MASENO UNIVERSITY
S.G.S.
ACC. NO.

CALCULATIONS FOR MUTATIONS OF QUIVERS WITH POTENTIAL

by

WAMBUA, VICTORIA MWIA

A thesis submitted in partial fulfilment of the requirements for the award of the degree of Master of Science in Pure Mathematics

Faculty of Science

MASENO UNIVERSITY

©2011

MASENO UNIVERSITY
S.G. S. LIBRARY

Abstract

From the work of Derksen, Weyman and Zelevinsky in [4], we get the definition for mutation of a quiver with potential. Stern in [18] describes some families of quivers corresponding to quivers with potential that can be mutated indefinitely. In this work, we use two of these families as our basic examples of study and find their appropriate potentials. Because the potentials get larger and larger with the mutations, we use two notations; first, a brief one and the second one providing useful information about the individual potentials. The results of this study offer explicit examples of quivers with potential which can be studied further.

Chapter 1

Introduction

A quiver Q consists of a set of vertices Q_0, a set of arrows Q_1 and two maps s and t, assigning to each arrow the starting and terminating vertices respectively. The main objects of our study are quivers with potential. A quiver with potential is a quiver Q together with the potential $\mathcal{S}$, which is a sum of cyclic paths of Q. The quivers in the families obtained and represented by tree diagrams by Stern in [18] are our main examples of quivers with potential. We obtain the appropriate potentials for some of the quivers in the two families given below. The quivers are related through mutation and we begin the mutations with the quiver A, at the centre of each tree diagram.

The following tree diagram represents quivers belonging to the P^2 family:

MASENO UNIVERSITY
S.G. S. LIBRARY

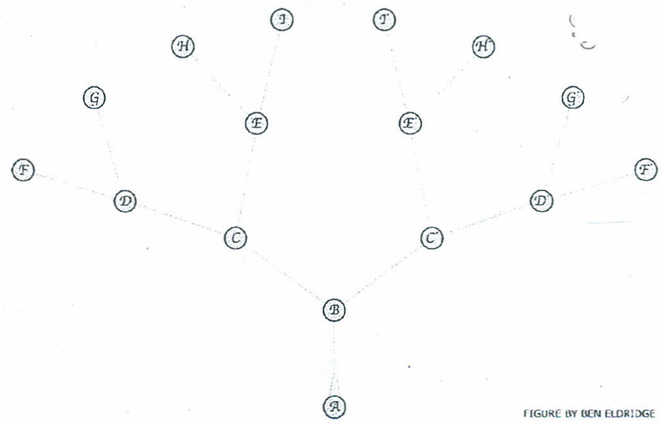


FIGURE BY BEN ELDRIDGE

The quivers corresponding to each of these points are:

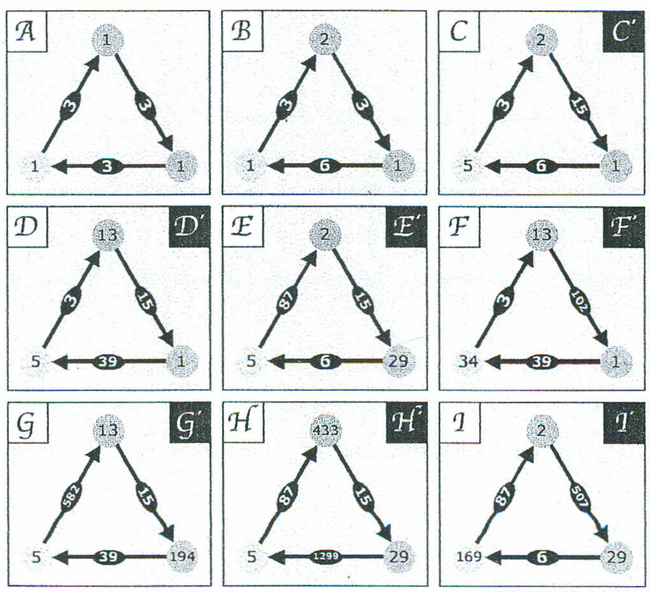
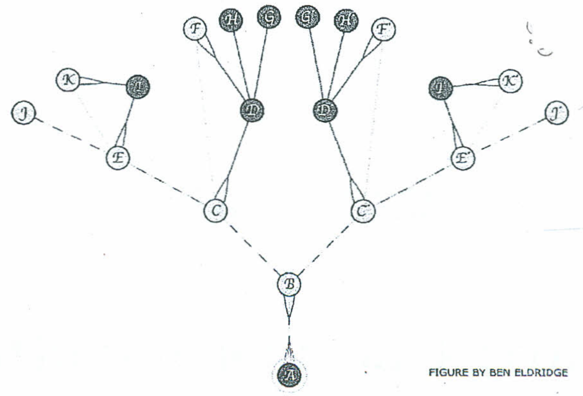


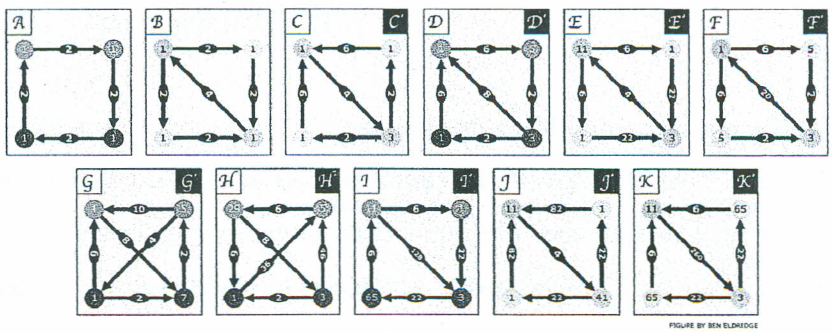
FIGURE BY BEN ELDRIDGE

In chapter 4 of this thesis, we calculate the potential for the quivers: B, C, D and E in section 4.2, then write them into sums in section 4.3.

The next tree diagram represents quivers belonging to the $P^1 \times P^1$ family.



The quivers corresponding to each of these points are:



In chapter 5, we calculate the potential for the quivers: B, C, D, E and F in section 5.2 and express each potential into sums in section 5.3.

Chapter 2

Mathematical background

2.1 Introduction

This chapter introduces the basic mathematical terms and concepts that are fundamental to the understanding of the entire thesis. We discuss the concept of quivers, giving some important examples and the notation used to represent them in this study in section 2.2. Paths and relations are dealt with in section 2.3 while potential, quivers with potential and cyclic derivatives are studied in section 2.4. A more algebraic approach to discussing these concepts is given in Obiero [13] and Owino [14].

Section 2.5 provides the sources of our study and in section 2.6, we state the problem area this study has attempted to solve. Section 2.7 gives the main objectives of our study while the main approach used in the study is stated in section 2.8. Finally, the significance of our study is covered in section 2.9.

2.2 Quivers and examples

In section 2.4, we introduce our main objects of study which are quivers with potential. A transformation called mutation is defined on these quivers with potential in chapter 3 and requires the quiver to have no loops or oriented 2-cycles. In this section we discuss some of these concepts as well as introduce the examples studied in chapters 4 and 5.

Definition 2.1. A **quiver** Q is a quadruple (Q_0, Q_1, s, t) where;

- Q_0 is a set of vertices,
- Q_1 is a set of arrows,
- $s : Q_1 \longrightarrow Q_0$ is a map taking an arrow to its starting vertex and
- $t : Q_1 \longrightarrow Q_0$ is a map taking an arrow to its terminating vertex.

The following examples introduce the notation used for the vertices and arrows in this study.

Example 2.2. Consider the quiver:

$$U \longrightarrow V$$

This quiver has vertices: $U, V \in Q_0$ and an arrow $a_{VU} \in Q_1$ with $s(a_{VU}) = U$ the starting vertex, and $t(a_{VU}) = V$ the terminating vertex.

Example 2.3. The quiver;

$$T \xrightarrow{2} U \xrightarrow{4} V$$

Example 2.6. The quiver

$$\begin{tikzcd} V \arrow[r, bend left] & W \arrow[l, bend left] \end{tikzcd}$$

is an example of a quiver with a 2-cycle. It consists of two vertices and two arrows oriented in opposite direction to each other.

Definition 2.7. A **finite quiver** is a quiver whose sets of vertices Q_0 and arrows Q_1 are both finite.

In this study, we deal with finite quivers. More specifically, we study two families of quivers related to P^2 and $P^1 \times P^1$ by Stern [18] in chapters 4 and 5 respectively. Given below are the central quivers in each case:

For the P^2 family, the central quiver is:

$$\begin{tikzcd} T \arrow[rr, "3"] & & U \arrow[dl, "3"] \\ & V \arrow[ul, "3"] & \end{tikzcd}$$

while in the $P^1 \times P^1$ family, the central quiver is:

$$\begin{tikzcd} T \arrow[r, "2"] & U \arrow[d, "2"] \\ V \arrow[u, "2"] & W \arrow[l, "2"] \end{tikzcd}$$

2.3 Paths and relations

The potential of quivers is defined in section 2.4 as a sum of the cyclic paths of the quiver. It is from these cyclic paths that we take the cyclic

MASENO UNIVERSITY
S.G. S. LIBRARY

derivatives discussed in the same section to obtain relations. In this section, we define paths, relations and a quiver with relations.

Definition 2.8. A **path** of length n of a quiver Q is a sequence of arrows:

$$a_{V_0V_1}a_{V_1V_2}\cdots\cdots\cdots a_{V_{n-1}V_n}$$

where the terminating vertex of any arrow is the starting vertex of the arrow to its right.

Example 2.9. The quiver:

$$\begin{tikzcd}
(T) \arrow[r, "b_{UT}"] & (U) \arrow[d, "c_{WU}"] \\
(V) & (W) \arrow[l, "d_{VW}"]
\end{tikzcd}$$

has a path: $d_{VW}c_{WU}b_{UT}$, of length 3.

Definition 2.10. A **trivial path** is a path of length zero.

For each vertex $V \in Q_0$, e_V denotes the trivial path which starts and terminates at the vertex V.

Example 2.11. The quiver:

$$T \xrightarrow{2} U \xrightarrow{4} V$$

has three trivial paths: e_T, e_U, e_V,

six paths of length one:

$$a_{UT}^{(1)}, a_{UT}^{(2)}, a_{VU}^{(1)}, a_{VU}^{(2)}, a_{VU}^{(3)}, a_{VU}^{(4)}$$

and eight paths of length two: $a^{(1)}_{VU}a^{(1)}_{UT}, a^{(2)}_{VU}a^{(1)}_{UT}, a^{(3)}_{VU}a^{(1)}_{UT}, a^{(4)}_{VU}a^{(1)}_{UT}, a^{(1)}_{VU}a^{(2)}_{UT}, a^{(2)}_{VU}a^{(2)}_{UT}, a^{(3)}_{VU}a^{(2)}_{UT}, a^{(4)}_{VU}a^{(2)}_{UT}$

Example 2.12. Consider the quiver below;

$$T \longrightarrow U \longleftarrow V \xleftarrow[2]{} W$$

This quiver has four trivial paths: e_T, e_U, e_V, e_W, four paths of length one: $a_{UT}, b_{UV}, c_{VW}, d_{VW}$, and two paths of length two: $b_{UV}c_{VW}$ and $b_{UV}d_{VW}$.

Definition 2.13. A path is **cyclic** if its starting and terminating vertices coincide. Otherwise, it is **acyclic**.

Definition 2.14. A cyclic path of length one is called a **loop**. A cyclic path of length n is an **n-cycle**.

Any cyclic path of length n can be written:

$$a_{V_nV_1}a_{V_1V_2}\dots a_{V_{n-1}V_n}$$

Definition 2.15. A cyclic path

$$a_{V_nV_1}a_{V_1V_2}\dots a_{V_{n-1}V_n}$$

is said to be **cyclically equivalent** to itself and any path:

$$a_{V_kV_{k+1}}\dots a_{V_nV_1}a_{V_1V_2}\dots a_{V_{k-1}V_k}$$

for $V_i := V_{i\,mod\,n}$.

Proposition 2.16. *Cyclic equivalence of cyclic paths is an equivalence relation.*

Proof. Let f and g be cyclic paths. Suppose f is cyclically equivalent to g, then either $g = f$ or $g \neq f$. If $g = f$, then f is cyclically equivalent to itself.

If$g \neq f$, then $f = a_{V_nV_1}a_{V_1V_2}\dots a_{V_{n-1}V_n}$ is cyclically equivalent to $g = a_{V_kV_{k+1}}\cdots a_{V_nV_1}a_{V_1V_2}\dots a_{V_{k-1}V_k}$ for $V_i := V_{i\,mod\,n}$.

With

$$U_i := \begin{cases} V_{k+i} & if \quad i \leq n-k \\ V_{k+i-n} & if \quad i > n-k \end{cases}$$

Then, $g = a_{U_nU_1}a_{U_1U_2}\cdots a_{U_{n-1}U_n}$ and $f = a_{U_{n-k}U_{n-k+1}}\cdots a_{U_nU_1}\cdots a_{U_{n-k-1}U_{n-k}}$ for $V_i := V_{i\,mod\,n}$. This implies g is cyclically equivalent to f.

Let $f = a_{V_nV_1}a_{V_1V_2}\dots a_{V_{n-1}V_n}$ be cyclically equivalent to $g = a_{V_kV_{k+1}}\cdots a_{V_nV_1}a_{V_1V_2}\cdots a_{V_{k-1}V_k}$, and $g = a_{U_nU_1}a_{U_1U_2}\cdots a_{U_{n-1}U_n}$ cyclically equivalent to $h = a_{U_lU_{l+1}}\cdots a_{U_nU_1}a_{U_1U_2}\cdots a_{U_{l-1}U_l}$

Then let

$$U_i := \begin{cases} V_{k+i} & if \quad i \leq n-k \\ V_{k+i-n} & if \quad i > n-k \end{cases}$$

$f = a_{V_nV_1}\cdots a_{V_{n-1}V_n}$ and $h = a_{V_{k+l}V_{k+l+1}}\cdots a_{V_nV_1}\cdots a_{V_{k+l-1}V_{k+l}}$ for $V_i := V_{i\,mod\,n}$. This implies f is cyclically equivalent to h. □

The cyclic paths dealt with in this study are defined to **cyclic equivalence**; we are thus not careful about where to start and end a cycle since whichever vertex we choose, we still have the same path.

Consider the following example:

Example 2.17. In the quiver:

$$\begin{tikzcd}
(T) \arrow[r, "b_{UT}"] & (U) \arrow[d, "c_{WU}"] \\
(V) \arrow[u, "a_{TV}"] & (W) \arrow[l, "d_{VW}"]
\end{tikzcd}$$

The cyclic path can be expressed in the following ways: $d_{VW}c_{WU}b_{UT}a_{TV}$, $b_{UT}a_{TV}d_{VW}c_{WU}$, $a_{TV}d_{VW}c_{WU}b_{UT}$, $c_{WU}b_{UT}a_{TV}d_{VW}$. However, cyclic equivalence says that all these expressions are one path and so any one of them suffices to represent the cycle in this quiver.

Definition 2.18. A **quiver with relations** is a quiver Q together with a set of relations. A relation is a linear combination of paths having the same starting point and the same end point equated to zero.

A path algebra is an algebra whose basis is all the paths of a quiver. The set of all relations imposed on a quiver Q generates an ideal with which we quotient a path algebra to obtain a path algebra of a quiver with relation, also called the quotient algebra. For more details on this, see Obiero [13] and Owino [14].

We discuss some special types of relations called potentials in the next section.

2.4 Quivers with potential

The relations introduced in the previous section can be obtained by taking cyclic derivatives on the potential of a quiver Q with respect to the single arrows in that quiver. This section thus gives the definitions for potential, a quiver with potential and an example illustrating cyclic derivatives.

Definition 2.19. A **quiver with potential** is a pair $(Q, \mathcal{S})$ consisting of a quiver Q and its potential $\mathcal{S}$. A **potential**, $\mathcal{S}$ of a quiver Q is a linear combination of cyclic paths of Q.

The quiver examples illustrating paths in section 2.3 have no potential since there are no cycles associated with them. However, the quivers in the two families introduced in section 2.2 have potentials. The potentials for the quivers at the center of each family is known from geometry and we give them as our examples of quivers with potential.

Example 2.20. Quiver:

$$\begin{tikzcd} T \arrow[rr, "3"] & & U \arrow[dl, "3"] \\ & V \arrow[ul, "3"] & \end{tikzcd}$$

with the potential:

$$\begin{aligned} \mathcal{S} \;=\; & a^{(1)}_{UT}a^{(2)}_{TV}a^{(3)}_{VU} - a^{(1)}_{UT}a^{(3)}_{TV}a^{(2)}_{VU} - a^{(2)}_{UT}a^{(1)}_{TV}a^{(3)}_{VU} + \\ & a^{(2)}_{UT}a^{(3)}_{TV}a^{(1)}_{VU} + a^{(3)}_{UT}a^{(1)}_{TV}a^{(2)}_{VU} - a^{(3)}_{UT}a^{(2)}_{TV}a^{(1)}_{VU} \end{aligned} \qquad (2.1)$$

and

Example 2.21. Quiver:

$$\begin{tikzcd} T \arrow[r, "2"] & U \arrow[d, "2"] \\ V \arrow[u, "2"] & W \arrow[l, "2"] \end{tikzcd}$$

with the potential:

$$\begin{aligned} \mathcal{S} = & a^{(1)}_{VW}a^{(1)}_{WU}a^{(1)}_{UT}a^{(1)}_{TV} + a^{(1)}_{VW}a^{(2)}_{WU}a^{(2)}_{UT}a^{(1)}_{TV} + \\ & a^{(2)}_{VW}a^{(1)}_{WU}a^{(1)}_{UT}a^{(2)}_{TV} + a^{(2)}_{VW}a^{(2)}_{WU}a^{(2)}_{UT}a^{(2)}_{TV} \end{aligned} \tag{2.2}$$

Calculation of the mutated potentials for some of the other quivers in these families is our main task in chapters 4 and 5.

Definition 2.22. The **cyclic derivative** on the potential $\mathcal{S}$, is the partial derivative δ_a of $\mathcal{S}$ with respect to an arrow $a \in Q_1$ on the potential. It is given by the equation below,

$$\delta_a(a_1....a_d) = \sum_{k=1}^{d} \delta_a(a_k)a_{k+1}....a_d a_1....a_{k-1} \tag{2.3}$$

where

$a_k \in Q_1, \forall k = 1, ..., d$

and

$$\delta_a(a_k) := \begin{cases} 1 & if \quad a_k = a \\ 0 & if \quad a_k \neq a \end{cases}$$

Example 2.23. Consider the quiver below;

$$\begin{tikzcd}
(T) \arrow[r, "b_{UT}"] & (U) \arrow[d, "c_{WU}"] \\
(V) \arrow[u, "a_{TV}"] & (W) \arrow[l, "d_{VW}"]
\end{tikzcd}$$

We choose the potential for the quiver as:

$$\mathcal{S} = d_{VW}c_{WU}b_{UT}a_{TV}$$

We now proceed to find the cyclic derivative with respect to an arrow of this quiver say, a_{TV}.

Using the formulae:

$$\delta_a(a_1....a_d) = \sum_{k=1}^{d} \delta_a(a_k)a_{k+1}....a_d a_1....a_{k-1} \tag{2.4}$$

We have:

$$\begin{aligned}
\delta_{a_{TV}}(d_{VW}c_{WU}b_{UT}a_{TV}) &= \delta_{a_{TV}}(d_{VW})c_{WU}b_{UT}a_{TV} + \delta_{a_{TV}}(c_{WU})b_{UT}a_{TV}d_{VW} \\
&+ \delta_{a_{TV}}(b_{UT})a_{TV}d_{VW}c_{WU} + \delta_{a_{TV}}(a_{TV})d_{VW}c_{WU}b_{UT}
\end{aligned}$$

From the condition :

$$\delta_a(a_k) := \begin{cases} 1 \quad if \quad a_k = a \\ 0 \quad if \quad a_k \neq a \end{cases}$$

We get:

$$\delta_{a_{TV}}(d_{VW}c_{WU}b_{UT}a_{TV}) = 0 + 0 + 0 + d_{VW}c_{WU}b_{UT}$$

Thus:

$$\delta_{a_{TV}}(d_{VW}c_{WU}b_{UT}a_{TV}) = d_{VW}c_{WU}b_{UT}$$

Giving the relation:

$$d_{VW}c_{WU}b_{UT} = 0$$

2.5 Literature review

In the 1970s, a theorem of Gabriel [8] changed the way algebra were studied. The theorem states that every basic algebra over an algebraically closed field is isomorphic to the path algebra of a quiver with relations. This theorem led to a change in the way algebras can be visualized and discussed, since many algebras could be represented using quivers.

Since then a number of connections of the representation theory of quivers to other algebraic topics have come up, in particular to Lie algebra [9, 11] , Hall algebras [1, 15, 17] and quantum groups [5, 12] and more recently to cluster algebras [6, 7].

Mutation of quivers was first introduced as a case study in the study of cluster algebras by Fomin and Zelevinsky in [6, 7]. Skew-symmetric integer matrices encode quivers without loops and oriented 2-cycles and their procedure for mutating a quiver is a special case of the mutation of matrices.

The idea of mutating quivers was further extended to mutation of quivers with special types of relations called potentials by Derksen, Weyman and Zelevinsky in [4]. Their work gives the definition for mutation of a quiver with potential and the procedure involved provided the quiver has no loops or oriented 2-cycles.

Defining mutation of quivers with potential raised a lot of interesting questions, especially since the mutation was not an operation on quivers with potential. Questions include: When is mutation an operation? If it is an operation, what happens to the potentials as we move from one mutation to another? How does this concept relate to other algebraic topics? What other ideas can be taken from related subjects like physics?

Studies have been done in attempt to provide partial solutions to some of the above issues. For instance, Stern [18] with strong exceptional collections showed that tilting mutation is an operation on some families of quivers corresponding to quivers with potential. He used family tree diagrams to represent his mutations. Based on Stern's examples, Obiero [13] related mutation of quivers with potential to R-charges in physics and Owino [14] studied blocks of exceptional collections from geometry. Both studies were done without knowing what the explicit potentials are for these quivers, and this is the aim of our study.

2.6 Statement of the problem

Derksen, Weyman and Zelevinsky in [4] initiated the mutation theory of a quiver with potential. Their study requires the quiver with potential to have no loops or oriented 2-cycles. In Stern [18] we have a rich source of families of quivers that correspond to quivers with potential which can be mutated indefinitely. The potentials for these quivers can be useful in a number of studies but we have not found any provisions for them in the literature. Thus, our main task is to obtain and write these potentials consistently, particularly for the cases that Stern [18] related to the P^2 and $P^1 \times P^1$.

2.7 Objective of the study

In this study, we obtain the potentials for some of the quivers with potential related to P^2 and $P^1 \times P^1$ found in Stern [18]. Through relabeling, we write the potentials consistently so as to determine whether or not prediction of the next mutated is possible using our notation.

2.8 Research methodology

An understanding of the mutated diagrams in Stern [18] is a guideline for this work. An ability to effectively and carefully calculate the mutated potential for a given quiver following the steps based on the work of Derksen, Weyman and Zelevinsky in [4] and writing them in a brief manner is the backbone of this study. These steps entail;

1. Obtaining an unreduced quiver through the first few steps of mutation.

2. Reducing this quiver using relations.

3. Relabeling of the arrows so to write the reduced potential in a short and a consistent way.

2.9 Significance of the study

Any result in the study of quivers with potential is important in the development of this new and active area of study. Our calculations provide explicit examples of quivers with potentials which can be worked on further. This study is already of use to my colleagues working on related areas. For instance, relabeling of arrows yielded patterns in which the symmetry of potentials expected in blocked quivers was seen in the relevant examples.

Chapter 3

Mutation of quivers with potential

3.1 Introduction

The key reference for this section is the paper by Derksen, Weyman and Zelevinsky [4]. They defined mutation of a quiver with potential provided the quiver has no loops or 2-cycles. A quiver is transformed to a mutated one with aid of the rules governing the transformation. This transformation is reversible by repeating mutation at the same vertex to obtain the original quiver.

3.2 Mutation of a quiver with potential

In mutating a quiver with potential, it is a consequence of Proposition 2.16 in section 2.3 that no two cyclically equivalent paths appear in the expansion of $\mathcal{S}$. However, it is also required that no cyclic path in this expansion starts (and terminates) at the vertex of mutation.

Construction 1. (See [4]) Mutation of a QP $(Q, \mathcal{S})$ at a vertex $V \in Q_0$ can be defined if Q satisfies the following conditions:

i) Q has no loops.

ii) Q has no oriented 2-cycles.

This construction involves seven steps which are described and illustrated by the example below:

Example 3.1. Consider the quiver :

$$\begin{tikzcd}
(T) \arrow[rr, "a_{UT}^{(i)}", "3"'] & & (U) \arrow[dl, "3"', "a_{VU}^{(i)}"] \\
& (V) \arrow[ul, "3"', "a_{TV}^{(i)}"] &
\end{tikzcd}$$

with the potential:

$$\begin{aligned} \mathcal{S} \;=\; & a_{UT}^{(1)}a_{TV}^{(2)}a_{VU}^{(3)} - a_{UT}^{(1)}a_{TV}^{(3)}a_{VU}^{(2)} - a_{UT}^{(2)}a_{TV}^{(1)}a_{VU}^{(3)} + \\ & a_{UT}^{(2)}a_{TV}^{(3)}a_{VU}^{(1)} + a_{UT}^{(3)}a_{TV}^{(1)}a_{VU}^{(2)} - a_{UT}^{(3)}a_{TV}^{(2)}a_{VU}^{(1)} \end{aligned} \tag{3.1}$$

Let's mutate it at vertex V,

1. The new quiver has the same vertices as the old one.

$$\begin{tikzcd}
(T) & & (U) \\
& (V) &
\end{tikzcd}$$

2. Arrows into vertex V become arrows out of V.

$$\begin{tikzcd}
(T) & & (U) \\
& (V) \arrow[ur, "3", "a^{(i)*}_{VU}"'] &
\end{tikzcd}$$

3. Arrows out of V become arrows into V.

$$\begin{tikzcd}
(T) \arrow[dr, "3", "a^{(i)*}_{TV}"'] & & (U) \\
& (V) \arrow[ur, "3", "a^{(i)*}_{VU}"'] &
\end{tikzcd}$$

4. Arrows not into or from V remain unchanged.

$$\begin{tikzcd}
(T) \arrow[rr, "a^{(i)}_{UT}", "3"'] \arrow[dr, "3", "a^{(i)*}_{TV}"'] & & (U) \\
& (V) \arrow[ur, "3", "a^{(i)*}_{VU}"'] &
\end{tikzcd}$$

5. All paths of length two through vertex V, denoted $a^{(i)}_{?V}a^{(j)}_{V?}$, give arrows, $(a^{(i)}_{?V}a^{(j)}_{V?})$, from the start to the terminating vertex of the original path.

$$\begin{tikzcd}
(T) \arrow[rr, "a^{(i)}_{UT}"'] \arrow[dr, "3", "a^{(i)*}_{TV}"'] & & (U) \arrow[ll, bend right, "(a^{(i)}_{TV}a^{(i)}_{VU})"'] \\
& (V) \arrow[ur, "3", "a^{(i)*}_{VU}"'] &
\end{tikzcd}$$

In this case we have 9 paths through V which give 9 new arrows namely:

$$(a^{(1)}_{TV}a^{(1)}_{VU}), (a^{(1)}_{TV}a^{(2)}_{VU}), (a^{(1)}_{TV}a^{(3)}_{VU})$$

$$(a^{(2)}_{TV}a^{(1)}_{VU}), (a^{(2)}_{TV}a^{(2)}_{VU}), (a^{(2)}_{TV}a^{(3)}_{VU})$$

$$(a^{(3)}_{TV}a^{(1)}_{VU}), (a^{(3)}_{TV}a^{(2)}_{VU}), (a^{(3)}_{TV}a^{(3)}_{VU})$$

This quiver is the unreduced quiver whose potential is given in the next step.

6. To obtain the unreduced potential, we use the formula:

$$\tilde{\mathcal{S}} = (\mathcal{S}) + \Delta_V, \tag{3.2}$$

where $(\mathcal{S})$ is obtained from $\mathcal{S}$ of the original quiver. We do not have paths of the form $a^{(i)}_{?V}a^{(j)}_{V?}$ in the unreduced quiver, but rather compositions for arrows of the form $(a^{(i)}_{?V}a^{(j)}_{V?})$. We thus substitute $(a^{(i)}_{?V}a^{(j)}_{V?})$ for each factor $a^{(i)}_{?V}a^{(j)}_{V?}$ with $s(a^{(i)}_{?V}) = t(a^{(j)}_{V?}) = V$, for any cyclic path occurring in the expansion of $\mathcal{S}$.

This element in our case is:

$$\begin{aligned}(\mathcal{S}) \quad = \quad & a^{(1)}_{UT}(a^{(2)}_{TV}a^{(3)}_{VU}) - a^{(1)}_{UT}(a^{(3)}_{TV}a^{(2)}_{VU}) - a^{(2)}_{UT}(a^{(1)}_{TV}a^{(3)}_{VU}) + \\ & a^{(2)}_{UT}(a^{(3)}_{TV}a^{(1)}_{VU}) + a^{(3)}_{UT}(a^{(1)}_{TV}a^{(2)}_{VU}) - a^{(3)}_{UT}(a^{(2)}_{TV}a^{(1)}_{VU}) +\end{aligned}$$

and

$$\Delta_V = \sum_{a,b\in Q_1 : s(b)=t(a)=V} (ba)a^*b^* \tag{3.3}$$

In this case this element is:

$$\begin{aligned}& (a^{(1)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(1)*}_{TV} + (a^{(1)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(1)*}_{TV} + (a^{(1)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(1)*}_{TV} + \\ & (a^{(2)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(2)*}_{TV} + (a^{(2)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(2)*}_{TV} + (a^{(2)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(2)*}_{TV} + \\ & (a^{(3)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(3)*}_{TV} + (a^{(3)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(3)*}_{TV} + (a^{(3)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(3)*}_{TV}\end{aligned}$$

Thus combining the two, the unreduced potential will be:

$$\begin{aligned}
\tilde{\mathcal{S}} \;=\;& a^{(1)}_{UT}(a^{(2)}_{TV}a^{(3)}_{VU}) - a^{(1)}_{UT}(a^{(3)}_{TV}a^{(2)}_{VU}) - a^{(2)}_{UT}(a^{(1)}_{TV}a^{(3)}_{VU}) + \\
& a^{(2)}_{UT}(a^{(3)}_{TV}a^{(1)}_{VU}) + a^{(3)}_{UT}(a^{(1)}_{TV}a^{(2)}_{VU}) - a^{(3)}_{UT}(a^{(2)}_{TV}a^{(1)}_{VU}) + \\
& (a^{(1)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(1)*}_{TV} + (a^{(1)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(1)*}_{TV} + (a^{(1)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(1)*}_{TV} + \\
& (a^{(2)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(2)*}_{TV} + (a^{(2)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(2)*}_{TV} + (a^{(2)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(2)*}_{TV} + \\
& (a^{(3)}_{TV}a^{(1)}_{VU})a^{(1)*}_{VU}a^{(3)*}_{TV} + (a^{(3)}_{TV}a^{(2)}_{VU})a^{(2)*}_{VU}a^{(3)*}_{TV} + (a^{(3)}_{TV}a^{(3)}_{VU})a^{(3)*}_{VU}a^{(3)*}_{TV}
\end{aligned}$$

The next step is the reduction process where the 2-cycles in the unreduced potential are removed using relations to give the reduced potential. This step is built on theorem 4.6 in [4] where we remove the trivial part; the linear combination of 2-cyclic paths of the unreduced potential to remain with the reduced part; the part involving only the n-cyclic paths, with $n \geq 3$.

7. To obtain the reduced potential, we take the cyclic derivatives on the unreduced potential with respect to the single arrows in the 2-cycles of the quiver appearing in the unreduced potential to get relations. In this case we have:

$$\delta_{a^{(1)}_{UT}}(\tilde{\mathcal{S}}) = (a^{(2)}_{TV}a^{(3)}_{VU}) - (a^{(3)}_{TV}a^{(2)}_{VU}) = 0, \Rightarrow (a^{(2)}_{TV}a^{(3)}_{VU}) = (a^{(3)}_{TV}a^{(2)}_{VU})$$

$$\delta_{a^{(2)}_{UT}}(\tilde{\mathcal{S}}) = (a^{(3)}_{TV}a^{(1)}_{VU}) - (a^{(1)}_{TV}a^{(3)}_{VU}) = 0, \Rightarrow (a^{(3)}_{TV}a^{(1)}_{VU}) = (a^{(1)}_{TV}a^{(3)}_{VU})$$

$$\delta_{a^{(3)}_{UT}}(\tilde{\mathcal{S}}) = (a^{(1)}_{TV}a^{(2)}_{VU}) - (a^{(2)}_{TV}a^{(1)}_{VU}) = 0, \Rightarrow (a^{(1)}_{TV}a^{(2)}_{VU}) = (a^{(2)}_{TV}a^{(1)}_{VU})$$

Notice that each relation reduces a 2-cycle and we thus end up with six arrows out of the initial nine arrows.

We relabel the arrows of the quiver for simplification and consistency as follows:

$$b^{(i)}_{VT} := a^{(i)*}_{TV}$$

$$b^{(i)}_{UV} := a^{(i)*}_{VU}$$

$$b^{(1)}_{TU} := (a^{(2)}_{TV}a^{(3)}_{VU}) = (a^{(3)}_{TV}a^{(2)}_{VU})$$

$$b^{(2)}_{TU} := (a^{(1)}_{TV}a^{(3)}_{VU}) = (a^{(3)}_{TV}a^{(1)}_{VU})$$

$$b^{(3)}_{TU} := (a^{(1)}_{TV}a^{(2)}_{VU}) = (a^{(2)}_{TV}a^{(1)}_{VU})$$

$$b^{(4)}_{TU} := (a^{(1)}_{TV}a^{(1)}_{VU})$$

$$b^{(5)}_{TU} := (a^{(2)}_{TV}a^{(2)}_{VU})$$

$$b^{(6)}_{TU} := (a^{(3)}_{TV}a^{(3)}_{VU})$$

The reduced quiver becomes:

$$\begin{tikzcd}
(T) \arrow[dr, "3"'] & & (U) \arrow[ll, "6"'] \\
& (V) \arrow[ur, "3"'] &
\end{tikzcd}$$

with the reduced potential:

$$\begin{aligned}
\overline{S} \;=\; & b^{(1)}_{TU}b^{(2)}_{UV}b^{(3)}_{VT} + b^{(1)}_{TU}b^{(3)}_{UV}b^{(2)}_{VT} + b^{(2)}_{TU}b^{(3)}_{UV}b^{(1)}_{VT} + \\
& b^{(2)}_{TU}b^{(1)}_{UV}b^{(3)}_{VT} + b^{(3)}_{TU}b^{(1)}_{UV}b^{(2)}_{VT} + b^{(3)}_{TU}b^{(2)}_{UV}b^{(1)}_{VT} + \\
& b^{(4)}_{TU}b^{(1)}_{UV}b^{(1)}_{VT} + b^{(5)}_{TU}b^{(2)}_{UV}b^{(2)}_{VT} + b^{(6)}_{TU}b^{(3)}_{UV}b^{(3)}_{VT}
\end{aligned} \tag{3.4}$$

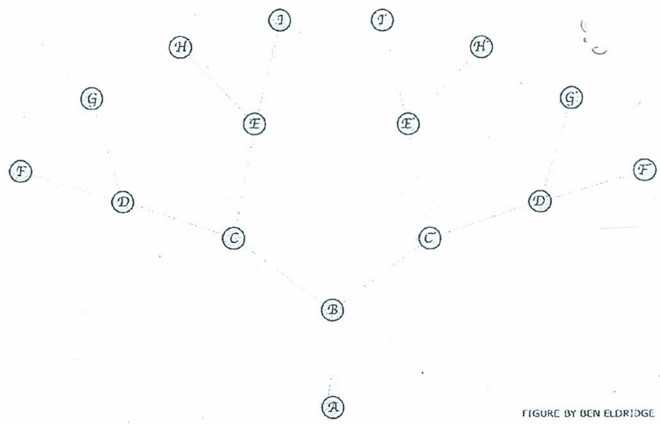
Chapter 4

Calculation of potentials for quivers with potential related to P^2

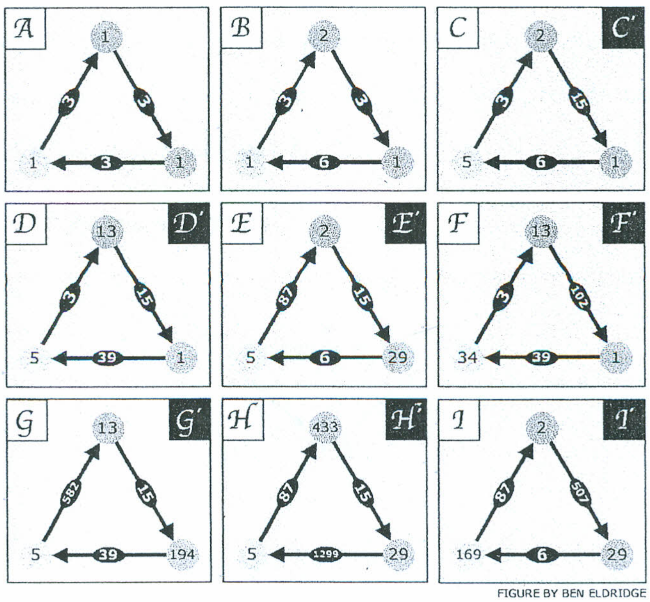
4.1 Introduction

In this chapter, we deal with a family of quivers corresponding to quivers with potential given in the tree diagram that Stern in [18] related to P^2.

The tree diagram is:



The quivers corresponding to each of these points are:



The quiver at the center of the tree diagram, called A was introduced in section 2.2 and its potential stated in section 2.4. It is a consequence of Stern [18] that this quiver together with all its mutants can be mutated indefinitely without giving loops or 2-cycles, and so the tree diagram shows how one quiver is related to another through mutation. However, there is no attempt in the literature to write the potentials for these mutations.

In section 4.2 we work out the potential for quiver C, and then provide the calculated potentials for the quivers D and E. Since these potentials are too big to write, section 4.3 provides the notation used to write them briefly.

4.2 Calculations

In this section we mutate quiver B using the steps discussed in section 3.2 to obtain quiver C and its potential. We then provide the mutated potentials for the quivers D and E.

Thus consider the quiver B:

$$\begin{tikzcd}
(T) \arrow[dr, "3"'] & & (U) \arrow[ll, "6"'] \\
& (V) \arrow[ur, "3"'] &
\end{tikzcd}$$

whose potential is:

$$\begin{aligned}
S \;=\; & b_{TU}^{(1)} b_{UV}^{(2)} b_{VT}^{(3)} + b_{TU}^{(1)} b_{UV}^{(3)} b_{VT}^{(2)} + b_{TU}^{(2)} b_{UV}^{(3)} b_{VT}^{(1)} + \\
& b_{TU}^{(2)} b_{UV}^{(1)} b_{VT}^{(3)} + b_{TU}^{(3)} b_{UV}^{(1)} b_{VT}^{(2)} + b_{TU}^{(3)} b_{UV}^{(2)} b_{VT}^{(1)} + \\
& b_{TU}^{(4)} b_{UV}^{(1)} b_{VT}^{(1)} + b_{TU}^{(5)} b_{UV}^{(2)} b_{VT}^{(2)} + b_{TU}^{(6)} b_{UV}^{(3)} b_{VT}^{(3)}
\end{aligned}$$

If we mutate it at U the unreduced quiver will be:

$$\begin{tikzcd}
(T) \arrow[rr, "6"] \arrow[dr, "3"] & & (U) \arrow[dl, "3"] \\
& (V) \arrow[ul, bend left, "18"] &
\end{tikzcd}$$

with the unreduced potential:

$$\begin{aligned}
\tilde{S} \;=\; & (b^{(1)}_{TU} b^{(2)}_{UV}) b^{(3)}_{VT} + (b^{(1)}_{TU} b^{(3)}_{UV}) b^{(2)}_{VT} + (b^{(2)}_{TU} b^{(3)}_{UV}) b^{(1)}_{VT} + \\
& (b^{(2)}_{TU} b^{(1)}_{UV}) b^{(3)}_{VT} + (b^{(3)}_{TU} b^{(1)}_{UV}) b^{(2)}_{VT} + (b^{(3)}_{TU} b^{(2)}_{UV}) b^{(1)}_{VT} + \\
& (b^{(4)}_{TU} b^{(1)}_{UV}) b^{(1)}_{VT} + (b^{(5)}_{TU} b^{(2)}_{UV}) b^{(2)}_{VT} + (b^{(6)}_{TU} b^{(3)}_{UV}) b^{(3)}_{VT} + \\
& (b^{(1)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(1)*}_{TU} + (b^{(1)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(1)*}_{TU} + (b^{(1)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(1)*}_{TU} + \\
& (b^{(2)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(2)*}_{TU} + (b^{(2)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(2)*}_{TU} + (b^{(2)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(2)*}_{TU} + \\
& (b^{(3)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(3)*}_{TU} + (b^{(3)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(3)*}_{TU} + (b^{(3)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(3)*}_{TU} + \\
& (b^{(4)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(4)*}_{TU} + (b^{(4)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(4)*}_{TU} + (b^{(4)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(4)*}_{TU} + \\
& (b^{(5)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(5)*}_{TU} + (b^{(5)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(5)*}_{TU} + (b^{(5)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(5)*}_{TU} + \\
& (b^{(6)}_{TU} b^{(1)}_{UV}) b^{(1)*}_{UV} b^{(6)*}_{TU} + (b^{(6)}_{TU} b^{(2)}_{UV}) b^{(2)*}_{UV} b^{(6)*}_{TU} + (b^{(6)}_{TU} b^{(3)}_{UV}) b^{(3)*}_{UV} b^{(6)*}_{TU}
\end{aligned}$$

Taking the cyclic derivatives with respect to the single arrows in the 2-cycles of the unreduced potential we obtain the relations below:

$$\delta_{b^{(3)}_{VT}}(\tilde{S}) = (b^{(1)}_{TU} b^{(2)}_{UV}) + (b^{(2)}_{TU} b^{(1)}_{UV}) + (b^{(6)}_{TU} b^{(3)}_{UV}) = 0 \Rightarrow (b^{(1)}_{TU} b^{(2)}_{UV}) = -(b^{(2)}_{TU} b^{(1)}_{UV}) - (b^{(6)}_{TU} b^{(3)}_{UV})$$

$$\delta_{b^{(1)}_{VT}}(\tilde{S}) = (b^{(2)}_{TU} b^{(3)}_{UV}) + (b^{(3)}_{TU} b^{(2)}_{UV}) + (b^{(4)}_{TU} b^{(1)}_{UV}) = 0 \Rightarrow (b^{(2)}_{TU} b^{(3)}_{UV}) = -(b^{(3)}_{TU} b^{(2)}_{UV}) - (b^{(4)}_{TU} b^{(1)}_{UV})$$

$$\delta_{b^{(2)}_{VT}}(\tilde{S}) = (b^{(3)}_{TU} b^{(1)}_{UV}) + (b^{(1)}_{TU} b^{(3)}_{UV}) + (b^{(5)}_{TU} b^{(2)}_{UV}) = 0 \Rightarrow (b^{(3)}_{TU} b^{(1)}_{UV}) = -(b^{(1)}_{TU} b^{(3)}_{UV}) - (b^{(5)}_{TU} b^{(2)}_{UV})$$

3 arrows out of the 18 have been reduced. We are left with 15 new arrows.

We rename the arrows as follows:

$$c_{UT}^{(i)} := b_{TU}^{(i)*}$$

$$c_{VU}^{(i)} := b_{UV}^{(i)*}$$

$$c_{TV}^{(1)} := (b_{TU}^{(3)} b_{UV}^{(2)})$$

$$c_{TV}^{(2)} := (b_{TU}^{(1)} b_{UV}^{(3)})$$

$$c_{TV}^{(3)} := (b_{TU}^{(2)} b_{UV}^{(1)})$$

$$c_{TV}^{(4)} := (b_{TU}^{(1)} b_{UV}^{(1)})$$

$$c_{TV}^{(5)} := (b_{TU}^{(2)} b_{UV}^{(2)})$$

$$c_{TV}^{(6)} := (b_{TU}^{(3)} b_{UV}^{(3)})$$

$$c_{TV}^{(7)} := (b_{TU}^{(4)} b_{UV}^{(1)})$$

$$c_{TV}^{(8)} := (b_{TU}^{(5)} b_{UV}^{(1)})$$

$$c_{TV}^{(9)} := (b_{TU}^{(6)} b_{UV}^{(1)})$$

$$c_{TV}^{(10)} := (b_{TU}^{(4)} b_{UV}^{(2)})$$

$$c_{TV}^{(11)} := (b_{TU}^{(5)} b_{UV}^{(2)})$$

$$c_{TV}^{(12)} := (b_{TU}^{(6)} b_{UV}^{(2)})$$

$$c_{TV}^{(13)} := (b_{TU}^{(4)} b_{UV}^{(3)})$$

$$c_{TV}^{(14)} := (b_{TU}^{(5)} b_{UV}^{(3)})$$

$$c_{TV}^{(15)} := (b_{TU}^{(6)} b_{UV}^{(3)})$$

$$-c_{TV}^{(3)} - c_{TV}^{(15)} = (b_{TU}^{(1)} b_{UV}^{(2)})$$

$$-c_{TV}^{(1)} - c_{TV}^{(7)} = (b_{TU}^{(2)} b_{UV}^{(3)})$$

$$-c_{TV}^{(2)} - c_{TV}^{(11)} = (b_{TU}^{(3)} b_{UV}^{(1)})$$

Thus the resulting quiver is C:

$$\begin{tikzcd}
(T) \arrow[rr, "6"] & & (U) \arrow[dl, "3"] \\
& (V) \arrow[ul, "15"] &
\end{tikzcd}$$

with the potential:

$$\begin{aligned}
\overline{S} = \; & c^{(1)}_{TV}c^{(2)}_{VU}c^{(3)}_{UT} - c^{(1)}_{TV}c^{(3)}_{VU}c^{(2)}_{UT} + c^{(2)}_{TV}c^{(3)}_{VU}c^{(1)}_{UT} - \\
& c^{(2)}_{TV}c^{(1)}_{VU}c^{(3)}_{UT} + c^{(3)}_{TV}c^{(1)}_{VU}c^{(2)}_{UT} - c^{(3)}_{TV}c^{(2)}_{VU}c^{(1)}_{UT} + \\
& c^{(4)}_{TV}c^{(1)}_{VU}c^{(1)}_{UT} + c^{(5)}_{TV}c^{(2)}_{VU}c^{(2)}_{UT} + c^{(6)}_{TV}c^{(3)}_{VU}c^{(3)}_{UT} + \\
& c^{(7)}_{TV}c^{(1)}_{VU}c^{(4)}_{UT} + c^{(8)}_{TV}c^{(1)}_{VU}c^{(5)}_{UT} + c^{(9)}_{TV}c^{(1)}_{VU}c^{(6)}_{UT} + \\
& c^{(10)}_{TV}c^{(2)}_{VU}c^{(4)}_{UT} + c^{(11)}_{TV}c^{(2)}_{VU}c^{(5)}_{UT} + c^{(12)}_{TV}c^{(2)}_{VU}c^{(6)}_{UT} + \\
& c^{(13)}_{TV}c^{(3)}_{VU}c^{(4)}_{UT} + c^{(14)}_{TV}c^{(3)}_{VU}c^{(5)}_{UT} + c^{(15)}_{TV}c^{(3)}_{VU}c^{(6)}_{UT} - \\
& c^{(7)}_{TV}c^{(3)}_{VU}c^{(2)}_{UT} - c^{(11)}_{TV}c^{(1)}_{VU}c^{(3)}_{UT} - c^{(15)}_{TV}c^{(2)}_{VU}c^{(1)}_{UT}
\end{aligned} \tag{4.1}$$

It is worth noting mutation of quiver A at any of its three vertices yields the same quiver B, that is why there are three lines from A that rejoin before entering B. Mutation of B at U and T gives quiver C and C' respectively. The difference between the primed and the unprimed quivers is that the arrows of one are reversed in the other.

We now give the worked out potentials for the other quivers with potentials obtained by mutations of quiver C.

- Mutation of quiver C above at V gives D:

$$\begin{tikzcd}
(T) \arrow[dr, "15"'] & & (U) \arrow[ll, "39"'] \\
& (V) \arrow[ur, "3"'] &
\end{tikzcd}$$

with the potential:

$$
\begin{aligned}
\overline{\mathcal{S}} \; = \; & d_{TU}^{(1)} d_{UV}^{(2)} d_{VT}^{(3)} + d_{TU}^{(1)} d_{UV}^{(3)} d_{VT}^{(2)} + d_{TU}^{(2)} d_{UV}^{(1)} d_{VT}^{(3)} + d_{TU}^{(2)} d_{UV}^{(3)} d_{VT}^{(1)} + \\
& d_{TU}^{(3)} d_{UV}^{(2)} d_{VT}^{(1)} + d_{TU}^{(3)} d_{UV}^{(1)} d_{VT}^{(2)} + d_{TU}^{(4)} d_{UV}^{(1)} d_{VT}^{(1)} + d_{TU}^{(5)} d_{UV}^{(2)} d_{VT}^{(2)} + \\
& d_{TU}^{(6)} d_{UV}^{(3)} d_{VT}^{(3)} + d_{TU}^{(7)} d_{UV}^{(1)} d_{VT}^{(4)} + d_{TU}^{(8)} d_{UV}^{(1)} d_{VT}^{(5)} + d_{TU}^{(9)} d_{UV}^{(1)} d_{VT}^{(6)} + \\
& d_{TU}^{(10)} d_{UV}^{(2)} d_{VT}^{(4)} + d_{TU}^{(11)} d_{UV}^{(2)} d_{VT}^{(5)} + d_{TU}^{(12)} d_{UV}^{(2)} d_{VT}^{(6)} + d_{TU}^{(13)} d_{UV}^{(3)} d_{VT}^{(4)} + \\
& d_{TU}^{(14)} d_{UV}^{(3)} d_{VT}^{(5)} + d_{TU}^{(15)} d_{UV}^{(3)} d_{VT}^{(6)} + d_{TU}^{(16)} d_{UV}^{(1)} d_{VT}^{(10)} + d_{TU}^{(17)} d_{UV}^{(1)} d_{VT}^{(11)} + \\
& d_{TU}^{(18)} d_{UV}^{(1)} d_{VT}^{(12)} + d_{TU}^{(19)} d_{UV}^{(1)} d_{VT}^{(13)} + d_{TU}^{(20)} d_{UV}^{(1)} d_{VT}^{(14)} + d_{TU}^{(21)} d_{UV}^{(1)} d_{VT}^{(15)} + \\
& d_{TU}^{(22)} d_{UV}^{(2)} d_{VT}^{(10)} + d_{TU}^{(23)} d_{UV}^{(2)} d_{VT}^{(11)} + d_{TU}^{(24)} d_{UV}^{(2)} d_{VT}^{(12)} + d_{TU}^{(25)} d_{UV}^{(2)} d_{VT}^{(13)} + \\
& d_{TU}^{(26)} d_{UV}^{(2)} d_{VT}^{(14)} + d_{TU}^{(27)} d_{UV}^{(2)} d_{VT}^{(15)} + d_{TU}^{(28)} d_{UV}^{(3)} d_{VT}^{(10)} + d_{TU}^{(29)} d_{UV}^{(3)} d_{VT}^{(11)} + \\
& d_{TU}^{(30)} d_{UV}^{(3)} d_{VT}^{(12)} + d_{TU}^{(31)} d_{UV}^{(3)} d_{VT}^{(13)} + d_{TU}^{(32)} d_{UV}^{(3)} d_{VT}^{(14)} + d_{TU}^{(33)} d_{UV}^{(3)} d_{VT}^{(15)} - \\
& c_{TU}^{(7)} d_{UV}^{(3)} d_{VT}^{(2)} - d_{TU}^{(11)} d_{UV}^{(1)} d_{VT}^{(3)} - d_{TU}^{(15)} d_{UV}^{(2)} d_{VT}^{(1)} - d_{TU}^{(22)} d_{UV}^{(1)} d_{VT}^{(7)} - \\
& d_{TU}^{(23)} d_{UV}^{(1)} d_{VT}^{(8)} - d_{TU}^{(24)} d_{UV}^{(1)} d_{VT}^{(9)} - d_{TU}^{(31)} d_{UV}^{(1)} d_{VT}^{(7)} - d_{TU}^{(32)} d_{UV}^{(1)} d_{VT}^{(8)} - \\
& d_{TU}^{(33)} d_{UV}^{(1)} d_{VT}^{(9)} + d_{TU}^{(17)} d_{UV}^{(2)} d_{VT}^{(1)} + d_{TU}^{(27)} d_{UV}^{(3)} d_{VT}^{(2)} + d_{TU}^{(37)} d_{UV}^{(1)} d_{VT}^{(3)} + \\
& d_{TU}^{(34)} d_{UV}^{(2)} d_{VT}^{(7)} + d_{TU}^{(35)} d_{UV}^{(2)} d_{VT}^{(8)} + d_{TU}^{(36)} d_{UV}^{(2)} d_{VT}^{(9)} + d_{TU}^{(37)} d_{UV}^{(3)} d_{VT}^{(7)} + \\
& d_{TU}^{(38)} d_{UV}^{(3)} d_{VT}^{(8)} + d_{TU}^{(39)} d_{UV}^{(3)} d_{VT}^{(9)} \qquad (4.2)
\end{aligned}
$$

- Mutation of quiver C at T gives quiver E:

$$
\begin{tikzcd}
(T) \arrow[dr, "15"'] & & (U) \arrow[ll, "6"'] \\
& (V) \arrow[ur, "87"'] &
\end{tikzcd}
$$

whose potential is:

$$
\begin{aligned}
\overline{S} \; = \; & e^{(1)}_{UV}e^{(2)}_{VT}e^{(3)}_{TU} + e^{(1)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} + e^{(2)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} + e^{(2)}_{UV}e^{(3)}_{VT}e^{(1)}_{TU} + e^{(3)}_{UV}e^{(1)}_{VT}e^{(2)}_{TU} + \\
& e^{(3)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} + e^{(4)}_{UV}e^{(1)}_{VT}e^{(1)}_{TU} + e^{(5)}_{UV}e^{(2)}_{VT}e^{(2)}_{TU} + e^{(6)}_{UV}e^{(3)}_{VT}e^{(3)}_{TU} + e^{(7)}_{UV}e^{(4)}_{VT}e^{(1)}_{TU} + \\
& e^{(8)}_{UV}e^{(5)}_{VT}e^{(1)}_{TU} + e^{(9)}_{UV}e^{(6)}_{VT}e^{(1)}_{TU} + e^{(10)}_{UV}e^{(7)}_{VT}e^{(1)}_{TU} + e^{(11)}_{UV}e^{(8)}_{VT}e^{(1)}_{TU} + e^{(12)}_{UV}e^{(9)}_{VT}e^{(1)}_{TU} + \\
& e^{(13)}_{UV}e^{(10)}_{VT}e^{(1)}_{TU} + e^{(14)}_{UV}e^{(11)}_{VT}e^{(1)}_{TU} + e^{(15)}_{UV}e^{(12)}_{VT}e^{(1)}_{TU} + e^{(16)}_{UV}e^{(13)}_{VT}e^{(1)}_{TU} + e^{(17)}_{UV}e^{(14)}_{VT}e^{(1)}_{TU} + \\
& e^{(18)}_{UV}e^{(15)}_{VT}e^{(1)}_{TU} + e^{(19)}_{UV}e^{(4)}_{VT}e^{(2)}_{TU} + e^{(20)}_{UV}e^{(5)}_{VT}e^{(2)}_{TU} + e^{(21)}_{UV}e^{(6)}_{VT}e^{(2)}_{TU} + e^{(22)}_{UV}e^{(7)}_{VT}e^{(2)}_{TU} + \\
& e^{(23)}_{UV}e^{(8)}_{VT}e^{(2)}_{TU} + e^{(24)}_{UV}e^{(9)}_{VT}e^{(2)}_{TU} + e^{(25)}_{UV}e^{(10)}_{VT}e^{(2)}_{TU} + e^{(26)}_{UV}e^{(11)}_{VT}e^{(2)}_{TU} + e^{(27)}_{UV}e^{(12)}_{VT}e^{(2)}_{TU} + \\
& e^{(28)}_{UV}e^{(13)}_{VT}e^{(2)}_{TU} + e^{(29)}_{UV}e^{(14)}_{VT}e^{(2)}_{TU} + e^{(30)}_{UV}e^{(15)}_{VT}e^{(2)}_{TU} + e^{(31)}_{UV}e^{(4)}_{VT}e^{(3)}_{TU} + e^{(32)}_{UV}e^{(5)}_{VT}e^{(3)}_{TU} + \\
& e^{(33)}_{UV}e^{(6)}_{VT}e^{(3)}_{TU} + e^{(34)}_{UV}e^{(7)}_{VT}e^{(3)}_{TU} + e^{(35)}_{UV}e^{(8)}_{VT}e^{(3)}_{TU} + e^{(36)}_{UV}e^{(9)}_{VT}e^{(3)}_{TU} + e^{(37)}_{UV}e^{(10)}_{VT}e^{(3)}_{TU} + \\
& e^{(38)}_{UV}e^{(11)}_{VT}e^{(3)}_{TU} + e^{(39)}_{UV}e^{(12)}_{VT}e^{(3)}_{TU} + e^{(40)}_{UV}e^{(13)}_{VT}e^{(3)}_{TU} + e^{(41)}_{UV}e^{(14)}_{VT}e^{(3)}_{TU} + d^{(42)}_{UV}e^{(15)}_{VT}e^{(3)}_{TU} + \\
& e^{(43)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} + e^{(44)}_{UV}e^{(2)}_{VT}e^{(4)}_{TU} + e^{(45)}_{UV}e^{(3)}_{VT}e^{(4)}_{TU} + e^{(46)}_{UV}e^{(4)}_{VT}e^{(4)}_{TU} + e^{(47)}_{UV}e^{(5)}_{VT}e^{(4)}_{TU} + \\
& e^{(48)}_{UV}e^{(6)}_{VT}e^{(4)}_{TU} + e^{(49)}_{UV}e^{(7)}_{VT}e^{(4)}_{TU} + e^{(50)}_{UV}e^{(8)}_{VT}e^{(4)}_{TU} + e^{(51)}_{UV}e^{(9)}_{VT}e^{(4)}_{TU} + e^{(52)}_{UV}e^{(10)}_{VT}e^{(4)}_{TU} + \\
& e^{(53)}_{UV}e^{(11)}_{VT}e^{(4)}_{TU} + e^{(54)}_{UV}e^{(12)}_{VT}e^{(4)}_{TU} + e^{(55)}_{UV}e^{(13)}_{VT}e^{(4)}_{TU} + e^{(56)}_{UV}e^{(14)}_{VT}e^{(4)}_{TU} + e^{(57)}_{UV}e^{(15)}_{VT}e^{(4)}_{TU} + \\
& e^{(58)}_{UV}e^{(1)}_{VT}e^{(5)}_{TU} + e^{(59)}_{UV}e^{(2)}_{VT}e^{(5)}_{TU} + e^{(60)}_{UV}e^{(3)}_{VT}e^{(5)}_{TU} + e^{(61)}_{UV}e^{(4)}_{VT}e^{(5)}_{TU} + e^{(62)}_{UV}e^{(5)}_{VT}e^{(5)}_{TU} + \\
& e^{(63)}_{UV}e^{(6)}_{VT}e^{(5)}_{TU} + e^{(64)}_{UV}e^{(7)}_{VT}e^{(5)}_{TU} + e^{(65)}_{UV}e^{(8)}_{VT}e^{(5)}_{TU} + e^{(66)}_{UV}e^{(9)}_{VT}e^{(5)}_{TU} + e^{(67)}_{UV}e^{(10)}_{VT}e^{(5)}_{TU} + \\
& e^{(68)}_{UV}e^{(11)}_{VT}e^{(5)}_{TU} + e^{(69)}_{UV}e^{(12)}_{VT}e^{(5)}_{TU} + e^{(70)}_{UV}e^{(13)}_{VT}e^{(5)}_{TU} + e^{(71)}_{UV}e^{(14)}_{VT}e^{(5)}_{TU} + e^{(72)}_{UV}e^{(15)}_{VT}e^{(5)}_{TU} + \\
& e^{(73)}_{UV}e^{(1)}_{VT}e^{(6)}_{TU} + e^{(74)}_{UV}e^{(2)}_{VT}e^{(6)}_{TU} + e^{(75)}_{UV}e^{(3)}_{VT}e^{(6)}_{TU} + e^{(76)}_{UV}e^{(4)}_{VT}e^{(6)}_{TU} + e^{(77)}_{UV}e^{(5)}_{VT}e^{(6)}_{TU} + \\
& e^{(78)}_{UV}e^{(6)}_{VT}e^{(6)}_{TU} + e^{(79)}_{UV}e^{(7)}_{VT}e^{(6)}_{TU} + e^{(80)}_{UV}e^{(8)}_{VT}e^{(6)}_{TU} + e^{(81)}_{UV}e^{(9)}_{VT}e^{(6)}_{TU} + e^{(82)}_{UV}e^{(10)}_{VT}e^{(6)}_{TU} + \\
& e^{(83)}_{UV}e^{(11)}_{VT}e^{(6)}_{TU} + e^{(84)}_{UV}e^{(12)}_{VT}e^{(6)}_{TU} + e^{(85)}_{UV}e^{(13)}_{VT}e^{(6)}_{TU} + e^{(86)}_{UV}e^{(14)}_{VT}e^{(6)}_{TU} + e^{(87)}_{UV}e^{(15)}_{VT}e^{(6)}_{TU} + \\
& e^{(38)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} + e^{(18)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} + e^{(22)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} - e^{(7)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} - e^{(20)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} - \\
& e^{(33)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} - e^{(49)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} - e^{(52)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} - e^{(55)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} - e^{(65)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} - \\
& e^{(68)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} - e^{(71)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} - e^{(81)}_{UV}e^{(3)}_{VT}e^{(2)}_{TU} - e^{(84)}_{UV}e^{(1)}_{VT}e^{(3)}_{TU} - \\
& e^{(87)}_{UV}e^{(2)}_{VT}e^{(1)}_{TU} \qquad (4.3)
\end{aligned}
$$

MASENO UNIVERSITY
S.G. S. LIBRARY

4.3 Notation introduced

The potentials in 4.2 are too big, this section writes down the calculated potentials for quivers: A, B, C, D and E into sums using two notations namely: **sum notation** and the **grouped sum notation**. In the former, we label the arrows in a way that all the cycles in the potentials can be summed together, while in the latter, the arrows are labeled in a manner that similar arrows between any two vertices appearing in given summation can be put together. The grouped sum notation follows from the sum notation.

- For the original quiver A:

$$\begin{tikzcd} (T) \arrow[rr, "3"] & & (U) \arrow[dl, "3"] \\ & (V) \arrow[ul, "3"] & \end{tikzcd}$$

  the potential 3.1 can be written as:

$$\mathcal{S} = \sum_{\sigma \in \Omega} \mathrm{sgn}(\sigma) a_{UT}^{\sigma_1} a_{TV}^{\sigma_2} a_{VU}^{\sigma_3}$$

  Where σ is denoted by $(\sigma_1 \sigma_2 \sigma_3)$:

$\sigma \in \Omega$	(123)	(132)	(231)	(213)	(312)	(321)
$\mathrm{sgn}(\sigma)$	+	-	+	-	+	-

  This notation can be found in [21] where the sign of a permutation σ denoted $\mathbf{sgn}(\sigma)$ is defined as $+1$ if σ is even and -1 if σ is odd. It can be explicitly expressed as

$$sgn(\sigma) = (-1)^{\mu(\sigma)}$$

where $\mu(\sigma)$ is the number of inversions of σ.

- The potential 3.4 for quiver B

$$\begin{tikzcd}
(T) \arrow[dr, "3"'] & & (U) \arrow[ll, "6"'] \\
& (V) \arrow[ur, "3"'] &
\end{tikzcd}$$

obtained through mutating A at V can be written:

$$\mathcal{S} = \sum_{\sigma \in \Omega} a_{TU}^{\sigma_1} a_{UV}^{\sigma_2} a_{VT}^{\sigma_3} + \sum_{i=1}^{3} a_{TU}^{(i+3)} a_{UV}^{(i)} a_{VT}^{(i)}$$

We relabel the arrows:

$a_{TU}^{(4)}, a_{TU}^{(5)}, a_{TU}^{(6)}$ by $b_{TU}^{(1)}, b_{TU}^{(2)}, b_{TU}^{(3)}$

and the potential becomes:

$$\mathcal{S} = \sum_{\sigma \in \Omega} a_{TU}^{\sigma_1} a_{UV}^{\sigma_2} a_{VT}^{\sigma_3} + \sum_{i=1}^{3} b_{TU}^{(i)} a_{UV}^{(i)} a_{VT}^{(i)}$$

Note that while we had the sign of permutation in the original quiver A, after mutation at vertex V the sign of permutation is not there. We see this behavior every time we move from one mutation to another. We can thus say that our sign of permutation in the potential of a quiver alternates between $+$ and $-$ with successive mutations of that quiver.

Another outcome is that all the composites of length two through the vertex of mutation appear in the potential. like in the case above, all paths of length 2 through V are in this potential. This is also a phenomenon encountered in all the potentials obtained in this case of P^2.

- The potential 4.1 for quiver C below

$$\begin{tikzcd}
(T) \arrow[rr, "6"] & & (U) \arrow[dl, "3"] \\
& (V) \arrow[ul, "15"] &
\end{tikzcd}$$

which was obtained after mutation of B at vertex U can be written as:

$$\begin{aligned}
\mathcal{S} &= \sum_{\sigma\in\Omega} \mathrm{sgn}(\sigma) a_{TV}^{\sigma_1} a_{VU}^{\sigma_2} a_{UT}^{\sigma_3} + \sum_{i=1}^{3} a_{TV}^{(i+3)} a_{VU}^{(i)} a_{UT}^{(i)} \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3} a_{TV}^{(i+3j+3)} a_{VU}^{(j)} a_{UT}^{(i+3)} - \sum_{i=1}^{3} a_{TV}^{(4i+3)} a_{VU}^{(i+2)_3} a_{UT}^{(i+1)_3}
\end{aligned}$$

Where $(a)_n$ is the mapping of $a \in \mathbb{Z}$ into the equivalence classes $[a]$ in $\{[1], ..., [n]\}$ under the equivalence mod n.

For instance;

$a_{VU}^{(3)_3}$ is the arrow $a_{VU}^{(3)}$,

$a_{VU}^{(4)_3}$ is the arrow $a_{VU}^{(1)}$, and

$a_{VU}^{(5)_3}$ is the arrow $a_{VU}^{(2)}$ e.t.c

We relabel the arrows:

$a_{TV}^{(4)}, a_{TV}^{(5)}, a_{TV}^{(6)}$ by $b_{TV}^{(1)}, b_{TV}^{(2)}, b_{TV}^{(3)}$
$a_{TV}^{(7)}, ..., a_{TV}^{(15)}$ by $c_{TV}^{(1)}, c_{TV}^{(2)}, ..., c_{TV}^{(9)}$
$a_{UT}^{(4)}, a_{UT}^{(5)}, a_{UT}^{(6)}$ by $b_{UT}^{(1)}, b_{UT}^{(2)}, b_{UT}^{(3)}$

The potential thus becomes:

$$
\begin{aligned}
\mathcal{S} &= \sum_{\sigma\in\Omega} \operatorname{sgn}(\sigma) a_{TV}^{\sigma_1} a_{VU}^{\sigma_2} a_{UT}^{\sigma_3} + \sum_{i=1}^{3} b_{TV}^{(i)} a_{VU}^{(i)} a_{UT}^{(i)} \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3} c_{TV}^{(i+3j-3)} a_{VU}^{(j)} b_{UT}^{(i)} - \sum_{i=1}^{3} c_{TV}^{(4i-3)} a_{VU}^{(i+2)_3} a_{UT}^{(i+1)_3}
\end{aligned}
$$

We now make observe that the component $\sum_{i=1}^{3} b_{TV}^{(i)} a_{VU}^{(i)} a_{UT}^{(i)}$ which is equivalent to $\sum_{i=1}^{3} b_{TU}^{(i)} a_{UV}^{(i)} a_{VT}^{(i)}$ already encountered in the grouped potential for the underlying quiver, B features here! From now onwards, this element is recognizable in all the grouped potentials for the quivers in this chapter.

- Mutation of quiver C at V gave the quiver D:

$$
\begin{tikzcd}
(T) \arrow[dr, "15"'] & & (U) \arrow[ll, "39"'] \\
& (V) \arrow[ur, "3"'] &
\end{tikzcd}
$$

whose potential 4.2 can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} &= \sum_{\sigma\in\Omega} a_{TU}^{\sigma_1} a_{UV}^{\sigma_2} a_{VT}^{\sigma_3} + \sum_{i=1}^{3} a_{TU}^{(i+3)} a_{UV}^{(i)} a_{VT}^{(i)} \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3} a_{TU}^{(i+3j+3)} a_{UV}^{(j)} a_{VT}^{(i+3)} + \sum_{i=1}^{6}\sum_{j=1}^{3} a_{TU}^{(i+6j+9)} a_{UV}^{(j)} a_{VT}^{(i+9)} \\
&- \sum_{i=1}^{3} a_{TU}^{(4i+3)} a_{UV}^{(i+2)_3} a_{VT}^{(i+1)_3} + \sum_{i=1}^{3} a_{TU}^{(10i+7)} a_{UV}^{(i+1)_3} a_{VT}^{(i)} \\
&- \sum_{i=1}^{3} (a_{TU}^{(i+21)} + a_{TU}^{(i+30)}) a_{UV}^{(1)} a_{VT}^{(i+6)} + \sum_{i=1}^{3} a_{TU}^{(i+33)} a_{UV}^{(2)} a_{VT}^{(i+6)} \\
&+ \sum_{i=1}^{3} a_{TU}^{(i+36)} a_{UV}^{(3)} a_{VT}^{(i+6)}
\end{aligned}
$$

We relabel these arrows as follows:

$a_{TU}^{(4)}, a_{TU}^{(5)}, a_{TU}^{(6)}$ by $b_{TU}^{(1)}, b_{TU}^{(2)}, b_{TU}^{(3)}$
$a_{TU}^{(7)}, ..., a_{TU}^{(15)}$ by $c_{TU}^{(1)}, ..., c_{TU}^{(9)}$
$a_{VT}^{(4)}, a_{VT}^{(5)}, a_{VT}^{(6)}$ by $b_{VT}^{(1)}, b_{VT}^{(2)}, b_{VT}^{(3)}$
$a_{VT}^{(7)}, a_{VT}^{(8)}, a_{VT}^{(9)}$ by $c_{VT}^{(1)}, c_{VT}^{(2)}, c_{VT}^{(3)}$
$a_{VT}^{(10)}, ..., a_{VT}^{(15)}$ by $d_{VT}^{(1)}, ..., d_{VT}^{(6)}$
$a_{TU}^{(16)}, ..., a_{TU}^{(33)}$ by $d_{TU}^{(1)}, d_{TU}^{(2)}, ..., d_{TU}^{(18)}$
$a_{TU}^{(34)}, ..., a_{TU}^{(39)}$ by $e_{TU}^{(1)}, e_{TU}^{(2)}, ..., e_{TU}^{(3)}$

and the potential now becomes:

$$
\begin{aligned}
\mathcal{S} \;=\;& \sum_{\sigma\in\Omega} a_{TU}^{\sigma_1} a_{UV}^{\sigma_2} a_{VT}^{\sigma_3} + \sum_{i=1}^{3} b_{TU}^{(i)} a_{UV}^{(i)} a_{VT}^{(i)} \\
+\;& \sum_{i=1}^{3}\sum_{j=1}^{3} c_{TU}^{(i+3j-3)} a_{UV}^{(j)} b_{VT}^{(i)} + \sum_{i=1}^{6}\sum_{j=1}^{3} d_{TU}^{(i+6j-6)} a_{UV}^{(j)} d_{VT}^{(i)} \\
-\;& \sum_{i=1}^{3} c_{TU}^{(4i-3)} a_{UV}^{(i+2)_3} a_{VT}^{(i+1)_3} + \sum_{i=1}^{3} d_{TU}^{(10i-8)} a_{UV}^{(i+1)_3} a_{VT}^{(i)} \\
-\;& \sum_{i=1}^{3} (d_{TU}^{(i+6)} + d_{TU}^{(i+15)}) a_{UV}^{(1)} c_{VT}^{(i)} + \sum_{i=1}^{3} e_{TU}^{(i)} a_{UV}^{(2)} c_{VT}^{(i)} \\
+\;& \sum_{i=1}^{3} e_{TU}^{(i+3)} a_{UV}^{(3)} c_{VT}^{(i)}
\end{aligned}
$$

In this potential, we recognize elements $\sum_{\sigma\in\Omega} a_{TU}^{\sigma_1} a_{UV}^{\sigma_2} a_{VT}^{\sigma_3}$, $\sum_{i=1}^{3} b_{TU}^{(i)} a_{UV}^{(i)} a_{VT}^{(i)}$, $\sum_{i=1}^{3}\sum_{j=1}^{3} c_{TU}^{(i+3j-3)} a_{UV}^{(j)} b_{VT}^{(i)}$ and $\sum_{i=1}^{3} c_{TU}^{(4i-3)} a_{UV}^{(i+2)_3} a_{VT}^{(i+1)_3}$ equivalent to $\sum_{\sigma\in\Omega} \operatorname{sgn}(\sigma) a_{TV}^{\sigma_1} a_{VU}^{\sigma_2} a_{UT}^{\sigma_3}$, $\sum_{i=1}^{3} b_{TV}^{(i)} a_{VU}^{(i)} a_{UT}^{(i)}$, $\sum_{i=1}^{3}\sum_{j=1}^{3} c_{TV}^{(i+3j-3)} a_{VU}^{(j)} b_{UT}^{(i)}$ and $\sum_{i=1}^{3} c_{TV}^{(4i-3)} a_{VU}^{(i+2)_3} a_{UT}^{(i+1)_3}$ in the grouped potential of the underlying quiver C.

Lets find out these elements in the next mutation potential.

- The potential for the quiver E

$$
\begin{tikzcd}
(T) \arrow[dr, "15"'] & & (U) \arrow[ll, "6"'] \\
& (V) \arrow[ur, "87"'] &
\end{tikzcd}
$$

after mutating C at vertex T given in 4.3 can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{\sigma\in\Omega} a_{UV}^{\sigma_1} a_{VT}^{\sigma_2} a_{TU}^{\sigma_3} + \sum_{i=1}^{3} a_{UV}^{(i+3)} a_{VT}^{(i)} a_{TU}^{(i)} \\
+ \; & \sum_{i=1}^{12}\sum_{j=1}^{3} a_{UV}^{(i+12j-6)} a_{VT}^{(i+3)} a_{TU}^{(j)} + \sum_{i=1}^{15}\sum_{j=1}^{3} a_{UV}^{(i+15j+27)} a_{VT}^{(i)} a_{TU}^{(j+3)} \\
- \; & \sum_{i=1}^{3}\sum_{i=1}^{3} a_{UV}^{(3i+16j+30)} a_{VT}^{(i+2)_3} a_{TU}^{(i+1)_3} - \sum_{i=1}^{3} a_{UV}^{(13i-6)} a_{VT}^{(i+2)_3} a_{TU}^{(i+1)_3} \\
+ \; & a_{UV}^{(18)} a_{VT}^{(1)} a_{TU}^{(3)} + a_{UV}^{(22)} a_{VT}^{(2)} a_{TU}^{(1)} \\
+ \; & a_{UV}^{(38)} a_{VT}^{(3)} a_{TU}^{(2)}
\end{aligned}
$$

We relabel the arrows as follows:

$a_{UV}^{(4)}, a_{UV}^{(5)}, a_{UV}^{(6)}$ by $b_{UV}^{(1)}, b_{UV}^{(2)}, b_{UV}^{(3)}$

$a_{UV}^{(7)}, \ldots, a_{UV}^{(42)}$ by $c_{UV}^{(1)}, \ldots, c_{UV}^{(36)}$

$a_{VT}^{(4)}, \ldots, a_{VT}^{(15)}$ by $b_{VT}^{(1)}, \ldots, b_{VT}^{(12)}$

$a_{UV}^{(43)}, \ldots, a_{UV}^{(87)}$ by $d_{UV}^{(1)}, \ldots, d_{UV}^{(45)}$

$a_{TU}^{(4)}, a_{TU}^{(5)}, a_{TU}^{(6)}$ by $b_{TU}^{(1)}, b_{TU}^{(2)}, b_{TU}^{(3)}$

Giving the potential:

$$
\begin{aligned}
\mathcal{S} &= \sum_{\sigma\in\Omega} a_{UV}^{\sigma_1} a_{VT}^{\sigma_2} a_{TU}^{\sigma_3} + \sum_{i=1}^{3} b_{UV}^{(i)} a_{VT}^{(i)} a_{TU}^{(i)} \\
&+ \sum_{i=1}^{12}\sum_{j=1}^{3} c_{UV}^{(i+12j-12)} b_{VT}^{(i)} a_{TU}^{(j)} + \sum_{j=1}^{3}(\sum_{i=1}^{3} d_{UV}^{(i+15j-15)} a_{VT}^{(i)} b_{TU}^{(j)} \\
&+ \sum_{i=1}^{12} d_{UV}^{(i+15j-12)} b_{VT}^{(i)} b_{TU}^{(j)}) - \sum_{i=1}^{3}\sum_{i=1}^{3} d_{UV}^{(3i+16j-12)} a_{VT}^{(i+2)_3} a_{TU}^{(i+1)_3} \\
&- \sum_{i=1}^{3} c_{UV}^{(13i-12)} a_{VT}^{(i+2)_3} a_{TU}^{(i+1)_3} + d_{UV}^{(12)} a_{VT}^{(1)} a_{TU}^{(3)} \\
&+ d_{UV}^{(16)} a_{VT}^{(2)} a_{TU}^{(1)} + d_{UV}^{(32)} a_{VT}^{(3)} a_{TU}^{(2)}
\end{aligned}
$$

In this potential, we recognize three elements: $\sum_{\sigma\in\Omega} a_{UV}^{\sigma_1} a_{VT}^{\sigma_2} a_{TU}^{\sigma_3}$, $\sum_{i=1}^{3} b_{UV}^{(i)} a_{VT}^{(i)} a_{TU}^{(i)}$ and $\sum_{i=1}^{3} c_{UV}^{(13i-12)} a_{VT}^{(i+2)_3} a_{TU}^{(i+1)_3}$ equivalent to those discovered in quiver C. However, the element $\sum_{i=1}^{12}\sum_{j=1}^{3} c_{UV}^{(i+12j-12)} b_{VT}^{(i)} a_{TU}^{(j)}$ is different from that in the underlying quiver C, in terms of the number of arrows c and b and hence posing a threat to our notation!.

NB With the trend observed above, it is clear that we are able to recognize only part of the elements in the new potential. This means, with this notation, we are not able to predict **fully** the next mutated potential.

MASENO UNIVERSITY
S.G. S. LIBRARY

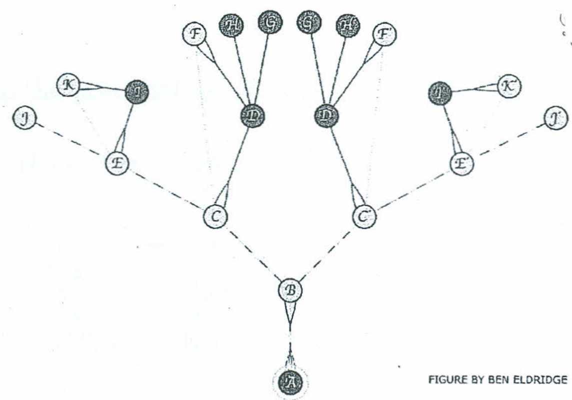
Chapter 5

Calculation of potentials for quivers with potential related to $P^1 \times P^1$

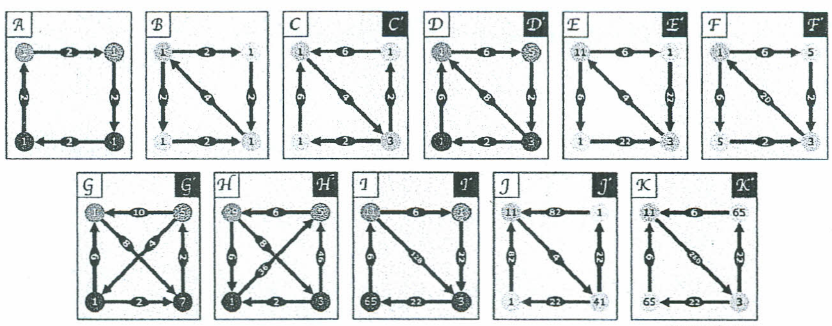
5.1 Introduction

This chapter deals with the family of quivers corresponding to quivers with potential in the tree diagram by Stern [18] which he related to $P^1 \times P^1$.

The tree diagram is:



The quivers corresponding to each of these points are:



Again, the potential for the central quiver in tree diagram A, is known from geometry and was stated in section 2.4. Mutation of this quiver and its mutants at various vertices yielded the tree diagram above. In other words, the tree diagram shows how one quiver is related to another through mutation. In section 5.2, we calculate the mutated potential for the quiver B, and then provide the worked out potentials for the quivers C, D, E, F and G. In section 5.3 we write all these potentials into sums.

5.2 Calculations

We calculate the potential for B using the steps in 3.2.

Consider the quiver A below:

$$\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[d, "2"] \\
(V) \arrow[u, "2"] & (W) \arrow[l, "2"]
\end{tikzcd}$$

with the potential:

$$\begin{aligned}
\mathcal{S} &= a^{(1)}_{VW}a^{(1)}_{WU}a^{(1)}_{UT}a^{(1)}_{TV} + a^{(1)}_{VW}a^{(2)}_{WU}a^{(2)}_{UT}a^{(1)}_{TV} + \\
&\quad a^{(2)}_{VW}a^{(1)}_{WU}a^{(1)}_{UT}a^{(2)}_{TV} + a^{(2)}_{VW}a^{(2)}_{WU}a^{(2)}_{UT}a^{(2)}_{TV}
\end{aligned} \tag{5.1}$$

Mutating it at W we get:

$$\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[dl, "4"] \\
(V) \arrow[u, "2"] \arrow[r, "2"'] & (W) \arrow[u, "2"']
\end{tikzcd}$$

Using the formula given in 3.2 for the unreduced potential we get:

$$\begin{aligned}
\tilde{\mathcal{S}} &= (a^{(1)}_{VW}a^{(1)}_{WU})a^{(1)}_{UT}a^{(1)}_{TV} + (a^{(1)}_{VW}a^{(2)}_{WU})a^{(2)}_{UT}a^{(1)}_{TV} + (a^{(2)}_{VW}a^{(1)}_{WU})a^{(1)}_{UT}a^{(2)}_{TV} + \\
&\quad (a^{(2)}_{VW}a^{(2)}_{WU})a^{(2)}_{UT}a^{(2)}_{TV} + (a^{(1)}_{VW}a^{(1)}_{WU})a^{(1)*}_{WU}a^{(1)*}_{VW} + (a^{(1)}_{VW}a^{(2)}_{WU})a^{(2)*}_{WU}a^{(1)*}_{VW} + \\
&\quad (a^{(2)}_{VW}a^{(1)}_{WU})a^{(1)*}_{WU}a^{(2)*}_{VW} + (a^{(2)}_{VW}a^{(2)}_{WU})a^{(2)*}_{WU}a^{(2)*}_{VW}
\end{aligned}$$

Note that we do not have 2-cycles in this expression, thus no reduction is needed.

Renaming the arrows:

$$b_{VU}^{(1)} := (a_{VW}^{(1)} a_{WU}^{(1)})$$

$$b_{VU}^{(2)} := (a_{VW}^{(1)} a_{WU}^{(2)})$$

$$b_{VU}^{(3)} := (a_{VW}^{(2)} a_{WU}^{(1)})$$

$$b_{VU}^{(4)} := (a_{VW}^{(2)} a_{WU}^{(2)})$$

We get the quiver B:

$$\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[dl, "4"] \\
(V) \arrow[u, "2"] \arrow[r, "2"'] & (W) \arrow[u, "2"']
\end{tikzcd}$$

having the potential:

$$\begin{aligned}
\overline{S} &= b_{VU}^{(1)} b_{UT}^{(1)} b_{TV}^{(1)} + b_{VU}^{(2)} b_{UT}^{(2)} b_{TV}^{(1)} + b_{VU}^{(3)} b_{UT}^{(1)} b_{TV}^{(2)} + \\
&\quad b_{VU}^{(4)} b_{UT}^{(2)} b_{TV}^{(2)} + b_{VU}^{(1)} b_{UW}^{(1)} b_{WV}^{(1)} + b_{VU}^{(2)} b_{UW}^{(2)} b_{WV}^{(1)} + \\
&\quad b_{VU}^{(3)} b_{UW}^{(1)} b_{WV}^{(2)} + b_{VU}^{(4)} b_{UW}^{(2)} b_{WV}^{(2)} \qquad (5.2)
\end{aligned}$$

Note that this resultant quiver B can be obtained by mutating A at the other three vertices except to see this as the same quiver some rotation of the resulting quivers will be required. There are four lines from A that rejoin before connecting to B. This shows that there are four ways of obtaining B from A.

The following are the worked out potentials for the other quivers obtained from mutations of B and some of its mutants.

- When we mutate the quiver B above at U we get the quiver C:

$$\begin{tikzcd}
(T) & (U) \arrow[l, "2"'] \arrow[d, "2"] \\
(V) \arrow[u, "6"'] \arrow[ur, "4"] & (W) \arrow[l, "6"]
\end{tikzcd}$$

with the potential:

$$\begin{aligned}
\overline{S} \;=\; & c^{(1)}_{VT}c^{(1)}_{TU}c^{(1)}_{UV} + c^{(2)}_{VT}c^{(1)}_{TU}c^{(3)}_{UV} + c^{(1)}_{VT}c^{(2)}_{TU}c^{(2)}_{UV} + \\
& c^{(2)}_{VT}c^{(2)}_{TU}c^{(4)}_{UV} + c^{(3)}_{VT}c^{(1)}_{TU}c^{(2)}_{UV} + c^{(4)}_{VT}c^{(2)}_{TU}c^{(1)}_{UV} + \\
& c^{(5)}_{VT}c^{(1)}_{TU}c^{(4)}_{UV} + c^{(6)}_{VT}c^{(2)}_{TU}c^{(3)}_{UV} + c^{(1)}_{VW}c^{(1)}_{WU}c^{(1)}_{UV} + \\
& c^{(2)}_{VW}c^{(1)}_{WU}c^{(3)}_{UV} + c^{(1)}_{VW}c^{(2)}_{WU}c^{(2)}_{UV} + c^{(2)}_{VW}c^{(2)}_{WU}c^{(4)}_{UV} + \\
& c^{(3)}_{VW}c^{(1)}_{WU}c^{(2)}_{UV} + c^{(4)}_{VW}c^{(2)}_{WU}c^{(1)}_{UV} + c^{(5)}_{VW}c^{(1)}_{WU}c^{(4)}_{UV} + \\
& c^{(6)}_{VW}c^{(2)}_{WU}c^{(3)}_{UV} \qquad (5.3)
\end{aligned}$$

Note that mutating the same quiver B at V gives a quiver C'. The difference between the primed and unprimed quivers is as explained in the P^2 case.

- When we again mutate the resulting quiver C above at T we get the quiver D:

$$\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[d, "2"] \arrow[dl, "8"'] \\
(V) \arrow[u, "6"] & (W) \arrow[l, "6"]
\end{tikzcd}$$

With the potential:

$$\begin{aligned}
\overline{S} \;=\; & d^{(1)}_{VU}d^{(1)}_{UT}d^{(1)}_{TV} + d^{(3)}_{VU}d^{(1)}_{UT}d^{(2)}_{TV} + d^{(2)}_{VU}d^{(2)}_{UT}d^{(1)}_{TV} + d^{(4)}_{VU}d^{(2)}_{UT}d^{(2)}_{TV} + \\
& d^{(2)}_{VU}d^{(1)}_{UT}d^{(3)}_{TV} + d^{(1)}_{VU}d^{(2)}_{UT}d^{(4)}_{TV} + d^{(4)}_{VU}d^{(1)}_{UT}d^{(5)}_{TV} + d^{(3)}_{VU}d^{(2)}_{UT}d^{(6)}_{TV} + \\
& d^{(5)}_{VU}d^{(1)}_{UT}d^{(4)}_{TV} + d^{(6)}_{VU}d^{(2)}_{UT}d^{(3)}_{TV} + d^{(7)}_{VU}d^{(1)}_{UT}d^{(6)}_{TV} + d^{(8)}_{VU}d^{(2)}_{UT}d^{(5)}_{TV} + \\
& d^{(1)}_{VW}d^{(1)}_{WU}d^{(1)}_{UT}d^{(1)}_{TV} + d^{(2)}_{VW}d^{(1)}_{WU}d^{(1)}_{UT}d^{(2)}_{TV} + d^{(1)}_{VW}d^{(2)}_{WU}d^{(2)}_{UT}d^{(1)}_{TV} + \\
& d^{(2)}_{VW}d^{(2)}_{WU}d^{(2)}_{UT}d^{(2)}_{TV} + d^{(3)}_{VW}d^{(1)}_{WU}d^{(1)}_{UT}d^{(3)}_{TV} + d^{(4)}_{VW}d^{(2)}_{WU}d^{(2)}_{UT}d^{(4)}_{TV} + \\
& d^{(5)}_{VW}d^{(1)}_{WU}d^{(1)}_{UT}d^{(5)}_{TV} + d^{(6)}_{VW}d^{(2)}_{WU}d^{(2)}_{UT}d^{(6)}_{TV}
\end{aligned} \tag{5.4}$$

- If we mutate the quiver C at vertex V we get the quiver E:

$$\begin{tikzcd}
(T) \arrow[r, "22"] & (U) \arrow[dl, "4"] \\
(V) \arrow[u, "6"] \arrow[r, "6"'] & (W) \arrow[u, "22"']
\end{tikzcd}$$

With the potential:

$$
\begin{aligned}
\overline{S} \;=\; & e^{(1)}_{UT}e^{(1)}_{TV}e^{(1)}_{VU} + e^{(2)}_{UT}e^{(2)}_{TV}e^{(1)}_{VU} + e^{(3)}_{UT}e^{(1)}_{TV}e^{(2)}_{VU} + e^{(4)}_{UT}e^{(2)}_{TV}e^{(2)}_{VU} + \\
& e^{(5)}_{UT}e^{(3)}_{TV}e^{(1)}_{VU} + e^{(6)}_{UT}e^{(4)}_{TV}e^{(1)}_{VU} + e^{(7)}_{UT}e^{(5)}_{TV}e^{(1)}_{VU} + e^{(8)}_{UT}e^{(6)}_{TV}e^{(1)}_{VU} + \\
& e^{(9)}_{UT}e^{(3)}_{TV}e^{(2)}_{VU} + e^{(10)}_{UT}e^{(4)}_{TV}e^{(2)}_{VU} + e^{(11)}_{UT}e^{(5)}_{TV}e^{(2)}_{VU} + e^{(12)}_{UT}e^{(6)}_{TV}e^{(2)}_{VU} + \\
& e^{(13)}_{UT}e^{(3)}_{TV}e^{(3)}_{VU} + e^{(14)}_{UT}e^{(4)}_{TV}e^{(3)}_{VU} + e^{(15)}_{UT}e^{(5)}_{TV}e^{(3)}_{VU} + e^{(16)}_{UT}e^{(6)}_{TV}e^{(3)}_{VU} + \\
& e^{(17)}_{UT}e^{(3)}_{TV}e^{(4)}_{VU} + e^{(18)}_{UT}e^{(4)}_{TV}e^{(4)}_{VU} + e^{(19)}_{UT}e^{(5)}_{TV}e^{(4)}_{VU} + e^{(20)}_{UT}e^{(6)}_{TV}e^{(4)}_{VU} + \\
& e^{(21)}_{UT}e^{(1)}_{TV}e^{(3)}_{VU} + e^{(22)}_{UT}e^{(1)}_{TV}e^{(4)}_{VU} + e^{(1)}_{UT}e^{(2)}_{TV}e^{(3)}_{VU} + e^{(3)}_{UT}e^{(2)}_{TV}e^{(4)}_{VU} + \\
& e^{(9)}_{UT}e^{(2)}_{TV}e^{(3)}_{VU} + e^{(19)}_{UT}e^{(2)}_{TV}e^{(3)}_{VU} + e^{(6)}_{UT}e^{(2)}_{TV}e^{(4)}_{VU} + e^{(16)}_{UT}e^{(2)}_{TV}e^{(4)}_{VU} + \\
& e^{(1)}_{UW}e^{(1)}_{WV}e^{(1)}_{VU} + e^{(2)}_{UW}e^{(2)}_{WV}e^{(1)}_{VU} + e^{(3)}_{UW}e^{(1)}_{WV}e^{(2)}_{VU} + e^{(4)}_{UW}e^{(2)}_{WV}e^{(2)}_{VU} + \\
& e^{(5)}_{UW}e^{(3)}_{WV}e^{(1)}_{VU} + e^{(6)}_{UW}e^{(4)}_{WV}e^{(1)}_{VU} + e^{(7)}_{UW}e^{(5)}_{WV}e^{(1)}_{VU} + e^{(8)}_{UW}e^{(6)}_{WV}e^{(1)}_{VU} + \\
& e^{(9)}_{UW}e^{(3)}_{WV}e^{(2)}_{VU} + e^{(10)}_{UW}e^{(4)}_{WV}e^{(2)}_{VU} + e^{(11)}_{UW}e^{(5)}_{WV}e^{(2)}_{VU} + e^{(12)}_{UW}e^{(6)}_{WV}e^{(2)}_{VU} + \\
& e^{(13)}_{UW}e^{(3)}_{WV}e^{(3)}_{VU} + e^{(14)}_{UW}e^{(4)}_{WV}e^{(3)}_{VU} + e^{(15)}_{UW}e^{(5)}_{WV}e^{(3)}_{VU} + e^{(16)}_{UW}e^{(6)}_{WV}e^{(3)}_{VU} + \\
& e^{(17)}_{UW}e^{(3)}_{WV}e^{(4)}_{VU} + e^{(18)}_{UW}e^{(4)}_{WV}e^{(4)}_{VU} + e^{(19)}_{UW}e^{(5)}_{WV}e^{(4)}_{VU} + e^{(20)}_{UW}e^{(6)}_{WV}e^{(4)}_{VU} + \\
& e^{(21)}_{UW}e^{(1)}_{WV}e^{(3)}_{VU} + e^{(22)}_{UW}e^{(1)}_{WV}e^{(4)}_{VU} + e^{(1)}_{UW}e^{(2)}_{WV}e^{(3)}_{VU} + e^{(3)}_{UW}e^{(2)}_{WV}e^{(4)}_{VU} + \\
& e^{(9)}_{UW}e^{(2)}_{WV}e^{(3)}_{VU} + e^{(19)}_{UW}e^{(2)}_{WV}e^{(3)}_{VU} + e^{(6)}_{UW}e^{(2)}_{WV}e^{(4)}_{VU} + \\
& e^{(16)}_{UW}e^{(2)}_{WV}e^{(4)}_{VU}
\end{aligned}
\tag{5.5}
$$

- Mutation of the quiver D above at W yields the quiver F:

$$
\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[dl, "20"] \\
(V) \arrow[u, "6"] \arrow[r, "6"'] & (W) \arrow[u, "2"']
\end{tikzcd}
$$

Whose potential is :

$$
\begin{aligned}
\overline{S} \; = \; & f^{(1)}_{VU} f^{(1)}_{UT} f^{(1)}_{TV} + f^{(2)}_{VU} f^{(2)}_{UT} f^{(1)}_{TV} + f^{(3)}_{VU} f^{(1)}_{UT} f^{(2)}_{TV} + f^{(4)}_{VU} f^{(2)}_{UT} f^{(2)}_{TV} + \\
& f^{(2)}_{VU} f^{(1)}_{UT} f^{(3)}_{TV} + f^{(1)}_{VU} f^{(2)}_{UT} f^{(4)}_{TV} + f^{(4)}_{VU} f^{(1)}_{UT} f^{(5)}_{TV} + f^{(3)}_{VU} f^{(2)}_{UT} f^{(6)}_{TV} + \\
& f^{(5)}_{VU} f^{(1)}_{UT} f^{(4)}_{TV} + f^{(6)}_{VU} f^{(2)}_{UT} f^{(3)}_{TV} + f^{(7)}_{VU} f^{(1)}_{UT} f^{(6)}_{TV} + f^{(8)}_{VU} f^{(2)}_{UT} f^{(5)}_{TV} + \\
& f^{(9)}_{VU} f^{(1)}_{UT} f^{(1)}_{TV} + f^{(10)}_{VU} f^{(1)}_{UT} f^{(2)}_{TV} + f^{(11)}_{VU} f^{(1)}_{UT} f^{(3)}_{TV} + f^{(13)}_{VU} f^{(1)}_{UT} f^{(5)}_{TV} + \\
& f^{(15)}_{VU} f^{(2)}_{UT} f^{(1)}_{TV} + f^{(16)}_{VU} f^{(2)}_{UT} f^{(2)}_{TV} + f^{(18)}_{VU} f^{(2)}_{UT} f^{(4)}_{TV} + f^{(20)}_{VU} f^{(2)}_{UT} f^{(6)}_{TV} + \\
& f^{(1)}_{VU} f^{(1)}_{UW} f^{(1)}_{WV} + f^{(2)}_{VU} f^{(2)}_{UW} f^{(1)}_{WV} + f^{(3)}_{VU} f^{(1)}_{UW} f^{(2)}_{WV} + f^{(4)}_{VU} f^{(2)}_{UW} f^{(2)}_{WV} + \\
& f^{(2)}_{VU} f^{(1)}_{UW} f^{(3)}_{WV} + f^{(1)}_{VU} f^{(2)}_{UW} f^{(4)}_{WV} + f^{(4)}_{VU} f^{(1)}_{UW} f^{(5)}_{WV} + f^{(3)}_{VU} f^{(2)}_{UW} f^{(6)}_{WV} + \\
& f^{(9)}_{VU} f^{(1)}_{UW} f^{(1)}_{WV} + f^{(10)}_{VU} f^{(1)}_{UW} f^{(2)}_{WV} + f^{(11)}_{VU} f^{(1)}_{UW} f^{(3)}_{WV} + f^{(12)}_{VU} f^{(1)}_{UW} f^{(4)}_{WV} + \\
& f^{(13)}_{VU} f^{(1)}_{UW} f^{(5)}_{WV} + f^{(14)}_{VU} f^{(1)}_{UW} f^{(6)}_{WV} + f^{(15)}_{VU} f^{(2)}_{UW} f^{(1)}_{WV} + f^{(16)}_{VU} f^{(2)}_{UW} f^{(2)}_{WV} + \\
& f^{(17)}_{VU} f^{(2)}_{UW} f^{(3)}_{WV} + f^{(18)}_{VU} f^{(2)}_{UW} f^{(4)}_{WV} + f^{(19)}_{VU} f^{(2)}_{UW} f^{(5)}_{WV} + \\
& f^{(20)}_{VU} f^{(2)}_{UW} f^{(6)}_{WV}
\end{aligned} \tag{5.6}
$$

- Further mutating the resulting quiver D above at U we get the quiver G:

$$
\begin{tikzcd}
(T) \arrow[d, "10"'] \arrow[dr, "4"] & (U) \arrow[l, "2"'] \\
(V) & (W) \arrow[l, "6"] \arrow[u, "2"']
\arrow[from=2-1, to=1-2, "8"']
\end{tikzcd}
$$

Whose potential has been found to be:

$$
\begin{aligned}
\overline{\mathcal{S}} \;=\;& g^{(1)}_{VT}g^{(1)}_{TU}g^{(1)}_{UV} + g^{(3)}_{VT}g^{(1)}_{TU}g^{(2)}_{UV} + g^{(2)}_{VT}g^{(1)}_{TU}g^{(3)}_{UV} + g^{(5)}_{VT}g^{(1)}_{TU}g^{(4)}_{UV} + \\
& g^{(4)}_{VT}g^{(1)}_{TU}g^{(5)}_{UV} + g^{(7)}_{VT}g^{(1)}_{TU}g^{(6)}_{UV} + g^{(6)}_{VT}g^{(1)}_{TU}g^{(7)}_{UV} + g^{(9)}_{VT}g^{(1)}_{TU}g^{(8)}_{UV} + \\
& g^{(4)}_{VT}g^{(2)}_{TU}g^{(1)}_{UV} + g^{(1)}_{VT}g^{(2)}_{TU}g^{(2)}_{UV} + g^{(6)}_{VT}g^{(2)}_{TU}g^{(3)}_{UV} + g^{(2)}_{VT}g^{(2)}_{TU}g^{(4)}_{UV} + \\
& g^{(8)}_{VT}g^{(2)}_{TU}g^{(5)}_{UV} + g^{(3)}_{VT}g^{(2)}_{TU}g^{(6)}_{UV} + g^{(10)}_{VT}g^{(2)}_{TU}g^{(7)}_{UV} + g^{(5)}_{VT}f^{(2)}_{TU}g^{(8)}_{UV} + \\
& g^{(1)}_{WT}g^{(1)}_{TU}g^{(1)}_{UW} + g^{(2)}_{WT}g^{(2)}_{TU}g^{(1)}_{UW} + g^{(3)}_{WT}g^{(1)}_{TU}g^{(2)}_{UW} + g^{(4)}_{WT}g^{(2)}_{TU}g^{(2)}_{UW} + \\
& g^{(1)}_{VW}g^{(1)}_{WT}g^{(1)}_{TU}g^{(1)}_{UV} + g^{(1)}_{VW}g^{(4)}_{WT}g^{(2)}_{TU}g^{(2)}_{UV} + g^{(2)}_{VW}g^{(1)}_{WT}g^{(1)}_{TU}g^{(3)}_{UV} + \\
& g^{(2)}_{VW}g^{(4)}_{WT}g^{(2)}_{TU}g^{(4)}_{UV} + g^{(3)}_{VW}g^{(1)}_{WT}g^{(1)}_{TU}g^{(2)}_{UV} + g^{(4)}_{VW}g^{(4)}_{WT}g^{(2)}_{TU}g^{(1)}_{UV} + \\
& g^{(5)}_{VW}g^{(1)}_{WT}g^{(1)}_{TU}g^{(4)}_{UV} + g^{(6)}_{VW}g^{(4)}_{WT}g^{(2)}_{TU}g^{(3)}_{UV}
\end{aligned}
\tag{5.7}
$$

5.3 Notation introduced

These potentials as the case was in 4.3 are very huge and as a result we use the notation introduced in the same section to write them briefly.

- The potential for the quiver A:

$$
\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[d, "2"] \\
(V) \arrow[u, "2"] & (W) \arrow[l, "2"]
\end{tikzcd}
$$

in 5.1 can be written in both notation as:

$$\mathcal{S} = \sum_{i,j=1}^{2} a^{(i)}_{VW} a^{(j)}_{WU} a^{(j)}_{UT} a^{(i)}_{TV}$$

- Mutation of quiver A at the vertex W yielded the quiver B:

$$\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[dl, "4"] \\
(V) \arrow[u, "2"] \arrow[r, "2"'] & (W) \arrow[u, "2"']
\end{tikzcd}$$

whose potential 5.2 can be written:

$$\mathcal{S} = \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UW}^{(j)} a_{WV}^{(i)}$$

Note the symmetry between T and W displayed by this potential. This is a very crucial behavior which is studied comprehensively by Owino in [14]. We find it in the potentials for quivers: C, E and F.

- The potential for the quiver C:

$$\begin{tikzcd}
(T) \arrow[d, "6"'] & (U) \arrow[l, "2"'] \arrow[d, "2"] \\
(V) \arrow[ur, "4"] & (W) \arrow[l, "6"]
\end{tikzcd}$$

in 5.3 can be written in the sum notation as:

$$\begin{aligned}
\mathcal{S} &= \sum_{i,j=1}^{2} a_{VT}^{(i)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} + \sum_{i,j=1}^{2} a_{VT}^{(2i+j)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)} \\
&+ \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WU}^{(j)} a_{UV}^{(2i+j-2)} + \sum_{i,j=1}^{2} a_{VW}^{(2i+j)} a_{WU}^{(j)} a_{UV}^{(2i-j+1)}
\end{aligned}$$

We rename the arrows as follows:

$a_{VT}^{(3)}, a_{VT}^{(4)}, a_{VT}^{(5)}, a_{VT}^{(6)}$ by $b_{VT}^{(1)}, b_{VT}^{(2)}, b_{VT}^{(3)}, b_{VT}^{(4)}$
$a_{VW}^{(3)}, a_{VW}^{(4)}, a_{VW}^{(5)}, a_{VW}^{(6)}$ by $a_{VW}^{(1)}, a_{VW}^{(2)}, a_{VW}^{(2)}, a_{VW}^{(2)}$

and the potential now reads:

$$
\begin{aligned}
\mathcal{S} &= \sum_{i,j=1}^{2} a_{VT}^{(i)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} + \sum_{i,j=1}^{2} b_{VT}^{(2i+j-2)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)} \\
&+ \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WU}^{(j)} a_{UV}^{(2i+j-2)} + \sum_{i,j=1}^{2} b_{VW}^{(2i+j-2)} a_{WU}^{(j)} a_{UV}^{(2i-j+1)}
\end{aligned}
$$

- The potential for Mutation of C at T gives the quiver D:

$$
\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[d, "2"] \arrow[dl, "8"'] \\
(V) \arrow[u, "6"] & (W) \arrow[l, "6"]
\end{tikzcd}
$$

whose potential 5.4 can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} &= \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UT}^{(j)} a_{TV}^{(2i+j)} \\
&+ \sum_{i,j=1}^{2} a_{VU}^{(2i+j+2)} a_{UT}^{(j)} a_{TV}^{(2i-j+3)} + \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WU}^{(j)} a_{UT}^{(j)} a_{TV}^{(i)} \\
&+ \sum_{i,j=1}^{2} a_{VW}^{(2i+j)} a_{WU}^{(j)} a_{UT}^{(j)} a_{TV}^{(2i+j)}
\end{aligned}
$$

Relabeling the arrows:

$a_{TV}^{(3)}, a_{TV}^{(4)}, a_{TV}^{(5)}, a_{TV}^{(6)}$ by $b_{TV}^{(1)}, b_{TV}^{(2)}, b_{TV}^{(3)}, b_{TV}^{(4)}$

$a_{VU}^{(5)}, a_{VU}^{(6)}, a_{VU}^{(7)}, a_{VU}^{(8)}$ by $b_{VU}^{(1)}, b_{VU}^{(2)}, b_{VU}^{(3)}, b_{VU}^{(4)}$

$a_{VW}^{(3)}, a_{VW}^{(4)}, a_{VW}^{(5)}, a_{VW}^{(6)}$ by $b_{VW}^{(1)}, b_{VW}^{(2)}, b_{VW}^{(3)}, b_{VW}^{(4)}$

We thus write the potential as:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UT}^{(j)} b_{TV}^{(2i+j-2)} \\
+ \; & \sum_{i,j=1}^{2} b_{VU}^{(2i+j-2)} a_{UT}^{(j)} b_{TV}^{(2i+j-2)} + \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WU}^{(j)} a_{UT}^{(j)} a_{TV}^{(i)} \\
+ \; & \sum_{i,j=1}^{2} b_{VW}^{(2i+j-2)} a_{WU}^{(j)} a_{UT}^{(j)} b_{TV}^{(2i+j-2)}
\end{aligned}
$$

Note that the symmetry we had in the original quiver C is lost but we have created paths of length 4.

- Mutation of quiver C at V gave the quiver E:

$$
\begin{tikzcd}
(T) \arrow[r, "22"] & (U) \arrow[dl, "4"] \\
(V) \arrow[u, "6"] \arrow[r, "6"'] & (W) \arrow[u, "22"']
\end{tikzcd}
$$

whose potential 5.5 can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{i,j=1}^{2} a_{UT}^{(2i+j-2)} a_{TV}^{(j)} a_{VU}^{(i)} + \sum_{i,j=1}^{4} a_{UT}^{(i+4j)} a_{TV}^{(i+2)} a_{VU}^{(j)} + \sum_{i=1}^{2} a_{UT}^{(i+20)} a_{TV}^{(1)} a_{VU}^{(i+2)} \\
+ \; & \sum_{i=1}^{2} a_{UT}^{(2i-1)} a_{TV}^{(2)} a_{VU}^{(i+2)} + (a_{UT}^{(9)} + a_{UT}^{(19)}) a_{TV}^{(2)} a_{VU}^{(3)} + (a_{UT}^{(6)} + a_{UT}^{(16)}) a_{TV}^{(2)} a_{VU}^{(4)} \\
+ \; & \sum_{i,j=1}^{2} a_{UW}^{(2i+j-2)} a_{WV}^{(j)} a_{VU}^{(i)} + \sum_{i,j=1}^{4} a_{UW}^{(i+4j)} a_{WV}^{(i+2)} a_{VU}^{(j)} + \sum_{i=1}^{2} a_{UW}^{(i+20)} a_{WV}^{(1)} a_{VU}^{(i+2)} \\
+ \; & \sum_{i=1}^{2} a_{UW}^{(2i-1)} a_{WV}^{(2)} a_{VU}^{(i+2)} + (a_{UW}^{(9)} + a_{UW}^{(19)}) a_{WV}^{(2)} a_{VU}^{(3)} + (a_{UW}^{(6)} + a_{UW}^{(16)}) a_{WV}^{(2)} a_{VU}^{(4)}
\end{aligned}
$$

Renaming the arrows of this potential:

$a_{UT}^{(5)}, a_{UT}^{(6)}, a_{UT}^{(7)}, \ldots, a_{UT}^{(20)}$ by $b_{UT}^{(1)}, \ldots, b_{UT}^{(16)}$

$a_{TV}^{(3)}, a_{TV}^{(4)}, a_{TV}^{(5)}, a_{TV}^{(6)}$ by $b_{TV}^{(1)}, b_{TV}^{(2)}, b_{TV}^{(3)}, b_{TV}^{(4)}$

$a_{UT}^{(21)}, a_{UT}^{(22)}$ by $c_{UT}^{(1)}, c_{UT}^{(2)}$

Since the potential is symmetrical, we apply the same relabeling to the other arrows in this potential where vertex T is replaced with W.

The potential in the grouped sum notation becomes:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{i,j=1}^{2} a_{UT}^{(2i+j-2)} a_{TV}^{(j)} a_{VU}^{(i)} + \sum_{i,j=1}^{4} b_{UT}^{(i+4j-4)} b_{TV}^{(i)} a_{VU}^{(j)} + \sum_{i=1}^{2} c_{UT}^{(i)} a_{TV}^{(1)} b_{VU}^{(i)} \\
& + \sum_{i=1}^{2} a_{UT}^{(2i-1)} a_{TV}^{(2)} b_{VU}^{(i)} + (b_{UT}^{(5)} + b_{UT}^{(15)}) a_{TV}^{(2)} b_{VU}^{(1)} + (b_{UT}^{(2)} + b_{UT}^{(12)}) a_{TV}^{(2)} b_{VU}^{(2)} \\
& + \sum_{i,j=1}^{2} a_{UW}^{(2i+j-2)} a_{WV}^{(j)} a_{VU}^{(i)} + \sum_{i,j=1}^{4} b_{UW}^{(i+4j-4)} b_{WV}^{(i)} a_{VU}^{(j)} + \sum_{i=1}^{2} c_{UW}^{(i)} a_{WV}^{(1)} b_{VU}^{(i)} \\
& + \sum_{i=1}^{2} a_{UW}^{(2i-1)} a_{WV}^{(2)} b_{VU}^{(i)} + (b_{UW}^{(5)} + b_{UW}^{(15)}) a_{WV}^{(2)} b_{VU}^{(1)} + (b_{UW}^{(2)} + b_{UW}^{(12)}) a_{WV}^{(2)} b_{VU}^{(2)}
\end{aligned}
$$

- Quiver F was obtained by mutating D at W:

$$
\begin{tikzcd}
(T) \arrow[r, "2"] & (U) \arrow[dl, "20"] \\
(V) \arrow[u, "6"] \arrow[r, "6"'] & (W) \arrow[u, "2"']
\end{tikzcd}
$$

its potential 5.6 can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} &= \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UT}^{(j)} a_{TV}^{(2i+j)} \\
&+ \sum_{i,j=1}^{2} a_{VU}^{(2i+j+2)} a_{UT}^{(j)} a_{TV}^{(2i-j+3)} + \sum_{i,j=1}^{2} a_{VU}^{(2i+j+10)} a_{UT}^{(j)} a_{TV}^{(i)} \\
&+ \sum_{i,j=1}^{2} a_{VU}^{(2i-j+17)} a_{UT}^{(j)} a_{TV}^{(2i+j)} + \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UW}^{(j)} a_{WV}^{(i)} \\
&+ \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UW}^{(j)} a_{WV}^{(2i+j)} + \sum_{i,j=1}^{2} a_{VU}^{(2i+j+2)} a_{UW}^{(j)} a_{WV}^{(2i-j+3)} \\
&+ \sum_{i,j=1}^{2} a_{VU}^{(2i+j+10)} a_{UW}^{(j)} a_{WV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i-j+17)} a_{UW}^{(j)} a_{WV}^{(2i+j)}
\end{aligned}
$$

Renaming the arrows of this potential:

$a_{TV}^{(3)}, a_{TV}^{(4)}, a_{TV}^{(5)}, a_{TV}^{(6)}$ by $b_{TV}^{(1)}, b_{TV}^{(2)}, b_{TV}^{(3)}, b_{TV}^{(4)}$

$a_{VU}^{(5)}, .., a_{VU}^{(8)}$ by $b_{VU}^{(1)}, .., b_{VU}^{(4)}$

$a_{VU}^{(13)}, ..., a_{VU}^{(16)}$ by $c_{VU}^{(1)}, ..., c_{VU}^{(4)}$

$a_{VU}^{(17)}, a_{VU}^{(18)}, a_{VU}^{(19)}, a_{VU}^{(20)}$ by $d_{VU}^{(1)}, d_{VU}^{(2)}, d_{VU}^{(3)}, d_{VU}^{(4)}$

Because of the symmetry between U and V of this quiver, we apply the renaming above to the remaining arrows of this potential where V is substituted with U.

The potential in the grouped sum notation as:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} + \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UT}^{(j)} b_{TV}^{(2i+j-2)} \\
+ & \sum_{i,j=1}^{2} b_{VU}^{(2i+j-2)} a_{UT}^{(j)} b_{TV}^{(2i-j+1)} + \sum_{i,j=1}^{2} c_{VU}^{(2i+j-2)} a_{UT}^{(j)} a_{TV}^{(i)} \\
+ & \sum_{i,j=1}^{2} d_{VU}^{(2i-j+1)} a_{UT}^{(j)} b_{TV}^{(2i+j-2)} + \sum_{i,j=1}^{2} a_{VU}^{(2i+j-2)} a_{UW}^{(j)} a_{WV}^{(i)} \\
+ & \sum_{i,j=1}^{2} a_{VU}^{(2i-j+1)} a_{UW}^{(j)} b_{WV}^{(2i+j-2)} + \sum_{i,j=1}^{2} b_{VU}^{(2i+j-2)} a_{UW}^{(j)} b_{WV}^{(2i-j+1)} \\
+ & \sum_{i,j=1}^{2} c_{VU}^{(2i+j-2)} a_{UW}^{(j)} a_{WV}^{(i)} + \sum_{i,j=1}^{2} d_{VU}^{(2i-j+1)} a_{UW}^{(j)} b_{WV}^{(2i+j-2)}
\end{aligned}
$$

- The potential for Mutation of D at U gave the quiver G:

$$
\begin{tikzcd}
(T) \arrow[d, "10"'] \arrow[dr, "8" near start] & (U) \arrow[l, "2"'] \\
(V) \arrow[ur, "4" near start] & (W) \arrow[l, "6"] \arrow[u, "2"']
\end{tikzcd}
$$

with potential 5.7 which can be written in the sum notation as:

$$
\begin{aligned}
\mathcal{S} \;=\; & \sum_{i,j=1}^{2} a_{WT}^{(2i+j-2)} a_{TU}^{(j)} a_{UW}^{(i)} + \sum_{i,j=1}^{2} a_{VT}^{(i)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} \\
+ & \sum_{i,j=1}^{2} a_{VT}^{(2i-j+3)} a_{TU}^{(j)} a_{UV}^{(2i+j+2)} + \sum_{i,j=1}^{2} a_{VT}^{(2i+j+4)} a_{TU}^{(j)} a_{UV}^{(2i-j+5)} \\
+ & \sum_{i,j=1}^{2} a_{VT}^{(2i+j)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)} + \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WT}^{(3j-2)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} \\
+ & \sum_{i,j=1}^{2} a_{VW}^{(2i+j)} a_{WT}^{(3j-2)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)}
\end{aligned}
$$

Relabeling the arrows:

$a_{VT}^{(3)}, a_{VT}^{(4)}, a_{VT}^{(5)}, a_{VT}^{(6)}$ by $b_{VT}^{(1)}, b_{VT}^{(2)}, b_{VT}^{(3)}, b_{VT}^{(4)}$

$a_{UV}^{(5)}, a_{UV}^{(6)}, a_{UV}^{(7)}, a_{UV}^{(8)}$ by $b_{UV}^{(1)}, b_{UV}^{(2)}, b_{UV}^{(3)}, b_{UV}^{(4)}$

$a_{VT}^{(7)}, a_{VT}^{(8)}, a_{VT}^{(9)}, a_{VT}^{(10)}$ by $c_{VT}^{(1)}, c_{VT}^{(2)}, c_{VT}^{(3)}, c_{VT}^{(4)}$

$a_{UV}^{(7)}, a_{UV}^{(8)}, a_{UV}^{(9)}, a_{UV}^{(10)}$ by $b_{UV}^{(1)}, b_{UV}^{(2)}, b_{UV}^{(3)}, b_{UV}^{(4)}$

$a_{VW}^{(3)}, a_{VW}^{(4)}, a_{VW}^{(5)}, a_{VW}^{(6)}$ by $b_{VW}^{(1)}, b_{VW}^{(2)}, b_{VW}^{(3)}, b_{VW}^{(4)}$

We thus write the potential in the grouped sum notation as:

$$\begin{aligned}
\mathcal{S} &= \sum_{i,j=1}^{2} a_{WT}^{(2i+j-2)} a_{TU}^{(j)} a_{UW}^{(i)} + \sum_{i,j=1}^{2} a_{VT}^{(i)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} \\
&+ \sum_{i,j=1}^{2} b_{VT}^{(2i-j+1)} a_{TU}^{(j)} b_{UV}^{(2i+j-2)} + \sum_{i,j=1}^{2} c_{VT}^{(2i+j-2)} a_{TU}^{(j)} b_{UV}^{(2i-j+1)} \\
&+ \sum_{i,j=1}^{2} b_{VT}^{(2i+j-2)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)} + \sum_{i,j=1}^{2} a_{VW}^{(i)} a_{WT}^{(3j-2)} a_{TU}^{(j)} a_{UV}^{(2i+j-2)} \\
&+ \sum_{i,j=1}^{2} b_{VW}^{(2i+j-2)} a_{WT}^{(3j-2)} a_{TU}^{(j)} a_{UV}^{(2i-j+1)}
\end{aligned}$$

Chapter 6

Summary and recommendations

In this work, we have calculated potentials for some quivers that Stern [18] related to P^2 and $P^1 \times P^1$. The number of terms in these potentials get too big with mutations. Even if an applet were to be used in doing the calculations, the potentials would still be big and thus difficult to understand. Our study has introduced some notation to write them briefly.

In using this notation, we discovered some interesting patterns in the potentials of the quivers in these families. For the P^2 family, certain patterns repeatedly occurred in the mutated potentials giving an idea of what the next mutated potential can be for a given quiver. This means that the notation used here is not the right one to use to fully predict the mutated potentials. We thus recommend that a study be done on a better notation to write these potentials. There is a relevant ongoing study by Crew and Velez in [2] which hopefully, might yield results that offer a better notation to write potentials.

The two families dealt with in this study are the simplest of the ten known del pezzo quivers. The fact that our notation has failed for these simple cases implies that there is much more work in studying the potentials for the other families namely: P^2 blown up to 1 point, P^2 blown up to 2 points, P^2 blown up to 3 points, P^2 blown up to 4 points, P^2 blown up to 5 points, P^2 blown up to 6 points, P^2 blown up to 7 points and P^2 blown up to 8 points.

References

[1] Andrew, W. H., *Lecture notes on Ringel-Hall algebras.* URL: http://www.maths.leeds.ac.uk/ ahubery/RHAlgs.pdf, 2007.

[2] Craw, A., Velez, A., *Cellular resolutions of noncummutative toric algebras form superpotentials.*arXiv:math/0401316v1[math.RT].23 jan 2004.

[3] Crawley-Boevey, W., *Lectures on representations of quivers.* URL: http://www.maths.leeds.ac.uk/ pmtwc/quivlecs.pdf., 1992.

[4] Derksen, H., Weyman, J. and Zelevinsky, A., *Quivers with potentials and their representations I:Mutations.* arXiv:0704.0649v2 [math.RA]. 18 april 2007.

[5] Du, J., *Quivers, their representations and quantum groups.* URL: http://www.math.virginia.edu/mathclub/Fall09/Nov6.pdf University of New South Wales, Sydney

[6] Fomin, S. and Zelevinsky, A., *Cluster algebras I: Foundations, J. Amer.* Math. Soc 15 (2002), 497-529.

[7] Fomin, S. and Zelevinsky, A., *Cluster algebras II: Finite type clasification, Invent.* Math. 154 (2003), 63-121.

[8] Gabriel, P., *Auslander-Reitensequences and representation-finite algebras.* Representation theory 1, Ottawa 1979. Lecture notes in Mathematics 831, Springer-verlag, Berlin/New-york 1980.

[9] Hall, B.C., *Lie Groups, Lie Algebras, and Representations*: An Elementary Introduction, Springer, 2003. ISBN 0-387-40122-9

[10] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory,* Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York, 1978. ISBN 0-387-90053-5

[11] Leites, D., Shchepochkina I., *Quivers and lie superalgebras* CZECHOSLOVAK JOURNAL OF PHYSICS Volume 47, Number 12, 1221-1229, DOI: 10.1023/A:1022873515587, 1997.

[12] Ringel, M.C., *From representations of quivers via Hall and Loewy algebras to quantum groups* URL: http://www.store4.math.uni-bielefeld.de/ ringel/opus/hall-via.pdf, 2005.

[13] Obiero, M.O., *Operation of Mutation on polar quivers.* MSc Thesis, Maseno University, Kenya. 2009.

[14] Owino, I. O., *Block quivers and their mutations.* MSC Thesis, Maseno University, Kenya. 2010.

[15] Ringel, C. M., *Hall algebras and quantum groups,* Invent. Math.101(1990):583-591,doi:10.1007/BF01231516,ISSN0020-9910, MR1062796.

[16] Savage, A., *Finite dimensional algebras and quivers.* arXiv:math/0505082v1 [math.RA]. 5 May 2005.

[17] Schiffmann, O., *Lectures on Hall algebras*, Course notes (ICTP, Trieste), arXiv:math/0611617, 2006.

[18] Stern, A.D., *A description of some autoequivalence classes of T-structures*. Ph.D Thesis, Sheffield University UK. 2008.

[19] Stern, A.D. and Obiero, M.O., *Arrows have a point!*, iSquared magazine, part 1, 2009.

[20] Stern, A.D. and Obiero, M.O., *Arrows have a point!*, iSquared magazine, part 2, 2009.

[21] Wikipedia, free encyclopedia *parity of permutation*, URL:http//en.wikipedia.org/wiki/parityofpermutation,Jan10,2001. Accessed on Jan 20,2010.