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ENUMERATION OF PLANE AND d-ARY TREE-LIKE STRUCTURES

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Abstract. Trees are generalized using various approaches such as considering tree-like structures. Some of the tree-like structures are Husimi graphs, cacti and oriented cacti. These graphs have been enumerated according to number of vertices, blocks, block types and degree sequences. Noncrossing and plane counterparts have also been enumerated by number of vertices, blocks and block types. In this paper, we enumerate plane Husimi graphs, cacti and oriented cacti according to root degree, outdegree of a given vertex and outdegree sequence. The d-ary tree like structures are also introduced in this paper and enumerated according to number of vertices, blocks, block types, outdegree sequence and number of leaves.

1. Introduction

One way of generalizing trees is by considering structures which posses properties of trees, i.e., connectedness and cycle-freeness. Various types of trees have been studied in literature. These include Cayley trees, plane trees, noncrossing trees and d-ary trees. Cayley trees were generalized to Husimi graphs and cacti by Kodi Husimi [9] in an attempt to solve a problem that arose from physics [13]. *Husimi graphs* are graphs obtained when edges in trees are replaced by complete graphs. These complete graphs are called *blocks*. If the complete graphs are replaced by cycles then we obtain *cacti*. If the blocks are oriented cycles then we get *oriented cacti* [17]. Husimi graphs, cacti and oriented cacti have been studied extensively in [1, 2, 7, 8, 12, 9, 13, 17] among other papers. The authors counted these tree-like structures according to number of vertices, blocks and block types.

A *degree* of a vertex i of a tree-like structure is the number of blocks that are incident to i and *degree sequence* is an ordered sequence of the degrees of the tree-like structure. Okoth, in his PhD thesis [14], enumerated Husimi graphs, cacti and oriented cacti with given degree sequences. He also related these structures to some families of connected cycle free set partitions.

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If the tree-like structure is drawn in the plane with vertices on the boundary of a circle such that the blocks do not cross inside the circle then we get *noncrossing tree-like structure*. These structures were introduced and studied by Okoth in [15]. Now, if a tree-like structure is drawn in the plane such that blocks are ordered then we have a *plane tree-like structure*. These structures were also introduced by Okoth in [15]. The statistics of enumeration used by the aforementioned author included number of vertices, blocks, block types and number of leaves. Recently, Kariuki and Okoth [10] constructed bijections between the set of plane Husimi graphs and the sets of plane trees, polygon dissections, standard Young tableaux, sequences satisfying certain conditions, Deutsch paths and restricted lattice paths.

Consider a plane tree like structure. Let b be a block attached to vertex i such that there is another vertex $j \neq i$ of b which is connected to i but lies at a lower level. Then j is a *child* of i and b is a *block child* of i. The *outdegree* of vertex i is the number of block children of i, and *outdegree sequence* is an ordered sequence of the outdegrees of the vertices of the tree-like structure. If the outdegree of each vertex is at most d then we get a d-ary tree-like structure. In Figure 1, we get a ternary cactus on 25 vertices. Bijections relating d-ary Husimi graphs to weakly

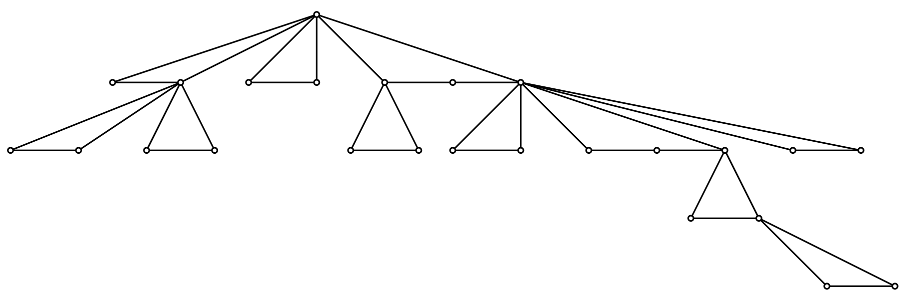

Figure 1. Ternary cactus on 25 vertices with 11 blocks.

labelled plane trees, generalised binary trees, plane trees in which each internal vertex has $d-1$ leaves as children and certain classes of Motzkin paths among other structures have been established by Kariuki and Okoth in [11].

The paper is organized as follows: In Section 2, we enumerate plane Husimi graphs according to outdegree sequence, outdegree of a vertex, root degree and total number of vertices with a given degree. Moreover, enumeration of d-ary tree-like structures according to the number of vertices, block types and number of blocks is achieved in Section 3. In the same section, we enumerate d-ary Husimi graphs, cacti and oriented cacti according to number of leaves and outdegree sequence. We conclude the paper in Section 4 and give directions in which the work could be extended.

2. Counting plane tree-like structures by outdegree sequences

In this section, we enumerate plane Husimi graphs, cacti and oriented cacti according to the degree of a vertex, outdegree of a vertex and outdegree sequences.

We also obtain a formula for the number of these graphs with a given root degree. To start us off, we prove the following important lemma:

Lemma 2.1. *Let $n, k \geq 1$ and let $n_1, n_2, \ldots$ be non-negative integers satisfying the conditions $n_1 + n_2 + \cdots = k$ and $n_1 + 2n_2 + \cdots = n - 1$. Then,*

$$\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{k!}{n_1!n_2!\cdots} = \binom{n-2}{n-1}.$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{1}{n_1!n_2!\cdots} &= [x^{n-1}y^k] \prod_{i\geq 1}\left(\sum_{j\geq 0}\frac{x^{ij}y^j}{j!}\right) \\
&= [x^{n-1}y^k] \prod_{i\geq 1} \exp(x^i y) \\
&= [x^{n-1}y^k] \exp(y(x + x^2 + \cdots)) \\
&= [x^{n-1}y^k] \exp\left(\frac{xy}{1-x}\right) \\
&= [x^{n-1}y^k] \sum_{i\geq 0} \frac{x^i y^i (1-x)^{-i}}{i!} \\
&= [x^{n-1}] \frac{x^k(1-x)^{-k}}{k!}.
\end{aligned}$$

By binomial theorem, we have

$$\begin{aligned}
\sum_{\substack{n_1+n_2+\cdots=k \\ n_1+2n_2+\cdots=n-1 \\ n_1,n_2,\ldots\geq 0}} \frac{1}{n_1!n_2!\cdots} &= [x^{n-k-1}]\frac{1}{k!}\sum_{i\geq 0}\binom{-k}{i}(-x)^i \\
&= [x^{n-k-1}]\frac{1}{k!}\sum_{i\geq 0}\binom{k+i-1}{i}x^i \\
&= \frac{1}{k!}\binom{n-2}{n-k-1}.
\end{aligned}$$

The result that follows. ◻

This following result was proved by Okoth in [15] from a generating function approach. We provide another proof.

Proposition 2.2. *The number of plane Husimi graphs on n vertices with k blocks is given by*

$$\frac{1}{n}\binom{n+k-1}{k}\binom{n-2}{k-1}. \tag{2.1}$$

Proof. Consider a plane Husimi graph on n vertices with k blocks of type $(n_2, n_3, \ldots)$ such that the vertices are labelled by integers from the set $\{1, 2, \ldots, n\}$. Adopting the reverse Prüfer algorithm of Seo and Shin [16] to the Prüfer algorithm for tree-like structures introduced by Collin Springer in [17], we find that there are

$$(k+1)(k+2)\cdots(n+k-1) = \frac{(n+k-1)!}{k!}$$

ways to put together blocks to form a labelled plane tree-like structure. Thus, there are

$$\frac{(n+k-1)!}{k!n!}$$

ways to put together blocks to form an unlabelled plane tree-like structure. We use Lemma 2.1 to sum over all the block types to get the required formula. ☐

Theorem 2.3. *The number of plane Husimi graphs on n vertices with k blocks and exactly d_i vertices of outdegree i is given by*

$$\frac{1}{n}\binom{n}{d_0, d_1, \ldots, d_k}\binom{n-2}{k-1}. \tag{2.2}$$

To prove the theorem, we require the concept of Łukasiewicz words [18]. Consider an alphabet $\mathcal{A}$ which consists of letters $l_0, l_1, \ldots$ and the empty word be denoted by 1. Let the weight of the letter l_i be $\varphi(l_i) = i - 1$. A word $w_1 w_2 \cdots w_m$ made of letters from $\mathcal{A}$ is a *Łukasiewicz word* if $\varphi(w_1) + \cdots + \varphi(w_j) \geq 0$ for $1 \leq j \leq m-1$ and $\varphi(w_1) + \cdots + \varphi(w_m) = -1$. Let $|\mathcal{A}| = n$ and let n_i be the multiplicity of letter l_i in a Łukasiewicz word. Then there are

$$\binom{n}{n_0, n_1, \ldots}$$

Łukasiewicz words on the alphabet $\mathcal{A}$.

Proof of Theorem 2.3. Consider any plane Husimi graph H on n vertices with k blocks such that there are n_i blocks of size $i \geq 2$. Label the vertices of the graph with integers $1, 2, \ldots n$ such that vertex i is the i^{th} vertex visited when H traversed in preorder. Let d_i be the outdegree of vertex i.

Let A be the set of words of length n and B the set of Łukasiewicz words of length n. In the book [18], Richard Stanley constructed a bijection $\phi : A \times [n] \longrightarrow B \times [r]$ by means of plane forests with r components where $[n] := \{1, 2, \ldots, n\}$. Setting $r = 1$, we obtain the necessary result since we are dealing with tree-like structures and not forest of tree-like structures. The set A is the set of words $x_{d_1} x_{d_2} \cdots x_{d_n}$ where d_i is the degree of vertex i.

From the bijection, we have

$$n|A| = \binom{n}{d_0, d_1, \ldots, d_k}$$

since $d_j = 0$ for all $j > k$. Making use of Lemma 2.1 to sum over all block types, the number of plane Husimi graphs on n vertices with k blocks is thus

$$|A|\binom{n-2}{k-1} = \frac{1}{n}\binom{n}{d_0, d_1, \ldots, d_k}\binom{n-2}{k-1}.$$

□

Corollary 2.4. *The total number of vertices of outdegree $i \geq 0$ over all plane Husimi graphs on $n \geq 1$ vertices with k blocks is*

$$\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}. \tag{2.3}$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{d_1+d_2+\cdots=n \\ d_1+2d_2+\cdots=k \\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{d_1!d_2!\cdots} &= [x^k y^n]\left(\sum_{j\geq 1}\frac{jx^{ij}y^j}{j!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right)\\
&= [x^k y^n]x^i y\left(\sum_{j\geq 1}\frac{x^{i(j-1)}y^{(j-1)}}{(j-1)!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right)\\
&= [x^k y^n]x^i y\left(\sum_{j\geq 0}\frac{x^{ij}y^j}{j!}\right)\prod_{\substack{m\geq 1\\ m\neq i}}\left(\sum_{j\geq 0}\frac{x^{mj}y^j}{j!}\right)\\
&= [x^k y^n]x^i y\exp(x^i y)\prod_{\substack{m\geq 1\\ m\neq i}}\exp(x^m y)\\
&= [x^k y^n]x^i y\prod_{m\geq 1}\exp(x^m y)\\
&= [x^k y^n]x^i y\exp(y(x+x^2+\cdots))\\
&= [x^{k-i}y^{n-1}]\exp\left(\frac{xy}{1-x}\right)\\
&= [x^{k-i}y^{n-1}]\sum_{i\geq 0}\frac{x^i y^i(1-x)^{-i}}{i!}\\
&= [x^{k-i}]\frac{x^{n-1}(1-x)^{-(n-1)}}{(n-1)!}.
\end{aligned}$$

By binomial theorem, we have

$$\sum_{\substack{d_1+d_2+\cdots=n\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{d_1!d_2!\cdots} = [x^{k-i-n+1}]\frac{1}{(n-1)!}\sum_{j\geq 0}\binom{-(n-1)}{j}(-x)^j$$

$$= [x^{k-i-n+1}]\frac{1}{(n-1)!}\sum_{j\geq 0}\binom{n+j-2}{j}x^j$$

$$= \frac{1}{(n-1)!}\binom{k-i-1}{k-i-n+1}.$$

Let d_i be the number of vertices of outdegree i. Now, we have the total number of vertices of outdegree $i>0$ in plane Husimi graphs on n vertices with k blocks as:

$$\sum_{d_0\geq 0}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_i}{n}\binom{n}{d_0,d_1,\ldots,d_k}\binom{n-2}{k-1}$$

$$= \sum_{d_0\geq 0}\frac{1}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}}\frac{d_i}{d_1!d_2!\cdots d_k!}$$

$$= \sum_{d_0\geq 0}\frac{1}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0-1)!}\binom{k-i-1}{n-d_0-2}$$

$$= \sum_{d_0\geq 0}\binom{n-1}{d_0}\binom{k-i-1}{n-d_0-2}\binom{n-2}{k-1}$$

$$= \binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.$$

The last equality follows by Vandermonde identity. Next, we obtain the number of vertices of outdegree 0, i.e., we get the sum

$$\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}} \frac{d_0}{n}\binom{n}{d_0,d_1,\ldots,d_k}\binom{n-2}{k-1}$$

$$= \frac{d_0}{n}\binom{n}{d_0}(n-d_0)!\binom{n-2}{k-1}\sum_{\substack{d_1+d_2+\cdots=n-d_0\\ d_1+2d_2+\cdots=k\\ d_1,d_2,\ldots\geq 0}}\frac{1}{d_1!d_2!\cdots d_k!}$$

$$= \binom{n-1}{d_0-1}(n-d_0)!\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0)!}\binom{k-1}{n-d_0-1}$$

$$= \binom{n-1}{d_0-1}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}.$$

The formula follows by summing over all d_0 making use of Vandermonde identity. $\square$

This corollary was also proved by Okoth [15] using generating functions.

Corollary 2.5. *There are*

$$\frac{1}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1} \tag{2.4}$$

plane Husimi graphs on n vertices with k blocks and d_0 leaves.

Proof. We sum over all d_i for $i = 1, 2, \ldots$ in Equation (2.2):

$$\begin{aligned}
&\sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0, d_2\geq 0,\ldots,d_k\geq 0}} \frac{1}{n}\binom{n}{d_0, d_1, \ldots, d_k}\binom{n-2}{k-1}\\
&\qquad = \frac{1}{n}\binom{n}{d_0}\sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0, d_2\geq 0,\ldots,d_k\geq 0}}\binom{n-d_0}{d_1, d_2, \ldots, d_k}\binom{n-2}{k-1}\\
&\qquad = \frac{(n-d_0)!}{n}\binom{n}{d_0}\binom{n-2}{k-1}\sum_{\substack{d_1+d_2+\cdots+d_k=n-d_0\\ d_1+2d_2+\cdots+kd_k=k\\ d_1\geq 0, d_2\geq 0,\ldots,d_k\geq 0}}\frac{1}{d_1!d_2!\cdots d_k!}.
\end{aligned}$$

From the proof of Lemma 2.1, we have the sum as:

$$\begin{aligned}
&\frac{(n-d_0)!}{n}\binom{n}{d_0}\binom{n-2}{k-1}\cdot\frac{1}{(n-d_0)!}\binom{k-1}{n-d_0-1}\\
&\qquad = \frac{1}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}.
\end{aligned}$$

This completes the proof. ∎

Summing over all d_0 in Equation (2.4), we obtain Equation (2.1). The total number of leaves in plane Husimi graphs on n vertices with k blocks is thus

$$\begin{aligned}
&\sum_{d_0=1}^{n-k}\frac{d_0}{n}\binom{n}{d_0}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}\\
&\qquad = \sum_{d_0=1}^{n-k}\binom{n-1}{d_0-1}\binom{k-1}{n-d_0-1}\binom{n-2}{k-1}\\
&\qquad = \binom{n+k-2}{n-2}\binom{n-2}{k-1}.
\end{aligned}$$

Lemma 2.6. *The number of plane Husimi graphs on n vertices with k blocks such that the root has degree r is given by*

$$\frac{r}{k}\binom{n+k-r-2}{k-r}\binom{n-2}{k-1}. \tag{2.5}$$

Proof. Let T be a plane Husimi graph on n vertices such that the root has degree r. Using Depth First Search (DFS), we label the vertices of the graph with integers $1, 2, \ldots, n$ such that the root is labelled 1. Let the degree of vertex i be r_i. Then $r + r_2 + r_3 + \cdots + r_n = k$. The number of nonnegative integer solutions of the equation $r_2 + r_3 + \cdots + r_n = k - r$ is $\binom{n+k-r-2}{k-r}$. Moreover, there are exactly r permutations of the total k cyclic permutations of $r_2, r_3, \ldots, r_n$ which are valid degree sequences.

Since there are a total of $\binom{n-2}{k-1}$ choices for block types (see Lemma 2.1) if there are n vertices in the plane tree-like structure with k blocks then the result follows immediately. $\square$

We provide another proof for Lemma 2.6 based on generating functions and making use of the following theorem which is proved in [6].

Theorem 2.7 (Lagrange-Bürmann Formula)**.** *Let $\phi(t)$ be a power series in t, not involving x. Then there is a unique power series $f = f(x)$ such that $f(x) = x\phi(f(x))$, and for any Laurent series $g(t)$, not involving x and for any integer $n \neq 0$ we have*

$$[x^n]g(f(x)) = \frac{1}{n}[t^{n-1}]\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right)\phi(t)^n.$$

Another proof of Lemma 2.6. Let $P(x)$ be the generating function for plane Husimi graphs, where x marks a vertex. Let y_i mark blocks of size i. Then

$$P = x + x\sum_{i\geq 1} y_{i+1}P^i + x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^2 + \cdots = \frac{x}{1 - \sum_{i\geq 1} y_{i+1}P^i}.$$

Thus the generating function for plane Husimi graphs with root degree r is $x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r$. We need $[x^n]x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r$, which we now compute:

$$[x^n]x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r = [x^{n-1}]\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r.$$

By Lagrange-Bürmann formula (Theorem 2.7), we have

$$
\begin{aligned}
[x^{n-1}]\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r &= \frac{1}{n-1}[t^{n-2}]r\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right)\\
&\qquad\left(\frac{x}{1-\sum_{i\geq 1} y_{i+1}t^i}\right)^{n-1}\\
&= \frac{r}{n-1}[t^{n-2}]\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right)\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{\ell}\\
&= \frac{r}{n-1}[t^{n-2}]\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell-1}\left(\sum_{i\geq 1} y_{i+1}it^{i-1}\right)\\
&= \frac{r}{n-1}[t^{n-2}]\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r+\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell}\\
&= \frac{r}{n-1}\sum_{\ell\geq 0}\binom{n+\ell-2}{\ell}(n-1)[t^{n-1}]\frac{1}{r+\ell}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell}\\
&= \sum_{\ell\geq 0}\frac{r}{r+\ell}\binom{n+\ell-2}{\ell}[t^{n-1}]\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{r+\ell}.
\end{aligned}
$$

We thus have,

$$
[x^n]x\left(\sum_{i\geq 1} y_{i+1}P^i\right)^r = \sum_{\ell\geq 0}\frac{r}{r+\ell}\binom{n+\ell-2}{\ell}\sum_{\substack{n_2+n_3+\cdots=r+\ell\\ n_2+2n_3+\cdots=n-1}}\frac{(r+\ell)!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\ldots}.
$$

So, the number of plane Husimi graphs on n vertices, k blocks and root degree r such that there are n_i blocks of size i is given by

$$
r\binom{n+k-r-2}{k-r}\frac{(k-1)!}{n_2!n_3!\ldots}
$$

where $n_2+n_3+\cdots=k$. By the proof of Lemma 2.1, we have that the number of plane Husimi graphs on n vertices, k blocks and root degree r is

$$
\begin{aligned}
r(k-1)!\binom{n+k-r-2}{\ell}\sum_{n_2+n_3+\cdots=k}\frac{1}{n_2!n_3!\ldots}\\
&= r(k-1)!\binom{n+k-r-2}{k-r}\cdot\frac{1}{k!}\binom{n-2}{n-k-1}\\
&= \frac{r}{k}\binom{n+k-r-2}{k-r}\binom{n-2}{k-1}.
\end{aligned}
$$

□

Summing over all r in Equation (2.5), we obtain the following result which was obtained by Okoth in an earlier paper [15].

Corollary 2.8. *There are a total of*

$$\frac{1}{n}\binom{n+k-1}{k}\binom{n-2}{k-1}$$

plane Husimi graphs on n vertices with k blocks.

Theorem 2.9. *Let $n, r \geq 1$ and $i \geq 0$. Then the total number of vertices with outdegree i among all plane Husimi graphs on n vertices with root degree r and k blocks is given by*

$$\begin{cases} r\binom{n+k-r-i-3}{n-3}\binom{n-2}{k-1}, & \textit{when} \quad i \neq r, \\ r\binom{n+k-2r-3}{n-3}\binom{n-2}{k-1} + \dfrac{r}{k}\binom{n+k-r-2}{n-2}\binom{n-2}{k-1}, & \textit{when} \quad i = r. \end{cases}$$

Proof. We consider the two cases:

Case 1: Consider $i \neq r$. Let T be a plane Husimi graph on n vertices with k blocks such that the root has degree r and a given vertex u has outdegree i. Let the outdegree of the remaining vertices be $d_1, d_2, \ldots, d_{n-2}$. Again these outdegrees of T are arranged as one traverses the Husimi graph by DFS. The total number of nonnegative integer solutions of the equation $d_1 + d_2 + \cdots + d_{n-2} = k - r - i$ is $\binom{n+k-r-i-3}{n-3}$. As in the proof of Theorem 3.9 in [4], there are r ways to insert r and i in the sequence $d_1, d_2, \ldots, d_{n-2}$ so that we recover the plane Husimi graph. Moreover, there are $\binom{n-2}{k-1}$ choices for block types. This proves the result.

Case 2: For $i = r$, we need to note that the roots are also counted. Thus the result follows by adding the result of Case 1 and Equation (2.5). $\square$

Corollary 2.10. *The total number of vertices of degree $i \geq 1$ over all plane Husimi graphs on $n \geq 1$ vertices with k blocks is*

$$\frac{n+k-1}{k}\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.$$

Proof. The desired formula is the sum of the total number of non-root vertices of degree $i-1$ and the number of roots of degree i in plane Husimi graphs on n vertices with k blocks. By Equations (2.3) and (2.5), we have the required formula as:

$$\begin{aligned}
&\left[\binom{n+k-i-1}{n-2} - \frac{i-1}{k}\binom{n+k-i-1}{n-2} + \frac{i}{k}\binom{n+k-i-2}{n-2}\right]\binom{n-2}{k-1} \\
&= \left[\frac{k-i+1}{k}\binom{n+k-i-1}{n-2} + \frac{i}{k}\binom{n+k-i-2}{n-2}\right]\binom{n-2}{k-1} \\
&= \left[\frac{n+k-i-1}{k}\binom{n+k-i-2}{n-2} + \frac{i}{k}\binom{n+k-i-2}{n-2}\right]\binom{n-2}{k-1} \\
&= \frac{n+k-1}{k}\binom{n+k-i-2}{n-2}\binom{n-2}{k-1}.
\end{aligned}$$

$\square$

3. Enumeration of d-ary tree-like structures

We begin by enumerating d-ary Husimi graphs according to number of blocks and block types.

Theorem 3.1. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number $dHG_n(n_2, n_3, \ldots)$ of d-ary Husimi graphs on n vertices having n_j blocks of size j is*

$$dHG_n(n_2, n_3, \ldots) = \frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 2} n_j!} \tag{3.1}$$

where k is the total number of blocks.

Proof. Let x mark a vertex. Let $D(x)$ be the generating function for d-ary Husimi graphs. Let y_i mark the number of vertices in each block. The generating function $D(x)$ satisfies the functional equation $D(x) = x(1 + \sum_{i\geq 2} y_{i+1} D^i)^d$. By the Lagrange inversion formula [18], we obtain

$$\begin{aligned}
[x^n]D(x) &= \frac{1}{n}[t^{n-1}](1 + \sum_{i\geq 2} y_{i+1}t^i)^{dn} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{dn}{k}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^k \\
&= \frac{1}{n}\sum_{k\geq 0}\binom{dn}{k}\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}}\frac{k! y_2^{n_2} y_3^{n_3}\cdots}{n_2! n_3! \cdots}.
\end{aligned}$$

Thus the required formula is

$$\frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 2} n_j!}.$$

□

Example 3.2. Consider binary Husimi graphs on 5 vertices with 2 blocks of type $(0, 2, 0, \ldots)$ satisfying the coherence conditions. There are nine such graphs as given in Figure 2.

Corollary 3.3. *The number of d-ary Husimi graphs on n vertices having k blocks is given by*

$$\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}. \tag{3.2}$$

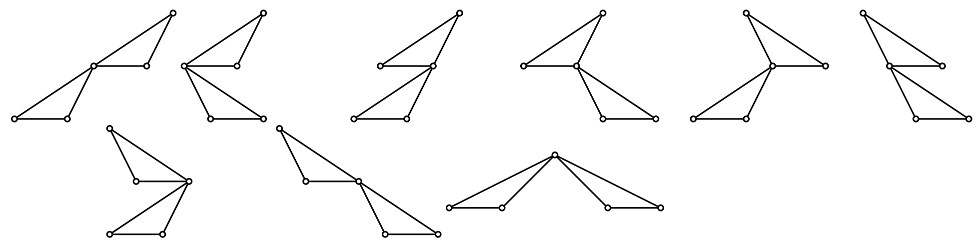

FIGURE 2. Binary Husimi graphs on 5 vertices with 2 blocks of type and block type $(0, 2, 0, \ldots)$.

Proof. From Equation (3.1), the required formula is given by

$$\frac{1}{n}\binom{dn}{k} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!}{n_2!n_3!\ldots}. \tag{3.3}$$

By Lemma 2.1, we have

$$\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!}{n_2!n_3!\ldots} = \binom{n-2}{k-1}. \tag{3.4}$$

Substituting Equation (3.4) in Equation (3.3), we find that the number of d-ary Husimi graphs on n vertices with k blocks is $\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}$. □

Summing over all k in Equation (3.2), we find that there are a total of

$$\frac{1}{n}\binom{(d+1)n-2}{n-1} \tag{3.5}$$

d-ary Husimi graphs on n vertices. Setting $d = 1$, we find that there are

$$\frac{1}{n}\binom{2n-2}{n-1}$$

unary Husimi graphs in which every vertex has outdegree 1 or 0. This is another manifestation of Catalan numbers.

Corollary 3.4. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number $dC_n(n_2, n_3, \ldots)$ of d-ary cacti on n vertices having n_j blocks of size j is*

$$dC_n(n_2, n_3, \ldots) = \frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 3} n_j!} \tag{3.6}$$

where k is the total number of blocks.

Proof. Since there is only one way to convert a complete graph into a cycle, the required equation follows from Equation (3.1) i.e, $dC_n(n_3, n_4, \ldots) = dHG_n(n_3, n_4, \ldots)$. □

Summing over all n_j and k in Equation (3.6), we find the total number of d-ary cacti on n vertices as

$$\sum_{k\geq 1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{1}{n}\binom{dn}{k}\frac{k!}{\prod_{j\geq 3}n_j!}.$$

Since there are exactly two orientations for each block in a cactus, then there are

$$\sum_{k\geq 1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{1}{n}\binom{dn}{k}\frac{2^{k-n_2}\cdot k!}{\prod_{j\geq 3}n_j!}$$

oriented cacti on n vertices.

In the following theorem, we get the number of d-ary Husimi graphs with a given number of leaves.

Theorem 3.5. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number of d-ary Husimi graphs on n vertices with ℓ leaves and having n_j blocks of size j is given by*

$$\sum_{m=0}^{n-\ell}\frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\binom{dm}{k}\frac{k!}{\prod_{j\geq 2}n_j!}, \tag{3.7}$$

where k is the number of blocks.

Proof. Let x and u mark vertices and leaves respectively. Also let y_i denote the number of vertices in each block. Then the bivariate generating function for the number of d-ary Husimi graphs with given number of vertices and leaves is given by $D(x,u) = x(u + (1+\sum_{i\geq 2}y_{i+1}D^i)^d - 1)$. We obtain the coefficients of x^n and u^ℓ in the generating function. By Lagrange inversion formula [18], we have

$$\begin{aligned}
[x^n u^\ell]D(x,u) &= \frac{1}{n}[u^\ell t^{n-1}]\left(u + \left(1+\sum_{i\geq 1}y_{i+1}t^i\right)^d - 1\right)^n\\
&= \frac{1}{n}[u^\ell t^{n-1}]\sum_{m=0}^{n}\binom{n}{m}u^m\left(\left(1+\sum_{i\geq 1}y_{i+1}t^i\right)^d - 1\right)^{n-m}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\left(1+\sum_{i\geq 1}y_{i+1}t^i\right)^{dm}(-1)^{n-\ell-m}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\sum_{k=0}^{dm}\binom{dm}{k}\left(\sum_{i\geq 1}y_{i+1}t^i\right)^k\\
&= \frac{1}{n}\binom{n}{\ell}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\sum_{k=0}^{dm}\binom{dm}{k}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{k!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\cdots}.
\end{aligned}$$

Thus the required formula is

$$\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}(-1)^{n-\ell-m}\binom{dm}{k}\frac{k!}{\prod_{j\geq 2} n_j!},$$

which completes the proof □

By summing over all n_j in Equation (3.7), we obtain the following corollary:

Corollary 3.6. *There are*

$$\frac{1}{n}\binom{n}{\ell}\binom{n-2}{k-1}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\binom{dm}{k}(-1)^{n-\ell-m}$$

d-ary Husimi graphs on n vertices with k blocks and having ℓ leaves

Corollary 3.7. *If $(n_2, n_3, \ldots)$ is a sequence of positive integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j + 1$, then the number of d-ary cacti with ℓ leaves and having n_j blocks of size j is given by*

$$\sum_{m=0}^{n-\ell} \frac{1}{n}\binom{n}{\ell}\binom{n-\ell}{m}\binom{dm}{k}(-1)^{n-\ell-m}\frac{k!}{\prod_{j\geq 3} n_j!}$$

where k is the total number of blocks.

Proof. The result follows by noting that that there is exactly one way of converting a d-ary Husimi graph to a d-ary cactus. □

In the sequel, we enumerate d-ary Husimi graphs by outdegree sequence.

Theorem 3.8. *The number of d-ary Husimi graphs on n vertices with k blocks such that there are r_i vertices of outdegree i is*

$$\frac{1}{n}\binom{n}{r_0, r_1, \ldots, r_d}\binom{n-2}{k-1}\binom{d}{0}^{r_0}\binom{d}{1}^{r_1}\binom{d}{2}^{r_2}\cdots\binom{d}{d}^{r_d} \tag{3.8}$$

if $\sum_{i=0}^{d} ir_i = k$. If the equation is not satisfied then there are no such d-ary Husimi graphs.

Proof. We consider a plane Husimi graph on n vertices with k blocks such that the maximum outdegree is d and that there are r_i vertices of outdegree i. The number of such graphs is given in Theorem 2.3. We can convert this graph to the required d-ary Husimi graph by selecting the positions of the block children for each vertex. If a vertex has j block children then there are $\binom{d}{j}$ positions for the block children in the d-ary Husimi graph. The result thus follows. □

By summing over all r_i satisfying the coherence conditions $r_0+r_1+\cdots+r_d = n$ and $r_1 + 2r_2 + \cdots = k$ we obtain the following corollary:

Corollary 3.9. *There are*

$$\frac{1}{n}\binom{dn}{k}\binom{n-2}{k-1}$$

d-ary Husimi graphs on n vertices and k blocks.

Corollary 3.10. *The number of binary Husimi graphs on n vertices such that there are r_0 vertices with no block child, r_1 vertices with 1 block child and r_2 children with 2 block children is given by*

$$\frac{2^{n+r_1-2}}{n}\binom{n}{r_0, r_1, r_2}.$$

Proof. The result follows from Equation (3.8) by setting $d = 2$, summing over all k and simple algebraic manipulations. □

4. Conclusion and future work

In this work, we have enumerated plane and d-ary tree like structures by the number of vertices and blocks. We have also used degree of the root to enumerate plane Husimi graphs. It would be interesting to obtain an equivalent result for d-ary Husimi graphs. We were also able to enumerate plane Husimi graphs and d-ary Husimi graphs by the outdegree of vertices. In [3], Dershowitz and Zaks enumerated plane trees by outdegree and level of a given vertex. Eu, Seo and Shin enumerated plane trees according to first children, non-first children and level in [5]. It would be interesting to obtain equivalent results for plane Husimi graphs and d-ary Husimi graphs. . We also obtained the formula for the number of d-ary Husimi graphs with a given number of leaves. An equivalent result for plane Husimi graphs was obtained earlier by Okoth in [15]. Forests of these plane and d-ary tree like structures can also be enumerated.

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