APPROXIMATIONS OF RUIN PROBABILITIES UNDER FINANCIAL CONSTRAINTS.

BY

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DECLARATION

This thesis is an original work and therefore has not been presented for award of any degree in any other institution.

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DEDICATION

This thesis is dedicated to my lovely son Anthony (Martial) Odhiambo, and more importantly to the Almighty God for His unending blessings.

ABSTRACT

This thesis studies the approximate ruin probabilities under financial constraints which include the rate of inflation, constant interest rate, and taxation. When the surplus falls below zero, the insurance company is technically considered ruined. The main objective of the study included; to establish a risk model which takes into account all the financial constraints,to establish analytically, the formula for the approximation of ruin probabilities for both exponentially and sub-exponentially distributed claims, to compare the approximate ruin probabilities from our model and those of the classical Cramér-Lundberg model, and finally to compare the convergence of Pareto and Log-normal distributions for the formulated model. An extensive review of literature is done and much attention is given to the research by Albrecher and Hipp whose research successfully formulates Lundberg's (classical) risk process in presence of tax. A risk model is formulated in the present study whose premium inflow is influenced by inflation and a constant interest rate. We thereafter invoke the Albrecher and Hipp loss-carried-forward tax scheme from which an approximation of probability of ruin for the light tailed (exponential) distribution is derived for an exact solution. Then, a suitable formula for the claims with sub-exponential distribution is also derived using the Pollaczek-Khintchine formula. Simulations are hence done using R and Microsoft Excel in this regard. The results show that approximating ruin probability when taking into account all the three financial constraints gives desirable results as compared to those of classical Lundberg model. The comparison between the two heavy-tailed distributions under the concept of limiting density ratio, shows that a Log-normal density exhibit a lighter tail, thus converges faster. However, the model is open for further improvements, specifically to incorporate a stochastic rates of interest. The results of this study will hence guide the policymakers and the insurance industry to make informed decisions to help guard against future ruin as witnessed in local insurance companies in Kenya and globally.

Contents

LIST OF ABBREVIATION

i.i.d: independent and identically distributed

R: It is a statistical Analysis Package

r.v: Random Variable

d.f : Density function of any r.v

P.D.F/p.d.f: Probability density function

P.M.F/p.m.f: Probability mass function

C.D.F/c.d.f: Cumulative distribution function

M.G.F or m.g.f: Moment generating function

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Chapter 1

Introduction

This chapter gives the background information, basic actuarial concepts used, mathematical tools needed to solve the problem,statement of the problem, objectives, and significance of the study.

1.1 Background Information and Preliminary Information

Risk theory, the concept which is the basis of the present study, was pioneered by Filip Lundberg in 1903 whose work was later reviewed, refined, and expanded further by Harald Cramér around 1930 and 1955. Their research and contributions marked the foundation of actuarial risk theory, a concept that is immensely driving research in actuarial mathematics. The classical risk model that came out of the mentioned research is the famous Cramér-Lundberg model whose details are discussed in chapter 3 of this thesis. This model generated a lot of research and extensions which have been extensively discussed in (Schmidli, 1995), (Cheung and Landriault, 2012), (Grandell, 1977), (Embrechts et al., 2013), and (Rolski et al., 2009). It is important to highlight that Cramér-Lundberg model and classical risk model or process are used interchangeably in the sequel.

Taylor (1979), studied the probability of ruin when the classical risk model was modified to take into account inflationary conditions and interest rate. Albrecher and Hipp (2007) pioneered the study of the risk model by considering the aspects of taxation. They investigated how the element of tax affects the ultimate ruin probability. Wei (2009) investigated the model termed a general risk in which probability of ruin is estimated in presence of Albrecher and Hipp tax scheme and a constant force of interest . Also, Dbicki et al. (2015) investigated the effect of financial factors including effects of the inflation and that of interest rates on Gaussian risk models.

The present research studies and reviews the classical risk processes outlined in (Cramér, 1930) and (Cramer, 1955) and extends the model to be more realistic by including the effects of economic factors such as taxation, inflation, and interest rates on the model. In this thesis, interest rates, inflation, and taxation will be referred to as financial constraints or economic factors. The probability of ruin is an indispensable technical tool which reflects the financial position of an insurance company. It is a tool which is used in the management of the capital of insurance portfolio. Though technical, a higher probability of ruin ultimately indicates instability and thereafter the panic by the management. This calls for consideration of factors such as attraction of extra capital by the insurance, lowering the premium rates to attract many policyholders. On the same note proper reinsurance should also be given a lot of consideration. It is imperative to note that the probability of ruin solely examines the risk of insurance but not corruption and mismanagement of the assets of the company, therefore a model that seeks to generate the accurate probabilities for a portfolio of insurance should specifically be used devoid of such vices.

1.2 Basic Concepts

At this point, we define some important and basic concepts that are very useful in this thesis. We highlight risk mathematics (actuarial) concepts, some probability distributions which are useful in the final chapters of the present research, and some mathematical concepts for example the monotone convergence theorem which aid the understanding of various concepts in the present study. Law of large numbers due to Poisson process is also highlighted.

1.2.1 Actuarial Science Concepts

(a) **An insurance** is the financial protection against losses from an insurance portfolio in exchange of regular premiums paid by the insured. An insurer is an entity or a business which develops a policy while insured is the policyholder who received financial protection upon taking the policy as outlined in (Wang et al., 2018) and (Kaas et al., 2008).

(b) **A premium** is the amount of money paid by policyholders for the protection from an anticipated risk as indicated in (David, 2016), (Promislow, 2006), (Feller,),(Ni, 2015), (Grandell, 1977), (Taylor, 1979), and (Klugman et al., 2012).

(c) **Claims** are the payments that are made to the insured for their insurance. The claim count process is usually denoted by $\{N(t), t \geq 0\}$ where $N(t)$ is the number of claims up to time t as outlined in (David, 2016), (Promislow, 2006), (Feller,), (Ni, 2015), (Grandell, 2012), (Embrechts et al., 2013), and (Klugman et al., 2012).

(d) **An epoch of a claim**, is the time when the claim arrive. Denoting the epochs by $\tau_1, \tau_2, \tau_3, ..., \tau_\infty$ then

$$T_k = \tau_k - \tau_{k-1}, \quad k \geq 1$$

are known as the arrival time between claims as in (Kaas et al., 2008), (Embrechts et al., 2013), and (Klugman et al., 2012).

(e) **A counting process** $N(t), t \geq 0$. It enumerates the total number of claims until time t. It is modelled using a Poisson process which is defined below,

Definition 1. *$N(t), t \geq 0$ is a Poisson process with rate $\lambda > 0$ lets the following mandatory conditions are met as illustrated in (Grandell, 2012), (Embrechts et al., 2013), and (Klugman et al., 2012):*

(i) The number of claims count at time zero is usually zero i.e $N(0) = 0$.

(ii) The process is stationary and has independent increments.

(iii) The number of claims $(N(t), t \geq 0)$ in an interval of length t is Poisson process which is distributed with the expected value λt. That is to say, $\forall\ s > 0,$ and $t > 0$ we have

$$\mathbb{P}\left(N(t+s) - N(s) = k\right) = \frac{(\lambda t)^k exp(-\lambda t)}{k!}, \quad k = 0, 1, ..., \quad t \geq 0, \quad \lambda > 0 \qquad (1.1)$$

Stationary increment as in (David, 2016), (Promislow, 2006), (Feller,), (Ni, 2015) and (Klugman et al., 2012), implies that the claim counts distribution in a fixed interval depends solely on the interval's length and not on the time of occurrence of the interval, i.e, there are no trend effects. Independent increments implies there is no overlapping between intervals. More importantly, Poisson process has the property that the times between arrival of claims follows an i.i.d exponential process, each with expected value λ^{-1}. The inter-arrival times of claims follows an exponentially distributed r.v. which when replaced by any arbitrary r.v. $\{X_1, X_2, ...\}$, leads to the generalization of the counting process follows;

Definition 2. *Consider the sequence of random variables $\{X_t\}_{t>0}$. $\{N(t),\ t \geq 0\}$ is a renewal process if the said sequence is i.i.d.*

Thus, we can then define a counting process as follows:

$$N(t) = \max\left\{n : S_n \leq t\right\}$$

where S_n is the sum of individual claims i.e. $S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$

(f) **A risk** is an unfavourable event. In insurance which is our subject of research, the typical risk is the possibility of a big claim which can cause ruin (bankruptcy/insolvency) of an insurance portfolio as highlighted in (David, 2016), (Promislow, 2006), (Feller,), and (Ni, 2015).

(g) **Inflation rate** is the rate of increase in the price of goods and services. Presence of inflation leaves us with less goods and services for the same amount of money. A real rate takes into consideration the effects of inflation as opposed to money or nominal rate.

1.2.2 Required Probability Distributions

A probability function that illustrates and outlines countable and possible values, likelihood, and moments that a r.v. can accommodate within a range of possible values and likelihoods that a random variable can take with a range of possible values (which can be discrete or continuous) is known as a probability distribution. The function has moments which are a set of parameter that measure a distribution. Four moments are commonly used and they include expected value (first moment);

$$\mathbb{E}[X] = \mu$$

variance (second moment);

$$\mathbb{V}ar(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

coefficient of skewness(third moment);

$$\text{Skew}(X) = \mathbb{E}[(X-\mu)^3]$$

and kurtosis (fourth moment)

$$\text{Kurt}(X) = \mathbb{E}[(X-\mu)^4]$$

. Thus the moment generating function (M.G.F) generates the moments (Feller,), (Shreve, 2004), and (Gupta and Kapoor, 1997).

Mathematically, M.G.F, $M_X(t)$ of a random variable X is given by $M_X(t) = \mathbb{E}\left[e^{tX}\right]$ for all values of t for which the expectation exists. Consider an arbitrary function $g(x)$. The expectation of a discrete random variable is

$$\mathbb{E}\left[g(x)\right] = \sum_x g(x)\mathbb{P}(X = x)$$

for a discrete random variable and

$$\mathbb{E}\left[g(x)\right] = \int_x g(x)f(x)dx$$

for a continuous random variable. where $\mathbb{P}(X = x)$ and $f(x)$ are the probability mass function p.m.f, and probability distribution function for the discrete and continuous r.vs respectively. The M.G.F is defined by

$$M_X(t) = \sum_x e^{tx}\mathbb{P}(X = x)$$

for discrete distribution and

$$M_X(t) = \int_x e^{tx} f(x)dx$$

, for continuous probability distributions.

Exponential Distribution

A continuous random variable X assumes a non-negative value is said to have an exponential probability distribution with parameter $\lambda > 0$ if its probability function is given by (Gupta and Kapoor, 1997) as

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } \ x > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{1.2}$$

The d.f is given by $F(x) = 1 - e^{-\lambda x}$, and the mean and variance of the distribution is given by $1/\lambda$, and $1/\lambda^2$ respectively. Finally the moment generation function (M.G.F) is given by

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \quad \lambda > t$$

Pareto Distribution

The p.d.f of a r.v. $X \sim \text{Par}(\alpha > 0,\ b > 0)$ is given by

$$f(x) = \begin{cases} \frac{\alpha b^\alpha}{(b+x)^{\alpha+1}}, & x > 0,\ \sigma > 0 \\ 0 & \text{elsewhere} \end{cases} \tag{1.3}$$

The tail of this distribution is given by

$$1 - F(x) = \left(\frac{b}{b + x}\right)^\alpha \tag{1.4}$$

Which decreases with power speed as outlined in (Klugman et al., 2012) and (Gupta and Kapoor, 1997). It's expected value and variance include

$$\mathbb{E}(X) = \frac{b}{\alpha - 1}, \quad \alpha > 1 \qquad \mathbb{V}ar(X) = \frac{\alpha b^2}{(\alpha - 1)(\alpha - 2)}, \quad \alpha > 2$$

The proof of the two properties of the distribution is as sketched below.

Proof. The M.G.F of a Pareto distribution is given by

$$(M_X(t))^{r=k} = \left(\int_x e^{tx} f(x) dx \right)^{r=k} = \left(\int_0^\infty e^{tx} \frac{\alpha b^\alpha}{(b+x)^{\alpha+1}} dx \right)^{r=k} \qquad k = 1, 2, ...$$

$$\Longleftrightarrow \frac{d^k}{dt^k} M_X(t)|_{t=0} = \mathbb{E}(X^t) = \frac{\Gamma(\alpha - t)\Gamma(1+t)}{\Gamma(\alpha)} b^t \quad t = 1, 2, ..., t < \alpha$$

The expected value and variance follows from the M.G.F, and this completes the proof. □

Log-Normal Distribution

Log-normal probability distribution function is given in (David, 2016) and (Marshall and Olkin, 2007) by

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{1}{2} \left(\frac{\log x - \mu}{\sigma} \right)^2 \right), & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.5}$$

for a random variable $X \sim LN(\mu, \sigma^2)$. The tail is given by,

$$1 - F(x) = 1 - \Phi\left(\frac{\log x - \mu}{\sigma} \right) \tag{1.6}$$

The expected value and variance include

$$\mathbb{E}(X) = \exp\left(\mu + \frac{1}{2}\sigma^2 \right), \qquad \mathbb{V}ar(X) = \exp(2\mu + \sigma^2)\left(e^{\sigma^2} - 1 \right)$$

The proof of the two properties of this distribution is sketched as below.

Proof.

$$M_X(t) = \int_x e^{tx} f(x) dx$$

$$\mathbb{E}(X^t) = \frac{d^k}{dt^k} M_X(t)|_{t=0} = e^{t\mu + \frac{1}{2}t^2\sigma^2} \qquad t = 1, 2, 3, ...$$

The expected value and the variance follows from the M.G.F, and this completes the proof.

□

1.2.3 Properties of Expectations

The following subsection follows the contents of (Shreve, 2004) closely,

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$. Then the following two important properties of expectation hold;

- If X, Y, and XY are integrable random variables, and X is $\mathcal{G}-$measurable, then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$$

- If in addition, X is independent of $\mathcal{G}$, then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$$

The latter property also hold if $X > 0$ (not necessarily integrable), although both sides are $+\infty$.

Proof. The proof can be obtained in (Shreve, 2004). □

1.2.4 The Law of Large Numbers

We state without proof the following important theorem as outlined in (Feller,) and (Gupta and Kapoor, 1997).

Assume $X_1, ..., X_n$ is a set of pairwise independent random variable with first and second moments given as $\mathbb{E}(X_i) = \mu$, and $\mathbb{V}ar(X_i) = \sigma^2$, respectively. Then for any $l > 0$,

$$\mathbb{P}(\mu - l \leq \bar{X} \leq \mu + l) \geq 1 - \frac{\sigma^2}{n^2 l}$$

Therefore, as it is evident that as $n \longrightarrow \infty$ the probability tends to 1 a.s (almost surely). Equivalently, given $S_n = n\bar{X}$, the population mean;

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| < \epsilon\right) \longrightarrow 1 \quad \text{a.s}$$

Proof. The proof is omitted, refer to (Gupta and Kapoor, 1997). □

Secondly, the strong law of large numbers for the Poisson states that

$$\lim_{t \longrightarrow \infty} \frac{N(t)}{t} = \lambda \quad \text{a.s}$$

and also that

$$\lim_{t \longrightarrow \infty} N(t) = \infty \qquad \text{a.s}$$

where $N(t)$, and λ are defined above.

Proof. See (Kyprianou, 2013). □

1.2.5 The Monotone Convergence Theorem

Let $f_n(x) > 0$ be measurable and integrable function on a measurable set X such that $f_n(x) \longrightarrow f(x)$ point-wise almost everywhere and $f_1(x) \leq f_2(x) \leq \cdots$, then

$$\int_x f_n(y)dy \sim \int_x f(y)dy \quad \text{as} \quad n \longrightarrow \infty$$

Proof. Refer to (Trench, 2013) for the outline of the proof. □

1.3 Statement of the Problem

Bankruptcy or insolvency is technically known as ruin of an insurance company. Ruin takes place when the total value of assets falls below that of liabilities. It is important to asses the ruin probability of an insurance portfolio so as to help ascertain the extent to which it can survive. Models such as Cramér-Lundberg have been developed to help determine the best approximations to ruin probabilities. However, the accuracy of the model is key for the existence of an insurance portfolio. The classical model does not take into account the effects of financial constraints, thus it is not an appropriate model for the approximation of ruin for an insurance portfolio. Various economic factors e.g., a high rate of inflation, have contributed negatively and certainly, the result is the decline in investments opportunities and loss of employment. Notable modifications have been made to make the approximations by the model more accurate by taking into account some economic conditions. In this study we modify the classical risk model to come up with an improved model which captures all the financial constraints.

1.4 Objectives of the Study

1.4.1 Core Objective

The main objective of the present research was to approximate ruin probabilities under the financial constraints.

1.4.2 Other Objectives

Specific objectives of this research include:

(i) To formulate a risk model which takes into account the financial constraints.

(ii) To establish analytically, the closed form formula for approximating ruin probabilities for both the exponentially and sub-exponentially distributed claims using the present model.

(iii) To simulate and compare the approximate probabilities for classical or Cramér-Lundberg's model and the formulated model.

(iv) To compare the convergence of tails of Pareto and log-normal distributions under the formulated model.

1.5 Significance of the Study

The results of this research are important since they accurately approximate the ruin probability of an insurance portfolio. The model developed gives better and reliable results as compared to the existing models, particularly, the Cramér-Lundberg model since our model takes into consideration the financial constraints. The insurance sector globally has experienced quite a lot of challenges which have led to the insolvency of some insurance companies, for example, the collapse of the Blue-Shield and Bima insurance companies in Kenya. Various economic factors, for example, a high rate of inflation, have contributed negatively and certainly, the result is the reduction in investments and loss of employment. Therefore, the effective modification of the classical risk model to incorporate the financial constraints is critical in the approximation of ruin probabilities. Though technical, the expected results obtained can be used to guide the insurance companies, through the regulatory authorities, to put proper structures and policies in place to guard against far-reaching consequences of ruin.

Chapter 2

Literature Review

The classical insurance risk model was proposed by Lundberg (1903). His research work laid the foundation of modern actuarial theory. This was succeeded by the improvements made by Cramér (1930), hence the invention of the Cramér-Lundberg model which is considered as classical risk model for approximating the ruin probabilities.

Taylor (1979) modified the classical risk model to incorporate inflationary conditions and interest rates. The author superimposed inflation in the model and realized that inflation does not affect free reserves. If the risk process was subjected to both inflation and interest accumulation, it was observed that the differences between the forces of inflation and those of interest are constant, however small and positive, the probability of ruin still holds. In their research, (Bohnert et al., 2016) asserted that there is a major concern on claims reserving if inflation risk is not given required attention in the risk process and consequently, it is of greater need in the non-life insurance.

(Debicki et al., 2015) investigated the aggregate claims as a Gaussian risk process as opposed to the usual compound Poisson risk process. The resulting risk process incorporated inflationary effect and interest rates. This was a commendable move to improve the classical risk process. Their results captured the finite time horizon ultimate probability of ruin for the non-classical model. They also came up with an estimate of the conditional ruin time as the initial capital of an insurance portfolio keeps growing by an exponential random variable.

(Zhu and Yang, 2008) considered a compound Poison risk model in which a credit interest adds, at some constant interest rate, to an insurance portfolio for some positive surplus, otherwise the debt interest is realized through some specified rate. It was also observed that an absolute ruin occurs when the surplus value first hit some constant, usually some negative critical value. The authors further investigated the ultimate ruin probability behaviour when the claim distribution are light-tailed and further extended it to the case when the distribution of claims distribution are heavy-tailed and motivated by (Cai, 2007). The model further presented the generalization of the ruin probability for the risk model with a constant force of interest although it does not take into account all the economic factors. (Konstantinides et al., 2010) studied the probability of absolute ruin with invariant premuim rate and a constant force of interest for the case of both definite and indefinite time horizon renewal model of risk. (Wang, 2018) also investigated the same renewal risk model which follows a Lévy process which is Geometric and with variant rate of return and a Brownian fluster. Their results indicated greater sensitivity to the said fluster, whenever heavy-tailed distributions are considered in the modelling of the claims process. Nevertheless, the studies mentioned above do not take into consideration the effects of taxation in the surplus models used.

The other economic factor that is of keen interest in the present thesis is the incorporation of taxation in the classical risk process. In their paper, (Albrecher and Hipp, 2007) considered an insurance portfolio and the insurance risk modelled by the Lundberg risk process and incorporated the tax rate in the classical risk model, specifically for the infinite time probability of ruin. In their model, it is assumed that tax is paid at an invariant rate $\gamma \in [0, 1)$ of the income due to premium inflow only if the insurance portfolio is stable in terms of profit inflow. Ultimately, the resulting non-ruin probability is the power of the non-ruin probability for the classical ruin model. The model nevertheless is far from reality as observed since it ignores very important economic factors like inflation, return, and dividend payments just to mention a few. This is done purposely to achieve simplicity and tractability as put by the authors.

Wei (2009) investigated the ruin probability in what the author described as a generalization of the risk model in which the probability of ruin generated by claims process which follows a compound Poisson process was derived. In their study, the authors outlined as a Corollary, the Albrecher and Hipp loss-carried-forward tax scheme. Further, an invariant force of interest is introduced presenting a special case. A closed form formula for the approximation of the probability of ruin with both light-tailed (exponential) and heavy-tailed (sub-exponential) claims distribution are derived. The authors, however, do not include the effects of taxation in their approach to approximation of ruin probability.

In general the results for the Cramér-Lundberg model with taxation presented by (Albrecher et al., 2008) in and the works of (Albrecher et al., 2008) in which the two-sided exit problem is properly solved using the fluctuation theory whereby the probability of ruin for the general spectral negative Lévy's model of risk with influence of taxation through a loss-carried-forward tax scheme is also investigated. The arbitrary moment of the discounted total tax and determination of the level of surplus to commence incorporation of taxation to maximize discounted average aggregate income for the authority in the model presents an adequate condition to come up with a unique optimal taxation. The model is nevertheless, far from reality since inflationary conditions and interest rates conditions are not taken into consideration.

(Wang et al., 2010) investigated a loss-carried forward tax scheme in the surplus-dependent risk process with a constant rate of investment interest. The authors stipulated that taxation affects both premium inflow and the rate of interest due to investment. On discounting, a closed form of the discounted tax payment and in presence of investment rate is explicitly presented, resulting into a proper condition for the constant rate of taxation for which there exists a starting level for levying a tax to an insurance portfolio. Clearly, the effects of inflation is not incorporated by the authors in their model.

(Cheung and Landriault, 2012) analysed the compound model of risk for the surplus process which depends on the premium inflow rate in the tax scheme introduced by (Albrecher and Hipp, 2007). Upon careful reading through the research and findings of (Albrecher et al., 2008), the generalization of the findings in the Gerber-Shiu work whereby a function for the maximum surplus before ruin occurs is obtained. The author demonstrated that there exists no significant differences between the said Gerber-Shiu's function in the tax-dependent risk model and the initial Gerber-Shiu function which is independent of taxation in the dividend barrier framework. Also closely captured in the paper is the level where taxation thrives and the discounted taxation moment. Nevertheless, the authors did not include inflationary conditions in the model.

(Sundt and Teugels, 1995) came up with a model which takes into consideration the effect of nominal force of interest on the risk reserve process. An integral equation for the survival probability is obtained explicitly under the said model, together with bounds on ruin probabilities for example Lundberg bound. However, the authors do not consider the effects of inflation and taxation in their analysis.

Clearly, the above-mentioned models and findings do not present a model which takes into consideration the effects of all the three economic factors, therefore, in this present study, we consider a more general risk process which factors in the three economic factors including rate of inflation, interest rates, and rate of taxation. This model will enable us to obtain approximate ruin probabilities which are considered more accurate.

Chapter 3

Research Methodology

3.1 Introduction

In this chapter, we consider in details, the classical risk model and outline the methods which we have used to help us achieve our main objective. We also discuss the classical approximation to the ruin probabilities by outlining an integro-differential equation from the key renewal theorem. We finally outline the properties of heavy-tailed (long-tailed or fat-tailed) distributions, specifically a class of sub-exponential, and regularly varying tails.

3.2 The Cramér-Lundberg Model

This model is known as the classical risk process because it's birth laid down the foundation of actuarial risk mathematics. It is from this model that we base our present study. A risk surplus is the amount by which an insurance portfolio's premium inflow exceeds its claims outgo. The classical risk process is usually described mathematically by a very significant equation in actuarial science as illustrated in (David, 2016),(Grandell, 2012), (Embrechts et al., 2013), (Promislow, 2006), and (Feller,). This is known also as the surplus equation, given by,

$$U(t) = u + ct - S(t), \quad t \geq 0 \tag{3.1}$$

Whereby, $u = U(0)$ denotes the initial surplus, ct denotes the premium income collected from time 0 to time t where it is also assumed that $c \geq 0$, and

$$S(t) = \begin{cases} \sum_{k=1}^{N(t)} X_k, & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t = 0 \end{cases} \tag{3.2}$$

denotes the aggregate claim amount in which $X_k,\ k = 1, 2, ..., N(t)$ are individual claim sizes, and of course $N(t)$ are claim count.

$\{S(t), t \geq 0\}$, is called Poisson process with Poisson(λ). The density function of $X_k's$ is henceforth denoted by $F(x)$ and it will be assumed for the entire research that $F(0) = 0$ so that there exists no negative amounts of claims as indicated in (Grandell, 2012), (Embrechts et al., 2013), and (Klugman et al., 2012).

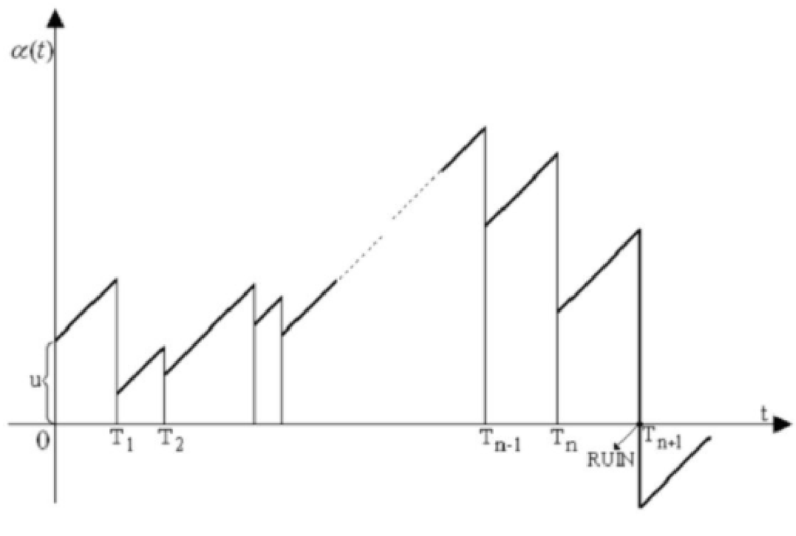

Figure 3.1: A trajectory of a Risk Process (Source: Internet)

It is important to understand what exactly Equation (3.1) means in practice. These parameters have the following interpretation; $U(t)$ is known mathematically as a risk process or the risk of insurance at time t. This value, identical to the current capital shows how risky the insurance portfolio is at time t. Increase (decrease) in this particular value brings a pleasant (unpleasant) news for the total insurance portfolio because its total capital tends to increase (decrease). If $U(t)$ hits 0 or becomes negative, the insurance company is ruined. $U(0) = u$ is termed the initial capital. It is the quantity of resources needed to establish an insurance portfolio. Finally, $S(t)$ is the claims process and its interpretation is explicit in Definition 1 of chapter 1.

3.2.1 Deriving the Rate of Premium inflow (ct)

This is called premium at time t, meaning the premium that an insurance portfolio asks from its policyholders to ensure its own survival, with survival rate corresponding to $\mathbb{E}[U(t)] \geq u, \forall\ t > 0$ i.e. the capital should be in the sense of expectation, no less than the initial capital. Derivation of ct is thus the most crucial part of this model. A natural question is how to figure out a reasonable ct i.e. how to determine premium at time t such that $\mathbb{E}[U(t)] \geq 0$. Clearly this is a mathematical question and we have, using the expected value principle (Rolski et al., 2009), that

$$\mathbb{E}(U(t)) = \mathbb{E}\left(u + ct - \sum_{k=1}^{N(t)} X_k\right) = u + ct - \mathbb{E}\left(\sum_{k=1}^{N(t)} X_k\right) \tag{3.3}$$

by conditional expectation, we have

$$u + ct - \mathbb{E}\left[\mathbb{E}\left(\sum_{k=1}^{N(t)} X_k | N(t)\right)\right] = u + ct - \mathbb{E}\left[\sum_{k=1}^{N(t)} \mathbb{E}\left(X_k | N(t)\right)\right]$$

by independence between X_k and $N(t)$,

$$= u + ct - \mathbb{E}\left[\left\{\sum_{k=1}^{N(t)} \mathbb{E}[X_k] = \sum_{k=1}^{N(t)} \mu = \mu N(t)\right\}\right] = u + ct - \mu \mathbb{E}[N(t)]$$

$$\mathbb{E}[U(t)] = u + ct - \mu\lambda t$$

Consequently, $\mathbb{E}[U(t)] \geq u \iff ct - \mu\lambda t \geq 0$. This condition can be interpreted that the average outflow (claims) being strictly smaller than the average income (premium inflow) in an insurance portfolio. This is also referred to as the net profit condition. Therefore a conscientious insurance company would go, after taking into account some business changes into consideration, for

$$ct = (1 + \theta)\mu\lambda t \tag{3.4}$$

where, θ is the relative safety loading.

3.2.2 Integro-differential Equation

This subsection follows closely the working as outlined in (Grandell, 1977). Consider

$$\Phi(u) = \mathbb{E}\left[\Phi(u) = (u + cT_1 - x_1)\right] = \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} \Phi(u + ct - x) dF(x) dt$$

where, T_1 is the epoch of the first claim, then we have $U(T_1) = cT_1 - X_1$. It is important to note that ruin cannot occur in the interval $(0, T_1)$ since the Poisson process is a special case of renewal process.

The function $\Phi(\star)$ is differentiable . By appropriate differentiation, integration, together with change of variable , the following famous integro-differential equation is obtained. Refer to [Appendix A2] for explicit steps.

$$\Phi(u) = \Phi(0) + \frac{\lambda}{c} \int_0^u \Phi(u - x)(1 - F(x)) dx \tag{3.5}$$

Where $\lambda, c, u,$ and $F(x)$ are as defined earlier and $\Phi(0)$ is the non-ruin probability when the surplus is zero. By monotone convergence theorem, it follows from Equation (3.5) that as $u \longrightarrow \infty$, we have

$$\Phi(\infty) = \Phi(0) + \frac{\lambda\mu}{c}\Phi(\infty) \tag{3.6}$$

From law of large numbers,

$$\mathbb{P}\left(\lim_{t \longrightarrow \infty} \frac{U(t)}{t} = c - \lambda\mu\right) = 1 \quad \text{a.s}$$

Therefore,

$$\Phi(\infty) = 1 \Longleftrightarrow \Phi(0) = 0 \tag{3.7}$$

Hence for exponentially distributed claims, it can be observed that,

$$\Psi(0) = \frac{\lambda\mu}{c} = \frac{1}{(1+\theta)}, \quad \text{where} \quad c > \mu\lambda \tag{3.8}$$

and finally for $u > 0$, it can be observed that

$$\Psi(u) = \frac{1}{(1+\theta)} \exp\left(-\frac{\theta u}{\mu(1+\theta)}\right) \tag{3.9}$$

Proof. See Appendix A1. □

3.3 The Definite and Indefinite Probabilities of Ruin

The classical risk process is defined by Equation (3.1). It is normally assumed that as time increases to infinity, so does the risk process. A critical quantity that is a subject of investigation in the present research is the level of possibility that at some time t, the reserve will be insufficient to overcome claims i.e. $U(t) < 0$. More formally;

Definition 3. *The probability of ruin in definite time is given by*

$$\Psi(u,T) = \mathbb{P}\left[U(t) < 0 \quad \textit{for some} \quad t \leq T\right], \qquad 0 < T < \infty, \quad u \geq 0 \tag{3.10}$$

Whenever, $T = \infty$ *we have ultimate ruin or infinite time ruin. It is denoted by* $\Psi(u) = \Psi(u,\infty),\ u \geq 0$. *The time to ruin is thus given by*

$$\tau(T) = \inf\left\{t : 0 \leq t \leq T \quad, U(t) < 0\right\}, \quad 0 < T < \infty \tag{3.11}$$

whereby by convention,

$$\inf(\emptyset) = \infty \tag{3.12}$$

We usually write $\tau = \tau(\infty)$ *for the ruin with infinite time horizon.*

Definition 4. *Given a risk process with a Poisson process* $\{N(t),\ t \geq 0\}$

$$S(t) = \sum_{k=0}^{N(t)} X_k - ct, \quad t \geq 0 \tag{3.13}$$

is termed a claim process which has a supremum $L = \sup_{t \geq 0} S(t)$. *Then equivalently*

$$\Psi(u) = \mathbb{P}\left(L > u | U(0) = u)\right) \tag{3.14}$$

3.4 The Adjustment Coefficient

Definition 5. *Consider an arbitrary claim size r.v* X. $t = r$ *is the smallest possible solution to the equation*

$$1 + (1+\theta)\mu t = M_X(t) \tag{3.15}$$

where $M_X(t)$ *is the M.G.F of* X. *We shall refer to such a variable as the adjustment coefficient.*

If $X \sim \text{Exp}(\mu)$, the adjustment coefficient is obtained from Equation (3.15) as follows.

$$1 + (1+\theta)\mu r = \frac{1}{(1-\mu r)} \tag{3.16}$$

An equation which r satisfies from Equation (3.15), and solving Equation (3.16) quadratically, $r = 0$ or $r = \frac{\theta}{\mu(1+\theta)}$. r can also be numerically solved by initial guess.

From Equation (3.15), we may write

$$1 + (1+\theta)\mu r = \mathbb{E}(e^{rX})$$

Expanding we obtain

$$1 + (1+\theta)\mu r = 1 + r\mu + \frac{1}{2}r^2\mathbb{E}(X^2)$$

Thus,

$$r < \frac{2\theta\mu}{\mathbb{E}(X^2)} \tag{3.17}$$

Another useful form for the adjustment coefficient is

$$1 + \theta = \int_0^\infty e^{rx}G(x)dx \quad \text{where} \quad G(x) = \frac{1-F(x)}{\mu}, \quad x > 0 \tag{3.18}$$

Also from Equations (3.15), and (3.18), we have

$$\int_0^\infty e^{rx}G(x)dx = \frac{(M_X(r) - 1)}{\mu r} \tag{3.19}$$

so that replacing $M_X(r)$ by $1 + (1+\theta)$, gives Equation (3.18).

3.4.1 Lundberg Inequality

Theorem 1. *Suppose that $r > 0$ is a solution to Equation (3.15). Then the probability of ruin satisfies*

$$\Psi(u) \leq e^{-ru}, \quad u \geq 0 \tag{3.20}$$

Proof. See Appendix A3. □

This is an important result since it presents an upper bound on the ruin probability on an insurance portfolio. Nevertheless, the condition does not exist for many practical scenarios i.e. the adjustment coefficient does not exist for many distributions especially the heavy tail distributions.

3.5 Cramér-Lundberg Approximation to the probability of Ruin

We state the key Renewal theorem which is of importance in the approximation of Cramér-Lundberg ruin probability.

Let $B(y)$ satisfy the following equation

$$B(y) = C(y) + \int_0^y B(y-x)dA(x) \tag{3.21}$$

Where, C is known, A is a given distribution function, $\lim_{y\longrightarrow\infty} B(y) = \lim_{y\longrightarrow\infty} C(y) + \frac{1}{\mu(A)}\int_0^\infty C(y)dy$, and $\mu(A) = \mathbb{E}(X(A)),\ 0 < \mu(A) < \infty$.

Proof. The proof can be found in (Feller,) and (Schmidli, 2017). □

We then assume that $\theta > 0$, or $c > \lambda\mu$. Consider the following equation

$$\Phi(u) = \Phi(0) + \frac{\lambda}{c}\int_0^u \Phi(u-x)[1-F(x)]dx$$

from Equation (3.8), we rewrite

$$1 - \Psi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u (1-\Psi(u-x))\,[1-F(x)]\,dx$$

$$\Psi(u) = \frac{\lambda}{c}\left(\mu - \int_0^u [1-F(x)]dx + \int_0^u \Psi(u-x)[1-F(x)]dx\right)$$

Next using the definition of the expected value, $\mu = \int_0^\infty [1-F(x)]dx$,

$$\Psi(u) = \frac{\lambda}{c}\left(\int_u^\infty [1-F(x)]dx + \int_0^u \Psi(u-x)[1-F(x)]dx\right) \tag{3.22}$$

To solve Equation (3.22) which is a renewal type equation. This equation resembles renewal one but

$$\int_0^\infty \frac{\lambda}{c}[1-F(x)]dx = \frac{\lambda\mu}{c} < 1$$

is not a probability distribution. For this function to be regarded as a density function of a distribution A, the integral must be equal to 1, which is a very important axiom of probability.

According to (Feller,), both sides of equation (3.22) is multiplied by e^{ru}, $r > 0$ and r is a properly chosen constant (Lundberg exponent) i.e.

$$\frac{\lambda}{c}\int_0^\infty e^{rx}[1 - F(x)]dx = 1$$

and that

$$e^{ru}\Psi(u) = \frac{\lambda}{c}\left(\int_u^\infty e^{ru}[1 - F(x)]dx + \int_0^u e^{r(u-x)}\Psi(u - x)[1 - F(x)]dx\right)$$

Which is now a proper renewal equation since

$$\int_0^\infty \frac{\lambda}{c}[1-F(x)]e^{ru}dx = \frac{\lambda}{c}\int_0^\infty\int_x^\infty e^{rx}dF(y)dx = \frac{\lambda}{c}\int_0^\infty\int_x^\infty e^{rx}dF(y)dx = \frac{\lambda(M_Y(r) - 1)}{cr} = 1$$

It then follows that

$$\lim_{t\longrightarrow\infty} e^{ru}\Psi(u) = \frac{C_1}{C_2} \tag{3.23}$$

where

$$C_1 = \frac{\lambda}{c}\int_0^\infty e^{ru}\int_u^\infty [1 - F(x)]dxdu \tag{3.24}$$

and

$$C_2 = \frac{\lambda}{c}\int_0^\infty xe^{rx}[1 - F(x)]dx \tag{3.25}$$

provided finite positive numbers r, C_1,and C_2 exist.

We then solve for C_1 as follows; we first change the order of integration in Equation (3.24) as indicated

$$C_1 = \frac{\lambda}{c}\int_0^\infty [1 - F(x)]\int_0^x e^{ru}dudx$$

since $\int_0^x e^{ru}du = \frac{e^{rx}}{r} - \frac{1}{r}$ and using the relationships $\mu = \int_0^\infty [1 - F(x)]dx$ and $\int_0^\infty e^{rx}[1 - F(x)]dx = \frac{c}{\lambda}$, we obtain

$$C_1 = \frac{\lambda}{rc}\int_0^\infty e^{rx}[1 - F(x)]dx - \frac{\lambda}{rc}\int_0^\infty [1 - F(x)]dx = \frac{1}{r} - \frac{\lambda\mu}{rc} = \frac{\theta}{r(1 + \theta)}$$

Next we obtain C_2, by first introducing the function

$$M_X(r) = \int_0^\infty e^{rx}dF(x) - 1$$

Where $M_X(r)$ is the M.G.F of the severity r.v X. Then using $\int_0^\infty e^{rx}[1-F(x)]dx = 1$, and also using integration by parts

$$\frac{c}{\lambda} = \int_0^\infty e^{rx}[1-F(x)]dx = -\frac{1}{r} + \frac{1}{r}\int_0^\infty e^{rx}dF(x) = \frac{M_X(r)-1}{r}$$

and we see that

$$M_X(r) = \frac{cr}{\lambda} + 1 \tag{3.26}$$

Note that, $M'_X(r) = \int_0^\infty xe^{rx}dF(x)$. Now using $\int xe^{rx} = \left(\frac{x}{r} - \frac{1}{r^2}\right)e^{rx} + C_3$ (anti-derivative of xe^{rx}), integrated by parts.

$$C_2 = \frac{\lambda}{c}\int_0^\infty xe^{rx}[1-F(x)]dx = \frac{\lambda}{c}[1-F(x)]\left[\left(\frac{x}{r} - \frac{1}{r^2}\right)e^{rx}\right]_0^\infty + \frac{\lambda}{c}\int_0^\infty \left(\frac{x}{r} - \frac{1}{r^2}\right)e^{rx}dF(x)$$

Where using integrable functions

$$\lim_{x\longrightarrow\infty}[1-F(x)]e^{rx} = 0$$

, and

$$\lim_{x\longrightarrow\infty}[1-F(x)]z\cdot e^{rx} = 0$$

$$C_2 = \frac{\lambda}{c}\cdot\frac{1}{r^2} + \frac{\lambda}{c}\int_0^\infty \left(\frac{x}{r} - \frac{1}{r^2}\right)e^{rx}dF(x)$$

Next, using expressions for $M_X(r) = \frac{cr}{\lambda} + 1$ and $M'_X(r) = \int_0^\infty xe^{rx}dF(x)$

$$C_2 = \frac{\lambda}{c}\left(\frac{1}{r^2} + \frac{M'_X(r)}{r} - \frac{M_X(r)}{r^2}\right) = \frac{\lambda}{c}\left(\frac{1}{r^2} + \frac{M'_X(r)}{r} - \frac{\frac{cr}{\lambda}+1}{r^2}\right)$$

$$C_2 = \frac{\lambda}{c}\left(\frac{1}{r^2} + \frac{M'_X(r)}{r} - \frac{M_X(r)}{r^2}\right) = \frac{\lambda}{c}\left(\frac{1}{r^2} + \frac{M'_X(r)}{r} - \frac{cr}{\lambda r^2} - \frac{1}{r^2}\right)$$

$$C_2 = \frac{\lambda}{c}\left(\frac{M'_X(r)}{r} - \frac{c}{\lambda r}\right) = \frac{\lambda\mu}{c}\frac{1}{r}\frac{1}{\mu}\left(M'_X(r) - \frac{c}{\lambda}\right)$$

Thus

$$\lim_{u\longrightarrow\infty} e^{ru}\Psi(u) = \frac{\frac{\theta}{r(1+\theta)}}{\frac{\lambda}{c}\left(\frac{M'_X(r)}{r} - \frac{c}{\lambda r}\right)} = \frac{\theta\mu}{M'_X(r) - \frac{c}{\lambda}} \tag{3.27}$$

Therefore, asymptotically we have

$$\Psi(u) \sim Ce^{-ru}, \quad \text{as} \quad u \longrightarrow \infty \tag{3.28}$$

where

$$C = \frac{\theta\mu}{M'_X(r) - \mu(1+\theta)} \tag{3.29}$$

(The Exponential Distribution): If $1 - F(x) = \exp\left(-\frac{x}{\mu}\right), x \geq 0$. The asymptotic classical ruin probability is given by

$$\Psi(u) \sim \frac{1}{1+\theta} \exp\left(-\frac{\theta u}{\mu(1+\theta)}\right), \quad \text{as} \quad u \longrightarrow \infty$$

Proof. using $r = \frac{\theta}{\mu(1+\theta)}$ and $M_X(t) = (1-\mu t)^{-1} \Longleftrightarrow M'_X(t) = \mu(1-\mu t)^{-2}$

$$M'_X(t) = \mu(1-\mu t)^{-2} = \mu[1-\theta(1+\theta)^{-1}]^{-2} = \mu(1+\theta)^2$$

Thus using Equations (3.28) and (3.29), we obtain $C = \frac{1}{1+\theta}$ which completes the proof. □

3.6 Heavy-tailed Claims Distributions

These are claims in which ”the actuary has to go and see one of the chief members of the company” [11]. Such claims may result in the technical ruin of an insurance company. For this reason, it is always observed that good models for claims distributions encompass heavy tail, i.e. claims distributions with tails which do not have an exponential factor (not bounded exponentially). In practice, among the heavy-tailed distribution, the most commonly used belong to a class of sub-exponential. It is important to note that this section follows closely the description from (Deelstra et al., 2014), (Schmidli, 2017) ,(Embrechts et al., 2013), (Wei, 2009), and (Rolski et al., 2009).

3.6.1 Sub-exponential Claims

Sub-exponential class denoted by $\mathcal{S}$ is an important class of heavy-tailed distributions. We define, a distribution F on $\mathbb{R}_{[0,\infty)}$ to be sub-exponential if

$$\lim_{x \longrightarrow \infty} \frac{\bar{F^{n*}}(x)}{\bar{F}(x)} = n, \quad \text{for} \quad n \geq 2 \tag{3.30}$$

where $\bar{F} = 1 - F(x) > 0 \ \forall \, x \geq 0$ is the tail of the distribution, and $\bar{F^{n*}}(x) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \leq x\right)$ is the $n-$fold convolution of F.

Given any two $i.i.d$ random variables X_1 and X_2 with common distribution function F, the convolution F^{2*} is defined by

$$\mathbb{P}[X_1 + X_2 \leq x] = F^{2*}(x) = \int_{-\infty}^{\infty} F(x-y)dF(y)$$

Therefore, a distribution F on the positive half-line is sub-exponential if

$$\bar{F^{2*}}(x) \sim 2\bar{F} \quad \text{as} \quad x \longrightarrow \infty$$

Then for $n-$fold convolutions is defined the same way, such that for any $n \geq 1$

$$\bar{F^{n*}}(x) \sim n\bar{F} \quad \text{as} \quad x \longrightarrow \infty$$

Thus $\bar{F^{n*}}(x)$ is the tail of the distribution of the maxima of n random variables, $X_1, ..., X_n$ and due to

$$(1-a)^n \sim 1 - na \quad \text{where} \quad a \longrightarrow 0$$

we obtain

$$\mathbb{P}[\max(X_1, .., X_n) > x] = 1 - F^n(x) = 1 - \left(1 - \bar{F}(x)\right)^n \sim n\bar{F}(x)$$

Finally, from the principle of single big jump for the sum of r.v's $X_1, ..., X_n$ with common distribution function, assuming independence of the variables,

$$\mathbb{P}\left(S_n = X_1 + X_2 + \cdots + X_n > x\right) \sim \mathbb{P}\left(M_n = \max(X_1, X_2, ..., X_n) > x\right) \quad \text{as} \quad x \longrightarrow \infty$$

The following are some of the important properties of the sub-exponential distribution (Embrechts et al., 2013), and (Kyprianou, 2013).

(i) $F \in \mathcal{S}$,

(ii) $1 - \Psi(u) = \Phi(u) \in \mathcal{S}$ and,

(iii) $\lim_{u \longrightarrow \infty} \frac{\Psi(u)}{1-F(u)} = \frac{1}{\theta}$

Proof. See (Schmidli, 2017), (Embrechts et al., 2013), and (Kyprianou, 2013) for proofs. □

3.6.2 Regularly Varying Tails

Definition 6. *A function $f(x) > 0$ is a regularly varying at ∞ if $\exists\ \alpha \in \mathbb{R}$ such that*

$$\lim_{x \longrightarrow \infty} \frac{f(t.x)}{f(x)} = t^{\alpha} \quad \forall\ t > 0 \tag{3.31}$$

Pareto distributions are regularly varying as opposed to the log-normal distribution. Indeed from the p.d.f of the latter distribution, we have that

$$\lim_{x \longrightarrow \infty} \frac{f(t.x)}{f(x)} = \lim_{x \longrightarrow \infty} \frac{1}{t} e^{-\frac{(\ln t)^2}{2\sigma^2}} e^{-\ln t \frac{(\ln x - \mu)}{\sigma^2}} = \begin{cases} 0 & t > 1 \\ 1 & t = 1 \\ \infty & t < 1 \end{cases}$$

A rapidly decreasing function at infinity results from this limit trait resulting to both Pareto and Log-normal distributions exhibiting quantitatively unique behaviours in their upper tails, (Kaas et al., 2008) and (Embrechts et al., 2013).

We have discussed the methods which are applied in the next chapter. These methods have aided in the formulation of the risk model in the present study. For example, the sub-exponential distribution discussed will help in the study of heavy tailed distribution and consequently, comparison between the convergence of the tails.

Chapter 4

Ruin Probabilities Under Financial Constraints

4.1 Introduction

This chapter constitutes the main part of this thesis. We state the theorems for two of the main results in the present thesis. We then outline the modified classical surplus process, in which it is assumed that the effects of real rate of interest only affects the premium inflow and not the claims outgo. It is also assumed that ruin does not occur as a result of mismanagement by the insurers. In this chapter, we outline the formulation of our surplus process in the presence of inflation and interest rate using the Fisher relation. The non-ruin probability in the presence of the two economic factors is hence obtained. Finally, the first main results are achieved by invoking the Albrecher and Hipp loss-carried-forward tax scheme as outlined in (Albrecher and Hipp, 2007) to allow us to include all the financial constraints in an approximation of our ruin probability, thus completing the proof of our first main result. The second main result then follows.

4.2 The Model Formulation

4.2.1 Fisher Relation for Real Force of Interest

Surplus Process in the Presence of Real rate of Interest The classical risk process is given by Equation (3.1). We modify the classical risk process to take into account the effects of interest rate and inflation. This is done by considering Fisher relation, which is defined by

$$\delta(M, i) = \frac{\log_e(1+M)}{\log_e(1+i)} \tag{4.1}$$

where $\delta(M, i)$, M, and i are the real force of interest, money rate of interest, and rate of inflation respectively.

4.2.2 Assumption for the model

Before outlining the model, the following assumptions are inevitable. First, the premium received for the present risk model is paid continuously at an invariant rate c, secondly, the company also receives investment interest with a constant real force of interest denoted by $\delta(M, i)$. The effects of inflationary conditions and the normal or money return on the total capital do not cancel out each other exactly.

4.2.3 Risk model in the Present Study

We modify Equation (3.1) and the risk reserve process outlined in (Klugman et al., 2012) to obtain the following model ((4.2)). Let $U_{\delta(\hat{M,i})}(t)$ be the reserve at some time t and from the above-mentioned assumptions,

$$dU_{\delta(\hat{M,i})}(t) = \left\{ c + dU_{\delta(\hat{M,i})}(t) \cdot \delta(M, i) \right\} dt - dS(t), \quad t \geq 0$$

$$U_{\delta(\hat{M,i})}(t) = ue^{\delta(M,i)t} + c \cdot \bar{s}_{\overline{t}|}^{\delta(M,i)} - \int_0^t dS(s), \quad t \geq 0 \tag{4.2}$$

Where $u = U(0)$, and

$$\bar{s}_{\overline{t}|}^{\delta(M,i)} = \int_0^t e^{\delta(M,i)s} ds = \begin{cases} t \ \text{ if } \ \delta(M,i) = 0 \\ \frac{e^{\delta(M,i)t}-1}{\delta(M,i)} \ \text{ if } \ \delta(M,i) > 0 \end{cases}$$

Now, discounting Equation (4.2), we obtain the following as the present value of the present model.

$$V_{\delta(M,i)}(t) = e^{-\delta(M,i)t} \hat{U_{\delta(M,i)}}(t) = e^{-\delta(M,i)t} \cdot u \cdot e^{\delta(M,i)t} + e^{-\delta(M,i)t} \cdot c \cdot \bar{s}_{\overline{t}|}^{\delta(M,i)} - e^{-\delta(M,i)t} \cdot \int_0^t dS(s), \ \ t \geq 0$$

But

$$e^{-\delta(M,i)t} \cdot \bar{s}_{\overline{t}|}^{\delta(M,i)} = \bar{a}_{\overline{t}|}^{\delta(M,i)} = \begin{cases} t \ \text{ if } \ \delta(M,i) = 0 \\ \frac{1-e^{\delta(M,i)t}}{\delta(M,i)} \ \text{ if } \ \delta(M,i) > 0 \end{cases}$$

Therefore,

$$V_{\delta(M,i)}(t) = u + c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)} - e^{-\delta(M,i)t} \cdot S(t), \quad t \geq 0 \tag{4.3}$$

From Equation (4.3), for a definite time $T > 0$, the definite time horizon probability of ruin is defined by

$$\Psi(u,t) = \mathbb{P}\left(V_{\delta(M,i)}(t) < 0 \quad \text{for some} \quad 0 \leq t \leq T\right)$$

$$= \mathbb{P}\left(\sup_{t\in[0,T]} \left[e^{-\delta(M,i)t} S(t) - c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}\right] > u\right)$$

Also, the indefinite time ruin probability is defined in this case by

$$\Psi(u) = \Psi(u,\infty) = \mathbb{P}\left(V_{\delta(M,i)}(t) < 0 \quad \text{for some} \quad t \geq 0\right)$$

4.3 Ruin Probability for the Exponentially distributed Claims

Theorem 2. *In the presence of all financial constraints, the surplus dependent ruin probability is expressed as*

$$\Psi_{\gamma,\delta(M,i)}(u) = 1 - \left[1 - \Psi_{\delta(M,i)}(u)\right]^{(1-\gamma)^{-1}} \tag{4.4}$$

where $\gamma \in [0,1)$ is the tax rate, $\Psi_{\delta(M,i)}(u)$ is the probability of ruin in the presence of inflation and interest rate, in which $\delta(M,i)$ is the real force of interest.

Proof. To prove the above theorem, the following are necessary,

Lemma 1. *Under equation (4.3) the premium loading factor is given by*

$$\theta_* = \frac{c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}}{e^{-\delta(M,i)t}\mu\lambda} - 1 \quad t > 0 \tag{4.5}$$

where $\theta_* > 0$ is the premium loading factor in the presence of real force of interest $\delta(M,i)$.

Proof. Using the expected value principle in Equation (4.3) as outlined in (Deelstra et al., 2014).

$$\mathbb{E}(V_{\delta(M,i)}(t)) = \mathbb{E}\left(u + c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)} - e^{-\delta(M,i)t}\sum_{k=0}^{N(t)} X_k\right)$$

$$\mathbb{E}(V_{\delta(M,i)}(t)) \geq u \Longleftrightarrow c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)} - e^{-\delta(M,i)t}\mu\lambda > 0$$

$$c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)} = (1+\theta_*)e^{-\delta(M,i)t}\mu\lambda$$

The result follows upon making θ_* the subject. □

We therefore prove Theorem (2) by first stating the following tax scheme. Albrecher and Hipp (2007), established the following loss-carried-forward tax scheme, a model that extended the classical risk model to incorporate taxation. The authors suggested that taxation is only payable if the insurance portfolio is in a "profitable situation". The authors obtained an important equation,

$$\Psi_\gamma(u) = 1 - [1 - \Psi_0(u)]^{(1-\gamma)^{-1}} \tag{4.6}$$

where $\Psi_\gamma(u)$ is the surplus-dependent probability of ruin in the presence of taxation, $\Psi_0(u)$ is the classical ruin probability, and $\gamma \in [0,1)$ is the rate of taxation. The reader is referred to (Albrecher and Hipp, 2007) for more details and corresponding proofs.

Thus invoking the said tax scheme as outlined in (Albrecher and Hipp, 2007), we obtain the following surplus and tax-dependent approximate ruin probability explicitly under the operation of all the financial constraints as follows,

$$\Psi_{\gamma,\delta(M,i)}(u) = 1 - \left[1 - \Psi_{\delta(M,i)}(u)\right]^{(1-\gamma)^{-1}} \tag{4.7}$$

which marks the end of the proof of Theorem (2). □

Example 1. *Suppose that the claim sizes exhibit an exponential distribution. The ruin probability in the presence of al financial constraints is given by Equation* (4.7) *where,*

$$\Psi_{\delta(M,i)}(u) = \frac{1}{1+\theta_*} \exp\left(-\frac{\theta_* u}{\mu(1+\theta_*)}\right) \tag{4.8}$$

is the corresponding ruin probability in the presence of inflation and constant interest rate.

Finally, asymptotic probability of ruin in the presence of all the financial constraints is given by

$$\Psi_{\gamma,\delta(M,i)}(u) \sim \frac{1}{1-\gamma}\Psi_{\delta(M,i)}(u), \quad u \longrightarrow \infty \tag{4.9}$$

as highlighted in (Albrecher and Hipp, 2007).

4.4 Ruin Probabilities for Claims with Sub-exponential Distribution

Theorem 3. *The probability of ruin for the claims with sub-exponential distribution in the presence of interest rate and inflation is given by*

$$\Psi_{\delta(M,i)}(u) \sim \frac{1}{\theta_*}\bar{F}(u) \quad as \quad u \longrightarrow \infty \tag{4.10}$$

and in the presence of all financial constraints(including tax), the ruin probability is thus given by

$$\Psi_{\gamma,\delta(M,i)}(u) = 1 - \left[1 - \Psi_{\delta(M,i)}(u)\right]^{(1-\gamma)^{-1}} \sim \frac{1}{1-\gamma}\Psi_{\delta(M,i)}(u) = \frac{\bar{F}(u)}{\theta_*(1-\gamma)} \tag{4.11}$$

where $\gamma \in [0,1)$ *and* $\delta(M,i)$ *are the rate of taxation, and force of interest respectively,* θ_* *is the premium loading factor in the presence of interest rate and rate of inflation.*

For a density function with a finite mean, μ,

$$F(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy \quad x > 0 \tag{4.12}$$

Proof. To prove the above theorem we first state without proof Lemma F.6 in Appendix of (Rolski et al., 2009) and Theorem 1.3 in (Kyprianou, 2013).

Lemma 2. *Let $F \in \mathcal{S}$. Then for any $\epsilon > 0$, there exist a $D \in \mathbb{R}$ such that*

$$\frac{1 - F^{n*}(x)}{1 - F(x)} \leq D(1+\epsilon)^n \quad \forall \quad x > 0 \quad \textit{and} \quad n \in \mathbb{N}$$

Proof. The proof of this theorem can be found in (Schmidli, 2017), pp. 219. □

Next we state the Pollaczek-Khintchine formula (Theorem 1.3 in (Kyprianou, 2013)) as a theorem below

It states that suppose $\frac{\lambda\mu}{c} < 1$. For all $u \geq 0$

$$\Phi(u) = \left(1 - \frac{\lambda\mu}{c}\right)\sum_{k\geq 0}\left(\frac{\lambda\mu}{c}\right)^k \eta^{*k}(u)$$

where $\eta(u) = \frac{1}{\mu}\int_0^u (1 - F(y)dy)\,,\ u \geq 0$ and for $k \geq 0$. It is understood that η^{*k} is the $k-$fold convolutions of η with the special understanding that

$$\eta^{*0}(du) = \delta_0(du)$$

The proof of this theorem is ignored and can be found in (Kyprianou, 2013).

From Equation (4.12), and choosing $\epsilon > 0$ such that $\lambda\mu e^{-\delta(M,i)t}(1+\epsilon) < c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}$

Recall that there exist a D such that

$$\frac{1 - F^{n*}(x)}{1 - F(x)} \leq D(1+\epsilon)^n \tag{4.13}$$

Now, from Pollaczek-Khintchine formula, we obtain

$$\frac{\Psi_{\delta(M,i)}(u)}{1 - F(u)} = \left(1 - \frac{e^{-\delta(M,i)t}\lambda\mu}{c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)\sum_{n=1}^{\infty}\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^n \frac{1 - F^{n*}(u)}{1 - F(u)}$$

$$\leq D\left(1-\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)\sum_{n=1}^{\infty}\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^{n}(1+\epsilon)^{n}<\infty$$

Interchanging sum and limits and recalling from (3.30) that

$$\lim_{u\longrightarrow\infty}\frac{1-F^{n*}(u)}{1-F(u)}=n$$

$$\lim_{u\longrightarrow\infty}\frac{\Psi_{i,r}(u)}{1-F(u)}=\left(1-\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)\sum_{n=1}^{\infty}n\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^{n}$$

$$=\left(1-\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^{n}$$

$$=\left(1-\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^{n}$$

$$=\sum_{m=1}^{\infty}\left(\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}\right)^{m}$$

this is a sum of infinite series so that from $s_{\infty}=\frac{a}{1-r}$, we obtain

$$\lim_{u\longrightarrow\infty}\frac{\Psi_{\delta(M,i)}(u)}{1-F(u)}=\frac{\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}}{1-\frac{e^{-\delta(M,i)t}\lambda\mu}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}}$$

$$=\frac{\lambda\mu e^{-\delta(M,i)t}}{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}-\lambda\mu e^{-\delta(M,i)t}}\quad t>0$$

Recall that $\theta_{*}=\frac{c\cdot\bar{a}_{\overline{t}|}^{\delta(M,i)}}{\lambda\mu e^{-\delta(M,i)t}}-1\quad t>0.$

This implies that

$$\lim_{u\longrightarrow\infty}\frac{\Psi_{i,r}(u)}{1-F(u)}=\frac{1}{\theta_{*}}\Longleftrightarrow\lim_{u\longrightarrow\infty}\Psi_{\delta(M,i)}(u)=\frac{\bar{F}(u)}{\theta_{*}}$$

so that

$$\Psi_{\delta(M,i)}(u)\sim\frac{\bar{F}(u)}{\theta_{*}}\quad\text{as}\quad u\longrightarrow\infty\tag{4.14}$$

Now in the presence of all the financial constraints, we obtain

$$
\begin{aligned}
\Psi_{\gamma,\delta(M,i)}(u) &= 1 - \left[1 - \Psi_{\delta(M,i)}(u)\right]^{(1-\gamma)^{-1}} \\
&\sim \frac{1}{1-\gamma}\Psi_{\delta(M,i)}(u) \\
&= \frac{\bar{F}(u)}{\theta_*(1-\gamma)}
\end{aligned}
$$

Which completes the proof. □

4.5 Simulations and Numerical Results

4.5.1 Simulation and Numerical results for claims with Exponential Distribution

In this section, we test for the accuracy of the formulated model with that of the conventional Cramér-Lundberg model. The probability of ruin is computed first when the model takes into account the effects of interest rates and inflation, also in the case when all the economic factors are taken into account.

The following important assumption is necessary; there is an average of 20 claims per period i.e. $\lambda = 20$, the expected size of claims per period is 600 i.e. $\mu = 600$, the nominal rate of interest is approximately 13.66%, and the rate of inflation is 5.8% for the said period, the value of the $\delta(M,i)|_{t=1} = 0.0717$, the periodic premium inflow is taken to be $c = 13,200$ giving the premium loading factor for the classical risk process is assumed to be $\theta = 0.1$. The premium loading factor for the new model is, therefore, $\theta_* = 0.14039$. We simulate for three values of rates of taxation, viz, $\gamma_1 = 0.1, \gamma_2 = 0.2$, and $\gamma_3 = 0.3$ in Tables 5.1 and 5.2, and 5.3, respectively.

We restate the following formulas which are used to arrive at the values presented in the tables. For classical risk process, the probability of ruin for exponential distribution is given by

$$\Psi(u) = \frac{1}{1+\theta} \exp\left(-\frac{\theta u}{\mu(1+\theta)}\right), \quad \text{where} \quad \theta = \frac{c}{\lambda\mu} - 1$$

For our model (which considers inflation and interest rate) given by Equation (4.3), the ruin probability is given by

$$\Psi_{\delta(M,i)}(u) = \frac{1}{1+\theta_*} \exp\left(-\frac{\theta_* u}{\mu(1+\theta_*)}\right), \quad \text{where} \quad \theta_* = \frac{c \cdot \bar{a}_{\overline{t}|}^{\delta(M,i)}}{\lambda\mu e^{-\delta(M,i)t}} - 1 \quad t > 0$$

Finally, ruin probability for our model and in the presence of all financial constraints (when taxation is also considered) we have

$$\Psi_{\gamma,\delta(M,i)}(u) = 1 - \left[1 - \Psi_{\delta(M,i)}(u)\right]^{(1-\gamma)^{-1}}, \quad \text{where} \quad \gamma \in [0,1)$$

Table 4.1: Approximate ruin probabilities for exponentially distributed claims ($\gamma_1 = 0.1$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.02059	0.00519	0.00577
30,000	0.00965	0.00186	0.00207

Table 4.2: Approximate ruin probabilities for exponentially distributed claims ($\gamma_2 = 0.2$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.02059	0.00519	0.00648
30,000	0.00965	0.00186	0.00232

Table 4.3: Approximate ruin probabilities for exponentially distributed claims ($\gamma_3 = 0.3$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.02059	0.00519	0.00741
30,000	0.00965	0.00186	0.00266

As indicated in tables 4.1, 4.2, and 4.3, the probability of ruin decreases when the economic factors are included in the computation. For instance in table (4.1), an investor with an initial capital of 25,000 has a probability of ruin of 0.02059. It is clear that when the interest rates and inflation are included in the model, the probability of ruin decreases to 0.00519. When the taxation rate of 10% are included, the probability of ruin increases to 0.00577, indicating the effects of taxation in the probability of ruin. The trend is evident in Tables 4.2 and 4.3, except the different taxation rates have varying effects on the computation of the ruin probabilities. For the same ruin probability of 0.00519, the rates of taxation of 20% and 30% produces ruin probabilities of 0.00648 and 0.00741 respectively. The data used here comprises a simulation of values of u from 0 to 30,000 in an interval of 100. Appropriate R codes and commands are used to accomplish this and the following observations are made. For exponential distribution, all the curves emanate from closely the same point and then diverges as u gradually increases with exponential decay, then ultimately converge as initial reserve increases. Graphs of all financial constraints with a taxation rate of 30 percent converges to zero faster as compared to that of the classical ruin model.

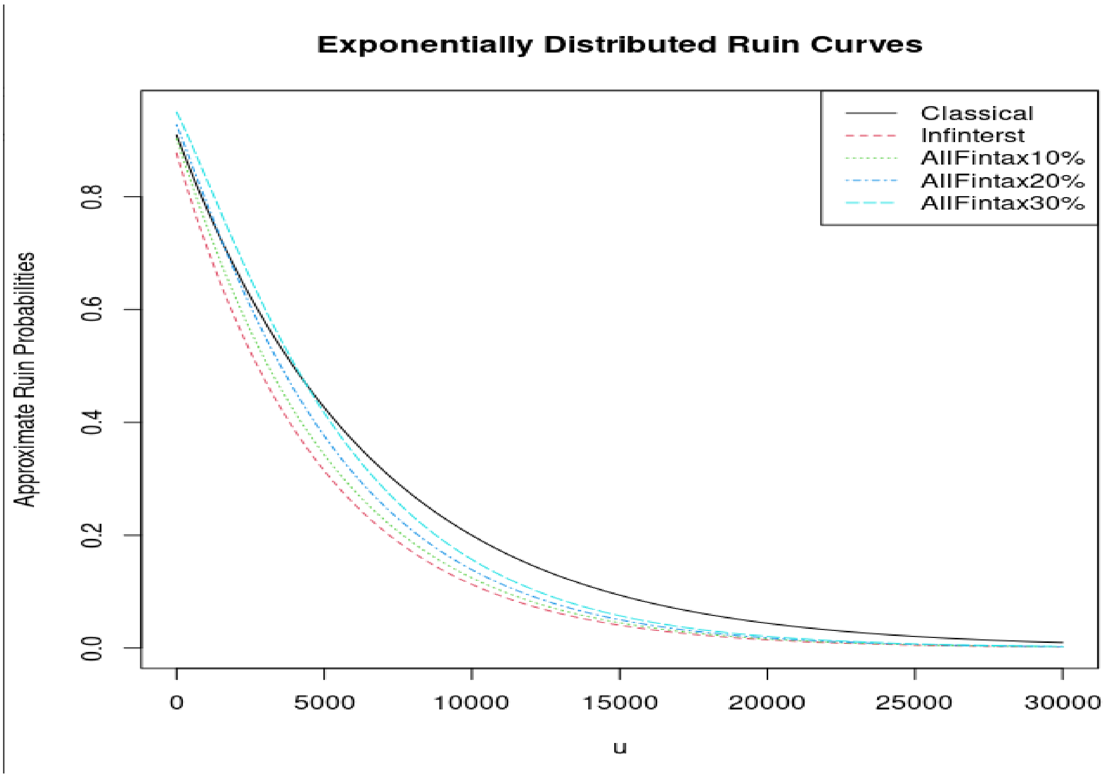

Figure 4.1: Exponentially Distributed Claims Curves

4.5.2 Numerical results and Simulation for Sub-exponential Claim Distribution

The following formulas are used in this section to get the exact values of the ruin probabilities for the sub-exponential distribution claims. For classical risk process, the ruin probability is given by

$$\Psi(u) \sim \frac{\bar{F}(u)}{\theta} \quad \text{as} \quad u \longrightarrow \infty, \quad \text{and} \quad \theta > 0$$

For our model, the corresponding ruin probabilities is represented as

$$\Psi_{\delta(M,i)}(u) \sim \frac{\bar{F}(u)}{\theta_*} \quad \text{as} \quad u \longrightarrow \infty, \quad \text{and} \quad \theta_* > 0$$

and

$$\Psi_{\gamma,\delta(M,i)}(u) \sim \frac{\bar{F}(u)}{\theta_*(1-\gamma)} \quad \text{as} \quad u \longrightarrow \infty, \quad \text{and} \quad \theta_* > 0$$

The assumptions made in this chapter for the constant remain as they are except some variable change for the computation of expected values of the Pareto and log-normal distribution since different parameter values are engaged.

The probability density function of a Pareto distribution is given by Equation (1.3), and its tail is given by Equation (1.4). In this section we assume that $\alpha = 2$, and $b = 600$ and the approximate ruin probabilities follow from the tables below.

Table 4.4: Approximate ruin probabilities for claims with Pareto distribution ($\gamma_1 = 0.1$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00549	0.00391	0.00352
30,000	0.00384	0.00274	0.00246

Table 4.5: Approximate ruin probabilities for claims with Pareto distribution ($\gamma_2 = 0.2$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00549	0.00391	0.00313
30,000	0.00384	0.00274	0.00219

Table 4.6: Approximate ruin probabilities for claims with Pareto distribution ($\gamma_3 = 0.3$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00549	0.00391	0.00274
30,000	0.00384	0.00274	0.00192

The tail of a log-normal distribution is given by Equation (1.6) and from the initial assumption its parameters include $\sigma = \sqrt{2}$, and $\mu = 5.4$.

For the Pareto distribution, the approximate ruin probabilities for the classical ruin model are the same. This probability is less than those obtained when the inflation and interest rates are included in the model. A slight increase is observed when the model takes into account all the financial constraints.

Table 4.7: Approximate ruin probabilities for claims with log-normal distribution ($\gamma_1 = 0.1$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00415	0.00296	0.00329
30,000	0.00259	0.00185	0.00205

The same is observed when dealing with the log-normal distribution. The difference between the probability of ruin in the classical model and the corresponding ones in the new model, specifically, when all the economic factors are taken into account is significantly smaller as evident in the tables. The difference is also very explicit when the comparison is further done to our model when only interest rates and inflation are taken into account.

Table 4.8: Approximate ruin probabilities for claims with log-normal distribution ($\gamma_2 = 0.2$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00415	0.00296	0.00370
30,000	0.00259	0.00185	0.00231

For the case of exponentially distributed claims, there is a positive difference between the probability of ruin in our model when other factors than taxation are taken into consideration. It is exactly opposite when we look at the cases of sub-exponential distributions. This implies that in such a case, the probability of ruin in the presence of all financial constraints is slightly higher as observed in Tables 4.7 and 4.8.

Table 4.9: Approximate ruin probabilities for claims with log-normal distribution ($\gamma_3 = 0.3$)

u	$\Psi(u)$	$\Psi_{\delta(M,i)}(u)$	$\Psi_{\gamma,\delta(M,i)}(u)$
25,000	0.00415	0.00296	0.00423
30,000	0.00259	0.00185	0.00264

These results are consistent with those obtained from simulation and numerical analysis from (Wei, 2009), especially for the sub-exponential distributions.

Finally we compare the thickness of the tails of Pareto and log-normal distributions. We calculate the relative values of the densities of both Pareto and Log-normal densities at the greatest end of the upper tail. Using the concept of the limiting density ratio, comparison between the two heavy tailed distributions, the presence of an exponential factor in the log-normal distribution results in a zero limiting density ratio, cementing the fact that the lighter-tail exhibited by a log-normal density in comparison with that of Pareto as represented in Figure (4.4). However with limited data, the tails of the Pareto and Log-normal distributions are indistinguishable as seen in the Figure, and this is due to the fact that both have regularly varying tails. The log-normal density goes to zero (decays at infinity) faster than Pareto

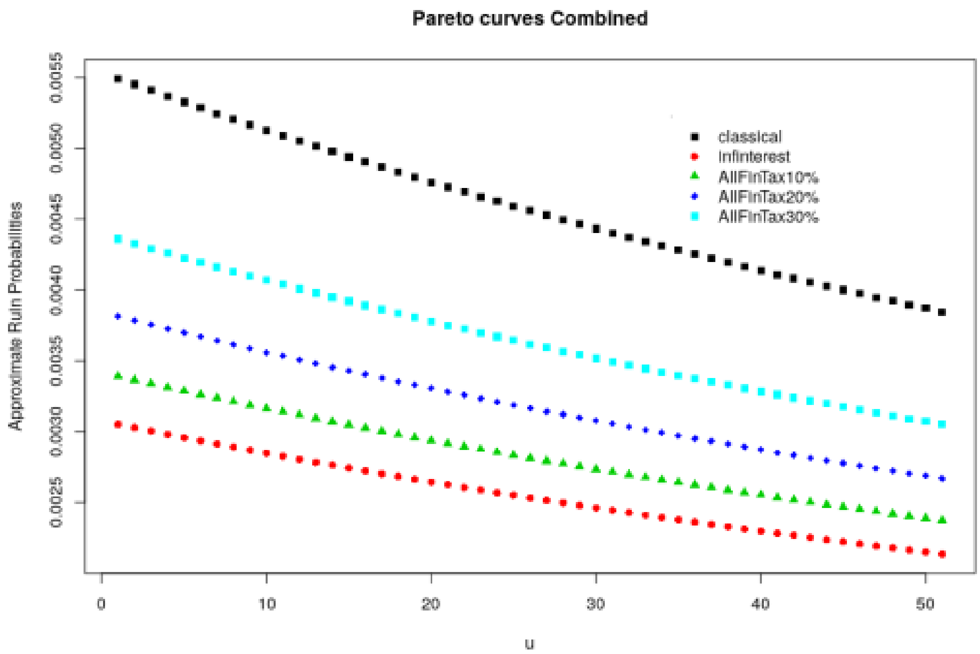

Figure 4.2: Ruin probabilities for Pareto Distributions

density even in the presence of all the three financial constraints. Such results are consistent with those from classical risk process as outlined in (Teugels, 1982) since log-normal density exhibit an exponential factor, resulting in a lighter tails.

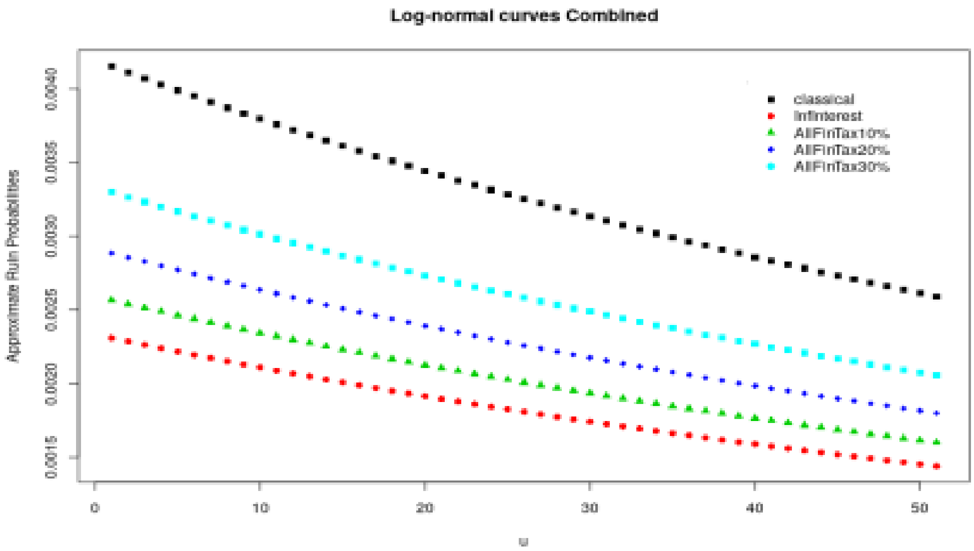

Figure 4.3: Ruin Probabilities for log-normal distributions

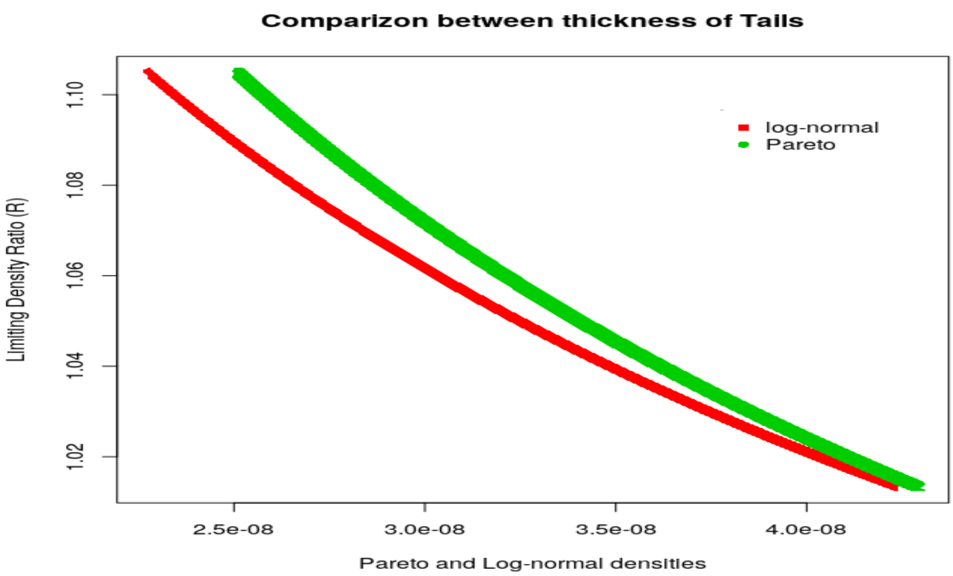

Figure 4.4: Comparison between Log-normal and Pareto Convergence

Chapter 5

Conclusions and Recommendations

This chapter covers the following; summary of findings, conclusions which are drawn, recommendations, and areas of further research. The conclusions made are based on simulations and numerical analysis, discussions, and results from the previous chapter.

5.1 Summary of the reseach findings

Based on the results of the simulation and numerical analysis, the model developed gives better approximations to the ruin probabilities. Claims with Pareto distribution in the present model give smaller approximate ruin probabilities as compared to those of the classical risk process. Further, it is observed that approximate ruin probabilities in the present model decrease as taxation increases. For the exponentially distributed claims, the approximate ruin probabilities for the classical risk process are greater than those obtained when our model is applied. Further, an increase in taxation rates results in increased ruin probabilities for a given initial reserve. The same trend is exhibited by the claims with a log-normal distribution. This is due to the presence of an exponential factor in the log-normal density.

5.2 Conclusions

In conclusion, the model developed in this thesis gives a better approximation to the ruin probabilities in comparison to those of classical ruin model both graphically by simulation and analytically by the exact formula. This is observed right away when financial factors such as inflation and interest are included in the classical risk model to come up with our ruin model. The results show greater improvements upon the inclusion of taxation so that the model takes into account all three financial factors (inflation rate, taxation, and constant rate of interest). Such results are supported further from numerical result and simulations of the values of initial reserves for the sub-exponential distribution, these results are very much consistent with the numerical analysis from (Wei, 2009), who came up with a model to approximate ruin probabilities in the presence of only interest rate and taxation. If log-normal and Pareto distributions are compared in terms of limiting density ratios (a way for measuring tail weight), the presence of exponential factor in log-normal density results in a limiting density of zero confirming that it exhibit a lighter tail and thus converges at a faster rate, thus this make it normally a good model for most non-life insurance.

5.3 Recommendations

Our model gives better and reliable approximate ruin probabilities in comparison to the classical ruin model, thus the study recommends that the insurance industry approximate their ruin probabilities with the model developed in the present research since it takes into consideration the three important economic factors which includes rate of taxation and real rate of interest. The present research was based on deterministic and particularly constant rates of inflation and interest. Similar research can be done when stochastic rates of interest and inflation are taken into consideration. Also claim count processes can be modelled with other processes such as non-homogeneous Poisson process, Cox process, pure renewal process, normal distribution, or any other suitable process in the presence of all the three economic factors discussed in the present research.

References

Albrecher, H. and Hipp, C. (2007). Lundberg's risk process with tax. *Blätter der DGVFM*, 28(1):13–28.

Albrecher, H., Renaud, J.-F., and Zhou, X. (2008). A lévy insurance risk process with tax. *Journal of Applied Probability*, 45(2):363–375.

Bohnert, A., Gatzert, N., and Kolb, A. (2016). Assessing inflation risk in non-life insurance. *Insurance: Mathematics and Economics*, 66:86–96.

Cai, J. (2007). On the time value of absolute ruin with debit interest. *Advances in Applied Probability*, 39(2):343–359.

Cheung, E. C. and Landriault, D. (2012). On a risk model with surplus-dependent premium and tax rates. *Methodology and Computing in Applied Probability*, 14(2):233–251.

Cramér, H. (1930). On the mathematical theory of risk, skandia jubilee volume, stockholm. *Cited on*, page 61.

Cramer, H. (1955). *Collective risk theory.* Skandia ins. Company.

David, C. (2016). *Insurance risk and ruin.* Cambridge University Press.

Debicki, K., Hashorva, E., and Ji, L. (2015). Gaussian risk models with financial constraints. *Scandinavian Actuarial Journal*, 2015(6):469–481.

Deelstra, G., Plantin, G., et al. (2014). *Risk theory and reinsurance.* Springer.

Embrechts, P., Klüppelberg, C., and Mikosch, T. (2013). *Modelling extremal events: for insurance and finance*, volume 33. Springer Science & Business Media.

Feller, W. An introduction to probability theory and its applications. *1957*.

Grandell, J. (1977). A class of approximations of ruin probabilities. *Scandinavian Actuarial Journal*, 1977(sup1):37–52.

Grandell, J. (2012). *Aspects of risk theory*. Springer Science & Business Media.

Gupta, S. and Kapoor, V. (1997). Fundamentals of mathematical statistics, ninth extensively revised edition, sultan chand & sons.

Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. (2008). *Modern actuarial risk theory: using R*, volume 128. Springer Science & Business Media.

Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss models: from data to decisions*, volume 715. John Wiley & Sons.

Konstantinides, D. G., Ng, K. W., and Tang, Q. (2010). The probabilities of absolute ruin in the renewal risk model with constant force of interest. *Journal of Applied Probability*, 47(2):323–334.

Kyprianou, A. E. (2013). *Gerber–Shiu risk theory*. Springer Science & Business Media.

Marshall, A. W. and Olkin, I. (2007). *Life distributions*, volume 13. Springer.

Ni, W. (2015). *Bonus-malus in insurance portfolios*. PhD thesis, University of Liverpool.

Promislow, S. D. (2006). Fundamentals of actuarial mathematics.

Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (2009). *Stochastic processes for insurance and finance*, volume 505. John Wiley & Sons.

Schmidli, H. (1995). Cramér-lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance: mathematics and economics*, 16(2):135–149.

Schmidli, H. (2017). *Risk theory*. Springer.

Shreve, S. E. (2004). *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media.

Sundt, B. and Teugels, J. L. (1995). Ruin estimates under interest force. *Insurance: Mathematics and Economics*, 16(1):7–22.

Taylor, G. C. (1979). Probability of ruin under inflationary conditions or under experience rating. *ASTIN Bulletin: The Journal of the IAA*, 10(2):149–162.

Teugels, J. L. (1982). Large claims in insurance mathematics. *ASTIN Bulletin: The Journal of the IAA*, 13(2):81–88.

Trench, W. F. (2013). Introduction to real analysis.

Wang, K., Chen, L., Yang, Y., and Gao, M. (2018). The finite-time ruin probability of a risk model with stochastic return and brownian perturbation. *Japan Journal of Industrial and Applied Mathematics*, 35(3):1173–1189.

Wang, S., Zhang, C., and Wang, G. (2010). A constant interest risk model with tax payments. *Stochastic models*, 26(3):384–398.

Wei, L. (2009). Ruin probability in the presence of interest earnings and tax payments. *Insurance: Mathematics and Economics*, 45(1):133–138.

Zhu, J. and Yang, H. (2008). Estimates for the absolute ruin probability in the compound poisson risk model with credit and debit interest. *Journal of Applied Probability*, 45(3):818–830.

Appendix a

Appendix

I Some Selected Proofs

a1: If $X_k \sim$ Exp $\left(\frac{1}{\mu}\right)$, then

$$\Psi(u) = \frac{1}{(1+\theta)} \exp\left(-\frac{\theta u}{\mu(1+\theta)}\right) \tag{a.1}$$

Proof. Consider the following equation

$$\Phi(u)' = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_0^u \Phi(u-x)dF(x)$$

$$u - x = v \implies dx = -dv$$

$$\Phi(u)' = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{\mu c}\int_0^u \Phi(x)e^{-\frac{(u-x)}{\mu}}dx$$

$$\Phi(u)' = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{\mu c}e^{\frac{-u}{\mu}}\int_0^u \Phi(x)e^{\frac{x}{\mu}}dx$$

$$\Phi(u)'' = \frac{\lambda}{c}\Phi(u)' + \frac{1}{\mu}\left(\frac{\lambda}{c}\Phi(u) - \Phi(u)'\right) - \frac{\lambda}{c\mu}\Phi(u)$$

$$\Phi(u)'' = \left(\frac{\lambda}{c} - \frac{1}{\mu}\right)\Phi(u)' = -\frac{\theta}{\mu(1+\theta)}\Phi(u)'$$

$$\Phi(u)'' = -\frac{\theta u}{\mu(1+\theta)}\Phi(u)' \implies \frac{\Phi(u)''}{\Phi(u)'} = -\frac{\theta u}{\mu(1+\theta)} = \ln\left(\Phi(u)'\right)'$$

$$\ln\Phi(u)' = \frac{-\theta u}{\mu(1+\theta)} + A_1$$

$$\Phi(u)' = A_2 \exp\left(\frac{-\theta u}{\mu(1+\theta)}\right) \implies \Phi(u) = A_3 \exp\left(\frac{-\theta u}{\mu(1+\theta)}\right) + A_4$$

From

$$\Phi(\infty) = 1 \implies A_4 = 1, \quad \text{and} \quad \Phi(0) = 1 - \frac{1}{(1+\theta)} \quad \implies A_3 = -\frac{1}{(1+\theta)}$$

Hence

$$\Phi(u) = 1 - \frac{1}{1+\theta} \exp\left(\frac{-\theta u}{\mu(1+\theta)}\right)$$

Therefore,

$$\Psi(u) = \frac{1}{1+\theta} \exp\left(\frac{-\theta u}{\mu(1+\theta)}\right)$$

□

a2: Let T_1 be the time of the first claim $X(T_1) = cT_1 - X_1$. At time T_1, the risk process starts like again, with only difference that the initial capital is $cT_1 - X_1$. Conditioning upon T_1 and X_1 whose d.f's are $F_{T_1}(t)$ and $F(x)$ respectively

$$\Phi(u) = \mathbb{E}\left[\Phi(u)|T_1, X_1\right] = \int_0^\infty \int_0^\infty \Phi(u+ct-x)F_{T_1}(t)dF(x) \quad \text{(a.2)}$$

Taking into account that ruin cannot occur in $(0, T_1)$, and also taking the distribution of T_1 to be exponential and replacing $dF_{T_1}(t) = \lambda e^{-\lambda t}dt$ and since large claims of size $x \geq u + ct$ implies ruin, we have

$$\Phi(u) = \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} \Phi(u + ct - x)dF(x)dt \qquad \text{(a.3)}$$

Proof. Let $y = u + ct \implies dt = dy/c$

$$\Phi(u) = \frac{\lambda}{c} e^{\lambda u/c} \int_u^\infty \lambda e^{-\lambda y/c} \int_0^y \Phi(y-x) dF(x) dy$$

Using product rule of differentiation and $\left[\int_0^u f(y)dy\right]' = f(y)$ we have

$$\Phi(u)' = \frac{\lambda}{c} \underbrace{\frac{\lambda}{c} e^{\lambda u/c} \int_u^\infty \lambda e^{-\lambda y/c} \int_0^y \Phi(y-x) dF(x) dy}_{\Phi(u)} - \frac{\lambda}{c} e^{\lambda u/c} e^{-\lambda u/c} \int_0^u \Phi(u-x) dF(x)$$

$$\Phi(u)' = \frac{\lambda}{c} \Phi(u) - \frac{\lambda}{c} \int_0^u \Phi(u-x) dF(x)$$

From

$dF(x) = -d(1 - F(x))$ and integration by parts

$$\Phi(u)' = \frac{\lambda}{c} \Phi(u) + \frac{\lambda}{c} \int_0^u \Phi(u-x) d(1 - F(x))$$

$$\Phi(u)' = \frac{\lambda}{c} \Phi(u) + \frac{\lambda}{c} \left[\Phi(0)(1 - F(u)) - \Phi(u)\right] + \frac{\lambda}{c} \int_0^u \Phi(u-x)' d(1 - F(x)) dx$$

$$\Phi(u)' = \frac{\lambda}{c} \Phi(0)(1 - F(u)) + \frac{\lambda}{c} \int_0^u \Phi(u-x)' d(1 - F(x)) dx$$

Integrating over $(0, t)$ yield.

$$\Phi(t) - \Phi(0) = \frac{\lambda}{c} \Phi(0) \int_0^t (1 - F(u)) + \frac{\lambda}{c} \int_0^t \int_0^u \Phi(u-x)' d(1 - F(x)) dx dt$$

Changing the order of integration in the double integral, we have

$$\Phi(u) - \Phi(0) = \frac{\lambda}{c} \Phi(0) \int_0^t (1 - F(u)) + \frac{\lambda}{c} \int_{x=0}^t (1 - F(x)) \int_{u=x}^t \Phi(u-x)' du dx$$

$$\Phi(t) - \Phi(0) = \frac{\lambda}{c} \Phi(0) \int_0^t (1 - F(u)) + \frac{\lambda}{c} \int_{x=0}^t (1 - F(x)) \left[\Phi(t-x) - \Phi(0)\right] dx$$

$$\Phi(t) - \Phi(0) = \frac{\lambda}{c} \int_0^t (1 - F(x)) \Phi(t-x) dx$$

hence,

$$\Phi(u) = \Phi(0) + \frac{\lambda}{c} \int_0^u (1 - F(x)) \Phi(u-x) dx$$

□

a3 Suppose that $r > 0$ is a solution to Equation (3.15). Then the probability of ruin satisfies

$$\Psi(u) \leq e^{-ru}, \quad u \geq 0 \tag{a.4}$$

The proof of this is sketched by induction as follows;

Proof. Ruin occurs before the n^{th} claim for $n = 0, 1, 2, ...$, with a probabiliy of $\Phi_n(u)$. Obviously $\Phi_0(u) = 0 = e^{-ru}$. We assume that $\Phi_n(u) \leq e^{-ru}$, we show that $\Phi_{n+1}(u) \leq e^{-ru}$. The inter-claim arrival time is exponentially distributed with density function $\lambda e^{-\lambda t}$. If the claim occurs at time $t > 0$, the surplus available at that time is $u + ct$, and ruin can only occur if the claim exceed this surplus with probability $1 - F(u + ct)$. If $x \in [0, u + ct]$ is the amount of claim in the event ruin does not occur upon the first claim, a srplus of $u + ct - x$ remains after the first claim. Thus,

$$\Phi_{n+1}(u) = \int_0^\infty \left[1 - F(u + ct) + \int_0^{u+ct} \Phi_n(u + ct - x) dF(x) \right] \lambda e^{-\lambda t}$$

$$\Phi_{n+1}(u) = \int_0^\infty \left[\int_{u+ct}^\infty F(x) + \int_0^{u+ct} \Phi_n(u + ct - x) dF(x) \right] \lambda e^{-\lambda t}$$

$$\Phi_{n+1}(u) \leq \int_0^\infty \left[\int_{u+ct}^\infty e^{r(u+ct-x)} dF(x) + \int_0^{u+ct} e^{-r(u+ct-x)} dF(x) \right] \lambda e^{-\lambda t}$$

Combining the two integrals, we obtain

$$\Phi_{n+1}(u) \leq \int_0^\infty \left[\int_0^\infty e^{r(u+ct-x)} dF(x) \right] \lambda e^{-\lambda t} = \lambda e^{-\lambda r} \int_0^\infty \int_0^\infty e^{-r(ct)} \left[e^{-rx} dF(x) \right] e^{-\lambda t} dt$$

$$\Phi_{n+1}(u) = \lambda e^{-\lambda r} \int_0^\infty e^{-r(\lambda + rc)t} \left[M_X(r) \right] dx = \lambda e^{-\lambda r} M_X(r) \int_0^\infty e^{-r(\lambda+rc)t} dt = \frac{\lambda M_X(r)}{\lambda + rc} e^{-ru}$$

From Equations (3.15) and (3.4) we obtain,

$$\lambda M_X(r) = \lambda \left[1 + (1 + \theta) r\mu \right] = \lambda + r(1 + \theta)\lambda\mu = \lambda + rc$$

so that $\Psi_{n+1}(u) \leq e^{-ru}$. Hence, $\Phi_n(u) \leq e^{-ru}$ for all n and finally

$$\lim_{t \longrightarrow \infty} \Psi_n(u) \leq e^{-ru} \tag{a.5}$$

□

II Codes Used

Codes for Exponentially Distributed Claims

```
> u=seq(0,3000000,100)# A sequence of numbers (initial capital + premiun inflow)
between 0 and 30,000 with an interval of 100

> nrp=function(u,theta,mu){
+ result=1/(1+theta)*exp(-(theta*u)/(mu*(1+theta)))
+ print(result)}

> y=nrp(u,0.1,600)# ruin probabilities for classical risk process

> y1=nrp(u,0.14039,600)# ruin probabilities for a risk process when
 force of inflation is taken into account

> y2=1-(1-y1)^(1/0.9) # ruin probabilities for all financial constraints
with tax rate of 10%

> y3=1-(1-y1)^(1/0.8)# ruin probabilities for all financial constraints
with tax rate of 20%

> y4=1-(1-y1)^(1/0.7)# ruin probabilities for all financial constraints
with tax rate of 30%

>matplot(u,cbind(y,y1,y2,y3,y4),xlab="u",ylab="Approximate Ruin Probabilities",
main="Exponentially Distributed Ruin Curves",type="l",col=c(1,2,3,4,5),
lty=c(1,2,3,4,5))

>legend(x="topright",legend=c("Classical","Infinterst","AllFintax10%",
```

```
"AllFintax20%","AllFintax30%"),col=c(1,2,3,4,5),lty=c(1,2,3,4,5))
```

b2: Claims with Pareto Distribution

```
> u=seq(0,3000,100)

> tpareto=function(u,alpha,b){
+ tail=(b/(b+u))^alpha
+ print(tail)}

> tpar=tpareto(u,2,600)
> y=tpar/0.1
> y1=tpar/0.14039
> y2=tpar/(.9*0.14039)
> y3=tpar/(.8*0.14039)
> y4=tpar/(.7*0.14039)

>matplot(u,cbind(y,y1,y2,y3,y4),xlab="u",ylab="Approximate ruin probabilities",
main="Pareto curves Combined",
type="l",col=c(1,2,3,4,5))

>legend(x="topright",legend=c("Classical","Infinterst","AllFintax10%",
"AllFintax20%","AllFintax30%"),col=c(1,2,3,4,5),lty=c(1,2,3,4,5))
```

b3: Codes Claims with Log-normal Distribution

```
>seq(0,3000,100)
> tlndist=1-plnorm(u,5.4,sqrt(2))

> y=tlndist/0.1
```

```
> y1=tlndist/0.14039
> y2=tlndist/(.9*0.14039)
> y3=tlndist/(.8*0.14039)
> y4=tlndist/(.7*0.14039)

>matplot(u,cbind(y,y1,y2,y3,y4),xlab="u",ylab="Approximate ruin probabilities",
main="Log-normal curves combined",
type="l",col=c(1,2,3,4,5))

>legend(x="topright",legend=c("Classical","Infinterst","AllFintax10%",
"AllFintax20%","AllFintax30%"),col=c(1,2,3,4,5),lty=c(1,2,3,4,5))
```

b4: Codes for Comparison of Tails of Pareto and Log-normal distributions

```
>x=1:1000

X1=dlnorm(x,5.4,sqrt(2))

> pdfparto=function(x,alpha,b){
+ result=(alpha*(b^alpha))/((b+x)^(alpha+1))
+ print(result)}

> X2=pdfparto(x,2,600)
>R=X1/X2
> matplot(cbind(X1,X2),R,xlab="Pareto and Log-normal densities",
ylab="Limiting Density Ratio (R)",main="Comparison between thickness of Tails")

>legend(x="topright",legend=c("Log-normal","Pareto"),col=c(4,5),lty=c(1,1))
```