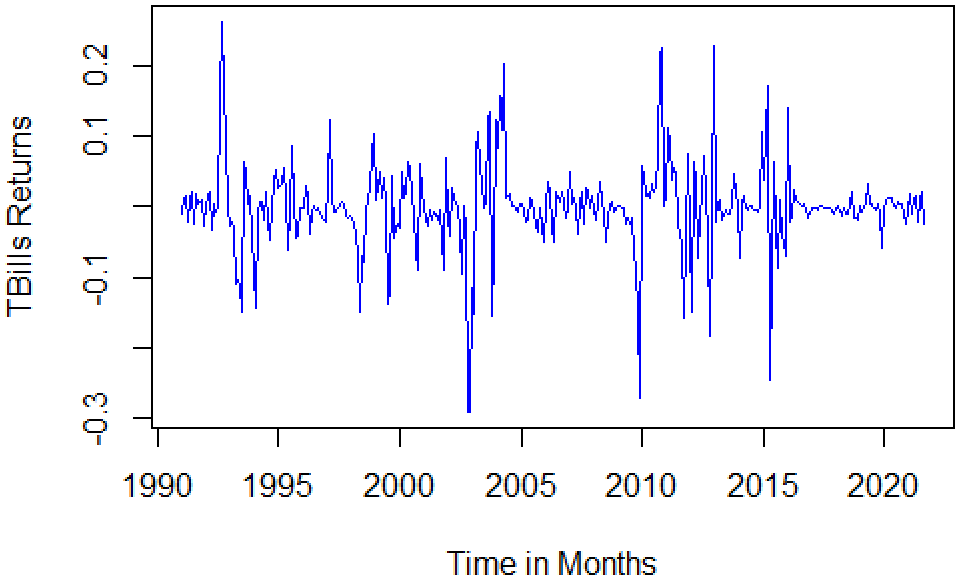


Figure 4.9.4: 91-day Treasury Bills returns.

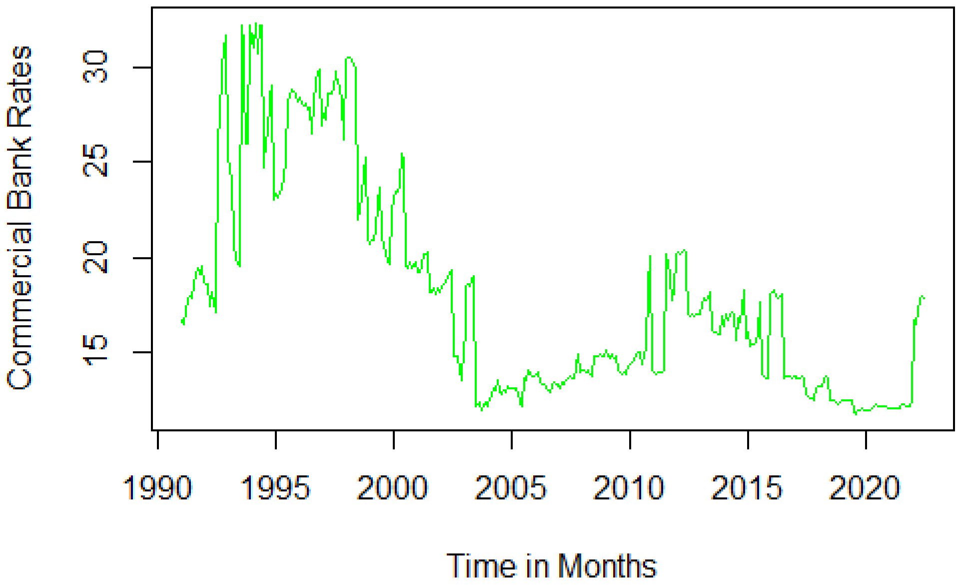


Figure 4.9.5: Commercial Bank Interest rates time series plot.

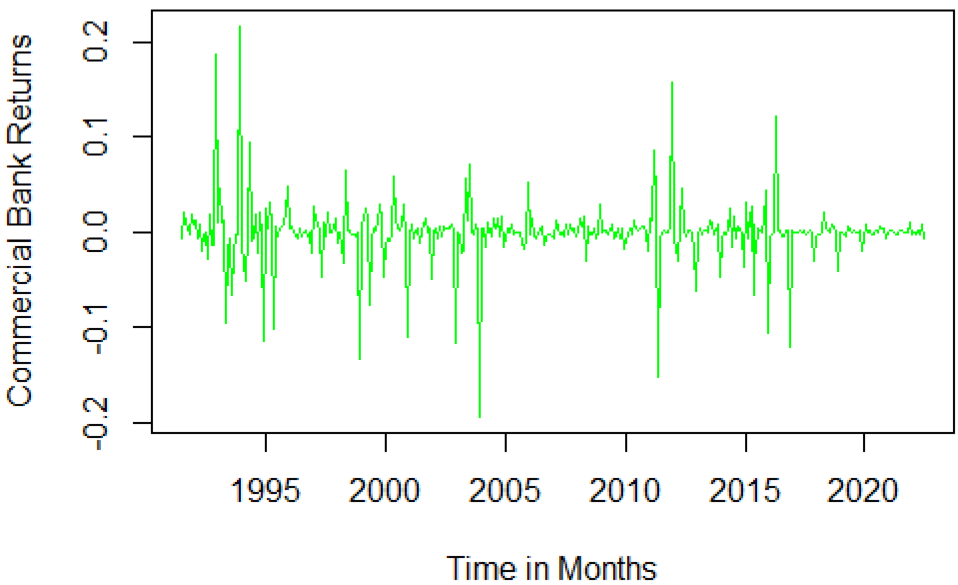


Figure 4.9.6: Commercial Bank Interest Rates returns.

### 4.10 Parameter Estimates

In this section, we shall estimate the parameters of the distributions studied above using mle which will be done fully in R programming Language.

#### 4.10.1 Parameter Estimates of the Gaussian Distribution

The parameters are as follows;

Table 4.10.1: Parameter Estimates of the Gaussian Distribution

| Parameters | 91-day Treasury Bills | Commercial Bank |
|---|---|---|
| $\mu$ | -0.001079472 | -0.0003724203 |
| $\sigma$ | 0.062602878 | 0.0324725290 |
| **Log-Lik** | 499.39 | 749.6415 |
| **AIC** | -994.7801 | -1495.283 |

From the model selection criteria, it is shown that the Gaussian distribution fits the Commercial Bank Weighted Average Interest rates better than the 91-day Treasury Bills Interest Rates. This is shown by the fact that it has the highest Log-Likelihood and the lowest AIC.

#### 4.10.2 Parameter Estimates for the Asymmetric NIG distribution

The parameters are as follows.

From the information in table 4.10.2 above, it is shown that the Asymmetric NIG fits the

Table 4.10.2: Parameter Estimates of the Asymmetric Normal Inverse Gaussian Distribution

| Parameters | 91-day Treasury Bills | Commercial Bank |
|---|---|---|
| $\alpha$ | 0.0684618635 | 0.0173190064 |
| $\mu$ | -0.0003035624 | 0.0004143653 |
| $\delta$ | 0.0719747059 | 0.0394937758 |
| $\beta$ | -0.0007691234 | -0.0007874793 |
| **Log-Lik** | 617.5179 | 1034.177 |
| **AIC** | -1227.036 | -2060.353 |

Commercial bank interest rates better than the 91-day Treasury Bills interest rates. This is due to the fact that it has the lowest AIC and the highest Log-Likelihood.

### 4.10.3 Parameter Estimates for the Asymmetric Student-t distribution

The estimates are as in the table.

Table 4.10.3: Parameter Estimates of the Asymmetric Student-t Distribution

| **Parameters** | **91-day Treasury Bills** | **Commercial Bank** |
|---|---|---|
| $\delta$ | 2.0000002797 | 2.0006884493 |
| $\mu$ | -0.0005295626 | 0.0006587517 |
| $\sigma$ | 64.8950735030 | 0.3942865031 |
| $\beta$ | 0.0367357193 | -0.0111227206 |
| **Log-Lik** | 603.6021 | 1008.648 |
| **AIC** | -1199.204 | -2009.296 |

According to the distribution optimization information in table 4.10.3 above (the Log-Likelihood and the AIC), it is sufficient to say that the Asymmetric Student-t distribution fits the Commercial Bank Weighted Average interest rates better than the 91-day Treasury Bill interest rates.

### 4.10.4 Parameter Estimates for the Asymmetric Generalized Hyperbolic distribution

The following table shows the MLE for the Asymmetric GH distribution. From the information

Table 4.10.4: Parameter Estimates of the Asymmetric G.H Distribution

| **Parameters** | **91-day Treasury Bills** | **Commercial Bank** |
|---|---|---|
| $\lambda$ | 0.1680857588 | -0.3313386045 |
| $\alpha$ | 0.0435677071 | 0.0164281275 |
| $\mu$ | -0.0001157169 | 0.0002918099 |
| $\sigma$ | 0.0637759052 | 0.0042322110 |
| $\gamma$ | -0.0009652545 | 0.0042322110 |
| **Log-Lik** | 625.9611 | 1033.994 |
| **AIC** | -1241.922 | -2057.988 |

in the table 4.10.4, it is evident that the Asymmetric G.H distribution fits the Commercial Bank Weighted Average interest rates better than the 91-day Treasury Bills interest rates.

### 4.10.5 Parameter Estimates for the Asymmetric Variance-Gamma distribution

The following are the MLE for the Asymmetric V-G distribution.

Table 4.10.5: Parameter Estimates of the Asymmetric Variance-Gamma Distribution

| **Parameters** | **91-day Treasury Bills** | **Commercial Bank** |
|---|---|---|
| $\lambda$ | 0.395714331 | 0.4217177019 |
| $\mu$ | -0.001181755 | -0.0012848100 |
| $\sigma$ | 0.064898080 | 0.0297106838 |
| $\gamma$ | 0.001904468 | 0.0008706809 |
| **Log-Lik** | 624.7602 | 994.1716 |
| **AIC** | -1241.52 | -1980.343 |

From the information in table 4.10.5 above, we see that the Asymmetriv Variance-Gamma distribution fits the Commercial Bank Weighted Average interest rates better then the 91-day Treasury Bills interest rates. This is according to the high Log-Likelihood and low AIC.

**From the Parameter Estimates above;**

We can conclude that the best model for the Commercial Bank Weighted Average interest rates is the Asymmetric Normal Inverse Gaussian Distribution. This is because, according to the optimization information provided by the models (in this case, the AIC and the Log-Likelihood of each model), the Asymmetric NIG has the highest Log-Likelihood and the lowest AIC. This makes it the best performing model. In a similar way, the best model for the 91-day Treasury Bills interest rates is the Asymmetric Generalized Hyperbolic distribution. This is because it provides the highest Log-Likelihood and the lowest AIC which makes it the best performing model with this data.

We now have two models that best describes the respective interest rates data, that is the Asymmetric Normal Inverse Gaussian Distribution (4.10.2) for the Commercial Bank Weighted Average interest rates and the Asymmetric Generalized Hyperbolic distribution (4.10.4) for the 91-day Treasury Bill interest rates. With this information, we can forecast the future interest rate returns in each category using special cases of the GARCH models (Generalized AutoRegressive Conditional Heteroskedasticity). In this case, we can integrate the GARCH-

"nig" to to model and hence forecast future returns of the Commercial Bank interest rates and GARCH-"ghyp" to model and hence forecast future returns of the 91-day Treasury Bills interest rates for a specified period, normally in months as shown in 4.10.1 and 4.10.2

The use of any GARCH model is limited to situations where the data only has ARCH and Auto-correlation effect. In order to show that GARCH is relevant in the modeling and forecasting using the NIG and the GHYP models assumptions, we will test for the ARCH effect and the Auto-correlation effect using Lagrange Multiplier and Box-Ljung tests respectively as follows.

## 4.11 Auto-Regressive Conditional Heteroskedasticity Test.

### 4.11.1 Box-Ljung Test for Auto-Correlation

In this test, the Ho: there is no Auto-Correlation while H1: there is presence of Auto-Correlation in the squared returns. The results are shown in the table below;

In both cases, the p-value < (0.05). We therefore reject the Ho and conclude that there is

Table 4.11.1: Box-Ljung test for Auto-Correlation

| | x-squared | p-value |
|---|---|---|
| 91-day Treasury Bill | 101.95 | 2.22e-16 |
| Commercial Bank | 84.207 | 6.463e-13 |

indeed the presence of Auto-Correlation in the squared returns. Therefore, Auto-Regressive Conditional Heteroskedasticity (ARCH) effect is present in the data. In simple terms, there is no constant volatility.

### 4.11.2 ARCH L-M Test

This is a Chi-squared hypothesis test for the presence of ARCH effect using Lagrange Multiplier (L-M). H0:there is no ARCH effect while H1: there is ARCH effect. The test results are as follows;

Since $p < 0.05$, we reject the H0 and conclude that there is ARCH effect in the data. This means

Table 4.11.2: ARCH L-M test

| | Chi-squared | p-value |
|---|---|---|
| 91-day Treasury Bill | 73.68 | 6.521e-11 |
| Commercial Bank | 68.151 | 7.092e-10 |

that there is no constant volatility.

### 4.12 Forecasting Returns of Interest Rates

The GARCH model will be used to forecast the returns of the Treasury Bill interest rates and the Commercial bank interest rates. The best model will be selected based on the Akaike Information Criteria from a number of alternatives. The ARMA model will be used as the mean model.

#### 4.12.1 Model Specification

Table 4.12.1 shows the model alternatives from which the best combination of the GARCH and ARMA model was obtained. The selection was only based on the model AIC.

From this table, the best combinations of the GARCH and ARMA models are GARCH (2,0) and ARMA (1,2) for the Treasury Bills interest rates and GARCH (1,1) with no mean model for the Commercial Bank interest rates. These combinations are selected because of their least values of the AIC. We shall then use these models for forecasting purposes.

#### 4.12.2 Model Parameters and Interpretation

The Maximum Likelihood Estimation method gave the GARCH-ghyp optimal parameters and the GARCH-nig optimal parameters as follows:

The $\mu$ parameter is negative and significant. This means that there is a possibility of negative returns in future. The ar1 parameter is positive and significant. It is the auto-regressive process to mean that the model predicts the present value of the time series using the immediate prior

Table 4.12.1: Table showing GARCH and ARMA orders

| GARCH (p,q) | ARMA (p,q) | AIC-GHYP | AIC-NIG |
|---|---|---|---|
| (1,1) | - | -3.6516 | -5.5852 |
| (1,1) | (1,0) | -3.7307 | -5.5816 |
| (1,1) | (1,1) | -3.7590 | -5.5761 |
| (1,1) | (1,2) | -3.7531 | -5.5755 |
| (1,1) | (2,0) | -3.7461 | -5.5768 |
| (1,1) | (2,1) | -3.7531 | -5.5758 |
| (1,1) | (2,2) | -3.7480 | -5.5804 |
| (1,2) | (1,1) | -3.7529 | -5.5710 |
| (1,2) | (1,2) | -3.7476 | -5.5701 |
| (1,2) | (2,0) | -3.7406 | -5.5712 |
| (1,2) | (2,1) | -3.7477 | -5.5703 |
| (1,2) | (2,2) | -3.7425 | -5.5749 |
| (2,0) | (1,1) | -3.7574 | -5.5069 |
| (2,0) | (1,2) | -3.8185 | -5.5045 |
| (2,0) | (2,0) | -3.7475 | -5.5083 |
| (2,0) | (2,1) | -3.7524 | -5.5035 |
| (2,0) | (2,2) | -3.7472 | -5.5007 |
| (2,1) | (1,0) | -3.7292 | -5.5768 |
| (2,1) | (1,1) | -3.7547 | -5.5688 |
| (2,1) | (1,2) | -3.7494 | -5.5657 |

value. Since the coefficient is greater than zero, it means that the function generate positively auto-correlated time series.

The ma1 is the moving average process to mean that the current value is linearly dependent on the current and past error terms. Since this coefficient is negative, the process is invertible, to mean that we can write MA(q) as an AR(∞) process.Therefore, the most recent observations in the time series have higher weights than observations from more distant past.

$\omega$ is positive and significant. This is the constant term in the GARCH model. $\alpha_1$ is the ARCH coefficient. It is positive and significant, showing that there is presence of volatility clustering. $\alpha_2$ is the laverage coefficient, is positive and significant, showing that volatility is impacted more by negative shock than by positive ones. $\beta_1$ is the GARCH coefficient (volatility parameter), is positive and significant and also greater than $\alpha$. This indicates that volatility clustering is persistent.

From the volatility parameters, $\alpha$ and $\beta$,we also note that $\alpha+\beta < 1$. This shows that the two

Table 4.12.2: GARCH-ghyp parameter estimates

| Parameters | MLE | P-Value |
|---|---|---|
| $\mu$ | -0.001220 | 0.017174 |
| ar1 | 0.939816 | 0.00000 |
| ma1 | -0.638837 | 0.00000 |
| $\omega$ | 0.000427 | 0.00000 |
| $\alpha_1$ | 0.724445 | 0.000012 |
| $\alpha_2$ | 0.274555 | 0.006553 |
| skew | -0.098449 | 0.000302 |
| shape | 0.250000 | 0.001597 |
| $\lambda$ | 0.173173 | 0.600545 |

Table 4.12.3: GARCH-nig Parameter Estimates

| Parameters | MLE | P-Value |
|---|---|---|
| $\mu$ | -0.000217 | 0.69760 |
| $\omega$ | 0.00000 | 1.00000 |
| $\alpha_1$ | 0.000022 | 0.55693 |
| $\beta_1$ | 0.99499 | 0.00000 |
| Skew | -0.123687 | 0.00000 |
| Shape | 0.064336 | 0.00000 |

models performs well.

### 4.12.3 The Forecast Values

The forecasts are as shown in the table below:

In 4.12.4, there is 0.99 probability that the monthly returns in October 2021 will be above 0.0867395. While in 4.12.5, there is 0.99 probability that the monthly returns for August 2022 will be above 0.0347193.

The forecasts are represented as follows:

Table 4.12.4: 12- Month ghyp Forecasts

| 0-roll | Returns, T0 = Sept 2021 | VaR(0.99) |
|---|---|---|
| T+1 | 0.03251 | 0.0867395 |
| T+2 | 0.03738 | 0.099733 |
| T+3 | 0.04159 | 0.110966 |
| T+4 | 0.04543 | 0.1212112 |
| T+5 | 0.04896 | 0.1306296 |
| T+6 | 0.05225 | 0.139408 |
| T+7 | 0.05534 | 0.147652 |
| T+8 | 0.05827 | 0.1554695 |
| T+9 | 0.06105 | 0.1628868 |
| T+10 | 0.06371 | 0.169984 |
| T+11 | 0.06627 | 0.1768142 |
| T+12 | 0.06872 | 0.183351 |

Table 4.12.5: 12- Month nig Forecasts

| 0-roll | Returns, T0 = Jul 2022 | VaR(0.99) |
|---|---|---|
| T+1 | 0.01285 | 0.0347193 |
| T+2 | 0.01282 | 0.0346383 |
| T+3 | 0.01279 | 0.0345572 |
| T+4 | 0.01275 | 0.0344491 |
| T+5 | 0.01272 | 0.0343681 |
| T+6 | 0.01269 | 0.034287 |
| T+7 | 0.01266 | 0.0342060 |
| T+8 | 0.01263 | 0.0341249 |
| T+9 | 0.01260 | 0.034044 |
| T+10 | 0.01256 | 0.033936 |
| T+11 | 0.01253 | 0.033855 |
| T+12 | 0.01250 | 0.033774 |

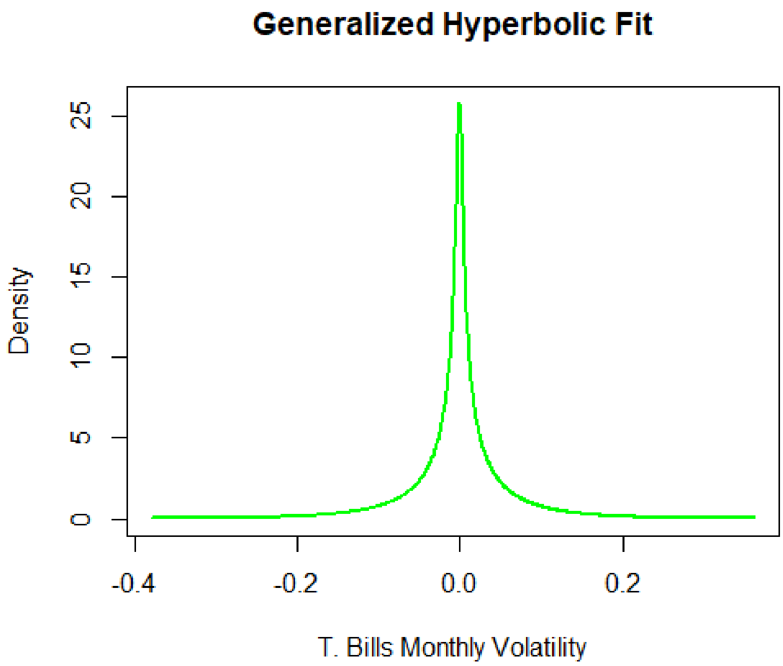


Figure 4.10.1: The generalized hyperbolic curve

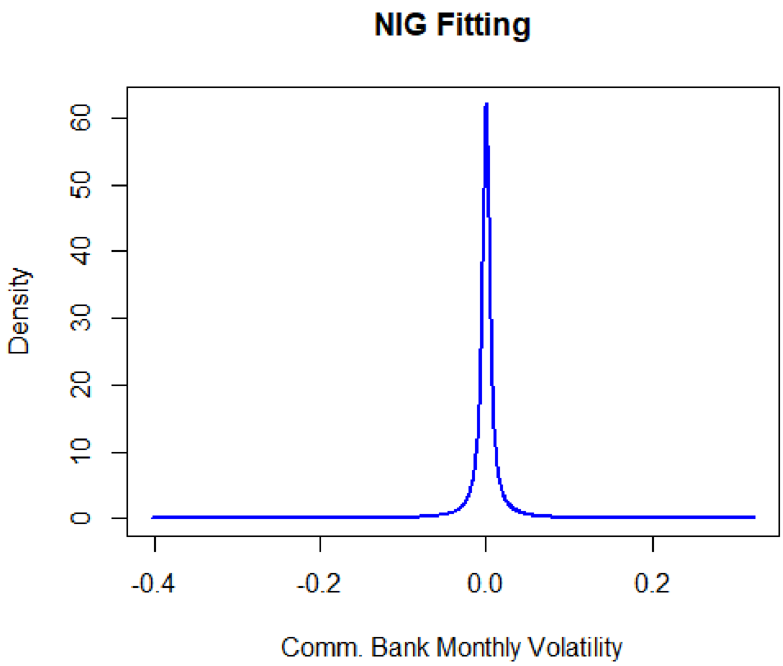


Figure 4.10.2: The NIG curve

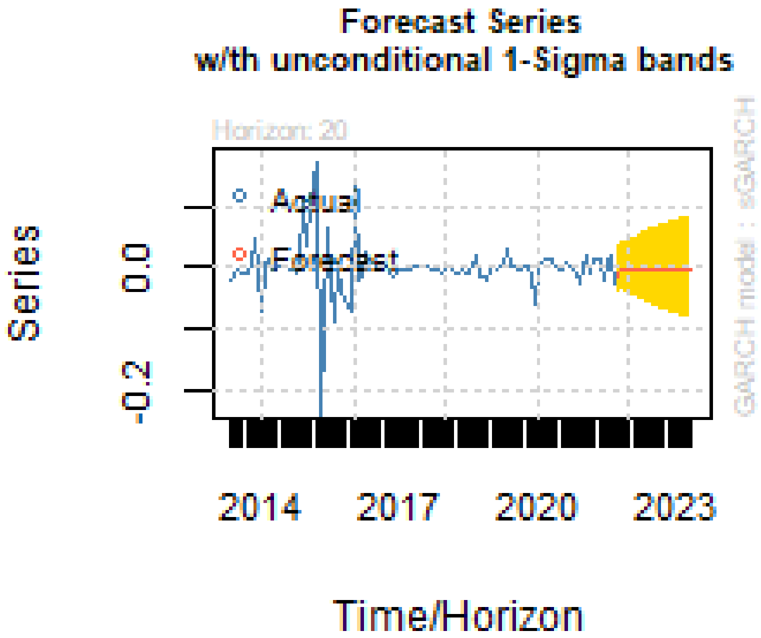


Figure 4.12.1: Forecast for Commercial Bank Interest rates

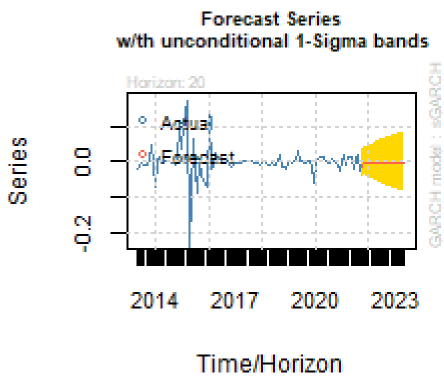


Figure 4.12.2: Forecast for Treasury Bills Interest Rates

# CHAPTER 5

## SUMMARY, CONCLUSION AND RECOMMENDATION

### Introduction

In this chapter, we present a summary of the research findings from the previous section, w provide conclusions and possible recommendations based on the data analysis. We also present the significance of this research in derivative pricing, interest rate dependent contracts, variance and volatility swaps.

### 5.1 Summary of the Research

This study observed the process of interest rate risk modeling using semi-heavy tail distributions of normal variance-mean mixtures. The risk measures under study are the volatility and the kurtosis of interest rates. Volatility in market investment is simply the term used to describe unpredictable and sometimes sharp movements in interest rates. Kurtosis is a measure of how often outliers occur. It is used to measure risk because high kurtosis is associated with high probabilities of extremely large and small returns (semi-heavy tails) and hence, high risk. On the other, low kurtosis is associated with low probabilities of extremely large and small returns and hence, low risk. In the literature review, we realize that most models assumed to describe the returns from interest rates have normal assumptions with constant variance. In the pricing of derivatives, this means that the variance of the underlying variable is constant throughout the life of the option, even if it is more than 20 years. In pricing of interest dependent contracts such as options and currencies, using the famous models such as Binomial Option pricing model as well as Black-Scholes model, this assumption can lead to wrong option prices and investment decisions. We have shown that this assumption is invalid.

This research used Interest rates from the Central Bank of Kenya: the 91-day Treasury Bill

Interest Rates and the Commercial Bank Interest rates, which were converted into daily continuously compounded returns. This was done by taking the natural log of the daily ratios. Flexible hyerbolic distributions such as the Normal Inverse Gaussian and the Reciprocal Inverse Gaussian distributions were constructed and properties studied. These were then used to fit the returns data from interest rates. Finally, the returns were forecasted using GARCH models, under the assumption that the error terms follow the respective hyperbolic distributions. With existing higher moments, the normal-variance mean mixtures shows good results in capturing the higher moments of the data and the tails. These mixtures captures the kurtosis which determines how risky they are as compared to the standard normal distribution. Under the normal assumption of [18], [41] and [4] in derivative pricing and [34], [35], and [11] in risk modeling, its is easier to underprice and in some cases overprice interest-dependent derivatives.

## 5.2 Conclusion

Interest rates both from the Treasury Bills and the Commercial Banks and economic variables that experience dynamic changes due to other economic factors such as economic conditions and demand for loans, inflation, government policy, and credit risk. This fluctuation can cause risk in investment in derivatives and commodities that use interest rates as fundamental economic variables and hence affect investments. These fluctuations are also known to be large at some point, highly volatile, or low other times, low volatility, depending on the strength of the economic factors. The volatility of the interest rates therefore are indication that these variables undergo dynamic change with time, to mean that they are not constant throughout the life of an option as assumed in derivative pricing. Understanding this and how they fluctuate is a very important aspect in financial investment sector.

The objective was to use flexible distributions to understand these fluctuations in the returns of interest rates and contrary to the normality assumption as seen in the literature review, we note that these distribution provide a better fit than the normal distribution. In addition to that, the data displays special properties that require the use of GARCH models with special error terms assumption in forecasting the volatility. It is also found that this models provide good forecast and are better under normal assumption. From the Goodness-of Fit test and the AIC, we note that these models provide a good fit.

### 5.3 Recommendation

From our findings in chapter 4, we recommend the use of semi-heavy tail distributions in modeling interest rate risk. This is because this family of distributions possess properties such as existence of higher moments and fat tails that can perform better than the normal distribution. This is important in the derivative pricing as it overrules the normality assumption. However, we recommend that in future studies;

- More distributions in this class explored in details and applied to the same data in order to determine the best performing distribution. This is because, this research did not explore all the distributions but four of the main ones. It also did not go into details in the properties except for the moments. More studies should also be done on proof of the properties of the tails.
- Other distributions outside the hyperbolic class should also be studied in details and subjected to the same conditions and data. Also, other heavy tail distributions and other approaches to model interest rate risk should be studied and result compared to ensure the best conditions for modeling interest rates.

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**APPENDIX**

In this section, we will outline the codes used during our data analysis in Chapter 4. The codes will be arranged to give the output as they are in the text.

```
#Declaring variables: x is a variable for 91-day Treasury Bills interest
 rates and q for Commercial Bank Interest rates and summary statistics.
x=CBK_RATESRev$Volatility
x
q=Rates$Returns
q
summary(x)
summary(q)

#Normality tests.
library(tseries)
ks.test(x,"pnorm",mean(x),sd=sd(x))
shapiro.test(x)
jarque.bera.test(x)
ks.test(q,"pnorm",mean(q),sd=sd(q))
shapiro.test(q)
jarque.bera.test(q)
qqnorm(Tbill, col = "blue")
qqline(Tbill)
qqnorm(CommBnk, col = "green")
qqline(CommBnk)

#Test for Mean Reversion.
adf.test(x)
adf.test(q)
hurstexp(x)
hurstexp(q)

```

```
#Plotting the Histogram with Normal Distribution.
h=hist(x, freq = F, 18, xlab= "T. Bills Monthly Volatility", ylab = "
  Density", col = "blue", breaks = 15, main = "Histogram and Gaussian
  Curve")
xfit<-seq(min(x), max(x), length =40)
yfit<-dnorm(xfit, mean = mean(x), sd = sd(x))
lines(xfit, yfit, col="red", lwd=2, df = 10, add = T)
h=hist(q, freq = F, 18, xlab= "Commercial Bank Monthly Volatility", ylab
  = "Density", col = "green", breaks = 15, main = "Histogram and Gaussian
  Curve")
x1fit<-seq(min(q), max(q), length =40)
y1fit<-dnorm(x1fit, mean = mean(q), sd = sd(q))
lines(x1fit, y1fit, col="red", lwd=2, df = 10, add = T)

#Time series analysis and ploting for both Interest Rates data using zoo
  library package.
library(zoo)
Tbill<-ts(x, start = c(1991,1), end = c(2021,9), frequency = 12)
Tbill
plot(Tbill, xlab="Time in Months", ylab="TBills Returns", col="blue")
CommBnk<-ts(q, start = c(1991,7), end = c(2022,7), frequency = 12)
CommBnk
plot(CommBnk, xlab="Time in Months", ylab="Commercial Bank Returns", col=
  "green")

# PARAMETER ESTIMATION USING MAXIMUM LIKELIHOOD ESTIMATION METHOD.
# Parameter Estimate for Normal Distribution.
normal.fit<-fit.gaussuv(data = Tbill)
summary(normal.fit)
normal1.fit<-fit.gaussuv(data = CommBnk)
summary(normal1.fit)

# Parameter Estimate for NIG distribution.
```

```
nig.fiT<-fit.NIGuv(data = Tbill)
summary(nig.fiT)
nig1.fiT<-fit.NIGuv(data = CommBnk)
summary(nig1.fiT)

# Parameter Estimate for Student-t distribution.
t.fit<-fit.tuv(data = Tbill)
summary(t.fit)
t.fitb<-fit.tuv(data = CommBnk)
summary(t.fitb)

# Parameter Estimate for Generalized Hyperbolic distribution.
hyp.fit<-fit.ghypuv(data = Tbill)
summary(hyp.fit)
hyp1.fit<-fit.ghypuv(data = CommBnk)
summary(hyp1.fit)

# Parameter Estimates for Variance-Gamma distribution
var.gr<- fit.VGuv(data = Tbill)
summary(var.gr)
var.grb<- fit.VGuv(data = CommBnk)
summary(var.grb)

# Fitting the Normal Inverse Gaussian and the Generalized Hyperbolic
  distributions.
plot(hyp.fit, type = "l", bg = "red", lwd = 2, xlab = "Monthly Returns",
  ylab = "Density", main = "G-hyp Distribution Fitting", sub = "Par(lambda
  , alpha, mu, sigma, gamma)", col = "blue")
plot(nig.fibk, type = "l", bg = "grey", lwd = 2, xlab = "Monthly Returns"
  , ylab = "Density", main = "NIG Distribution Fitting", sub = "Par(alpha,
   mu, sigma, gamma)", col = "green")

# Auto-Regressive Conditional Heteroskedasticity Tests.
```

```
library(FinTS)
ArchTest(coredata(Tbill))
ArchTest(coredata(CommBnk))
Box.test(coredata(Tbill^2), type = "Ljung-Box", lag = 12)
Box.test(coredata(CommBnk^2), type = "Ljung-Box", lag = 12)

# FORECASTING USING GARCH MODELS
# Model Specification.
library{rugarch}
Tbill_garch11_spec<-ugarchspec(variance.model = list(garchOrder = c(2,0))
  , mean.model = list(armaOrder = c(1,2)), distribution.model = "ghyp")
Tbill_garch11_spec
CommBnk_garch11_spec<-ugarchspec(variance.model = list(garchOrder = c
  (1,1)), mean.model = list(armaOrder = c(0,0)), distribution.model = "nig
  ")
CommBnk_garch11_spec

# Parameter Estimation using MLE
Tbill_garch11_fit<-ugarchfit(spec = Tbill_garch11_spec, data = Tbill)
Tbill_garch11_fit
CommBnk_garch11_fit<-ugarchfit(spec = CommBnk_garch11_spec, data =
  CommBnk)
CommBnk_garch11_fit

# Forecasting and VaR Computation.
Tbill_garch_forecast<-ugarchforecast(Tbill_garch11_fit, n.ahead = 12)
Tbill_garch_forecast
plot(Tbill_garch_forecast, which = "all")
CommBnK_garch_forecast<-ugarchforecast(CommB_garch_fit, n.ahead = 12)
CommBnK_garch_forecast
plot(CommBnK_garch_forecast)

# NIG and GHYP Quantiles Estimation at 99pc
```

```
qnig(0.99)
qghyp(0.99)

# VaR at 99pc Probability
quantiles*sigma
```