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Comparative Analysis of Spectral Theory of Second Order Difference and Differential Operators with Unbounded Odd Coefficient

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Abstract. We show that selfadjoint operator extensions of minimal second order difference operators have only discrete spectrum when the odd order coefficient is unbounded but grows or decays according to specific conditions. Selfadjoint operator extensions of minimal differential operator under similar growth and decay conditions on the coefficients have a absolutely continuous spectrum of multiplicity one.

1. Introduction

We consider the second order symmetric differential operators generated by

$$\tau y(x) = -(p_1(x)y'(x))' + i[q_1(x)y'(x) + (q_1(x)y(x))'] + p_0(x)y(x). \tag{1.1}$$

defined on $\mathcal{L}^2([0,\infty))$ and their discrete counterparts

$$\mathcal{L}y(t) = -\Delta[(p_1(t)\Delta y(t-1)] + i[q_1(t)\Delta y(t-1) + \Delta(q_1(t)y(t))] + p_0(t)y(t) \tag{1.2}$$

defined on $\ell^2(\mathbb{N})$. In (1.1), $y'(x)$ is the derivative of $y(x)$ with respect to x for $x \in [0,\infty)$ while in (1.2) $\Delta f(t) = f(t+1) - f(t)$ with $t \in \mathbb{N}$. Here, the coefficients p_0,

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p_1 and q_1, as functions of either x or t, are real valued functions that are either twice differentiable in the case of (1.1) or the second difference tends to zero as $t \to \infty$ for the case of (1.2). Throughout this text, the variable x will be assumed to be on the half-line $[0, \infty)$ while t will be assumed to be in $\mathbb{N}$. Many authors, including one of the authors in this paper (see the papers [4, 5] by Behncke and Nyamwala), have reported that second order operators have very similar spectral results and that significant differences can only be achieved in order four or more, and with unbounded coefficients. This conclusion is largely dependent on the analysis of order two operators with either bounded coefficients or unbounded even order coefficients. Actually, in Section 4.1 of [5], the degenerate second order case, conditions (4.1) and (4.13) imply that the results stated in Proposition 4.1 of the same reference is for bounded power coefficients. Even in the papers by Remling [7, 8], the analysis for the existence of absolutely continuous spectrum was done for even order coefficients, namely, the potential for the case of one-dimensional Schrödinger operators. His results included the Oracle theorem that predicts the potential and general results on the approach to certain limit potentials for the existence of absolutely continuous spectrum in the discrete case. For unbounded odd order coefficients, this is not the case as the results in this paper reveal. Here, we consider uniform growth conditions on the coefficients as follows:

$$|q_1(.)| \nearrow \infty, \qquad p_0, p_1 = o(q_1), \qquad \forall x \in [0, \infty) \text{ and } \forall t \in \mathbb{N}. \tag{1.3}$$

Further, we assume that the coefficients of (1.1) obey the following decay conditions.

$$\frac{f'}{f} \in \mathcal{L}^2, \quad \frac{f''}{f}, (\frac{f'}{f})^2 \in \mathcal{L}^1, \qquad f = p_0, p_1, q_1. \tag{1.4}$$

with their discrete counterparts, coefficients of (1.2), obeying

$$\frac{\Delta^2 f}{f}, \ \left(\frac{\Delta f}{f}\right)^2 \in \ell^1, \qquad \frac{\Delta f}{f} \in \ell^2, \qquad f = p_0, p_1, q_1. \tag{1.5}$$

In order to obtain deficiency indices and spectral results, we have solved the equations $\tau y(x) = zy(x)$ and $\mathcal{L}y(t) = zy(t)$ for (1.1) and (1.2) respectively. Here, z is the spectral parameter. Following Behncke and Hinton [3] as well as in the paper by Hinton and Schneider [6], we convert (1.1) into its first order system of the form

$$\mathcal{J}Y' = (\mathcal{A}z + \mathcal{B})Y, \ \mathcal{A} = \mathcal{A}^* > 0, \ \mathcal{B} = \mathcal{B}^*, \text{ with } \mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this particular case $\mathcal{A} = diag(1, 0)$. One, therefore, defines a symmetric operator T generated by (1.1) on a Hilbert space $\mathcal{L}^2_{\mathcal{A}}([0, \infty))$, the space of $\mathbb{C}^2$-valued $\mathcal{A}$-square integrable functions with scalar products given by

$$\langle g, h \rangle_{\mathcal{A}} = \int_0^\infty g^*(x)\mathcal{A}(x)h(x)dx.$$

So T is defined by $Ty = \mathcal{A}^{-1}(\mathcal{J}y' - \mathcal{B}y)$, and a regularity condition is required for the formal definition of T. If γ is a constant such that $\mathcal{J}y' - \mathcal{B}y = \gamma\mathcal{A}y$ for some y with $\|y\|_{\mathcal{A}} = 0$, then $y = 0$ and

$$\mathcal{J}y' - \mathcal{B}y = \mathcal{A}f, \text{ with } \|y\|_{\mathcal{A}} = 0, \ \|f\|_{\mathcal{A}} = 0. \tag{1.6}$$

This condition will hold for any constant γ and thus the deficiency indices of the minimal operator generated by (1.1) will be independent of the spectral parameter z. This regularity condition can be dropped by construction of a non-zero kernel of the operator generated by (1.1) and with added boundary conditions. The operator T is then constructed by restricting its resolvent to the orthogonal complement of the kernel. In line with Hinton and Schneider [6], let $y \in \mathcal{L}^2([0,\infty))$ then the maximal operator T^* generated by τ is defined by

$$D(T^*) = \left\{ y \in \mathcal{L}^2([0,\infty)) : \ y_1, y_2 \text{ are absolutely continuous in } [0,\infty) \right\}.$$

Here, we require that $\tau y \in \mathcal{L}^2([0,\infty)$ and $\tau y = T^* y$ for all $y \in D(T^*)$, and y_1 and y_2 are quasiderivatives of (1.1) as defined in Walker [9]. Restricting this domain to only functions of y with compact support within $[0,\infty)$ results in a pre-minimal operator whose closure is the required minimal operator that we will denote by T. If z is a spectral parameter with $\mathrm{Im} z > 0$, then define a set of indices (N_+, N_-) as the deficiency indices of T where $N_+ = \dim N_{T^* - \bar{z}}$ and $N_- = \dim N_{T^* - z}$ are the dimensions of the null spaces of $T^* - \bar{z}I$ and $T^* - zI$ respectively. By the von Neumann Theorems [10], if $N_- = N_+ \neq 0$, then T has selfadjoint operator H defined by

$$D(H) = D(T) \dotplus \{y + Vy : \ y \in N(T^* - zI)\},$$

where V is a uniquely determined isometric mapping such that $V : N(T^* - zI) \to N(T^* + zI)$. In the case of non-limit point case, boundary conditions are required at infinity. For more details, see [6].

A similar regularity condition is achieved for difference operators generated by (1.2) if we have a first order form of

$$\mathcal{J}\Delta Y(t)[z\mathcal{W}(t) + \mathcal{P}(t)]R(Y(t)), \ \ \mathcal{W}(t) = diag(1,0) \ \ \mathcal{P}(t) = \mathcal{P}^*(t),$$

where $Y(t) = (x(t), u(t))^{tr}$, tr means transpose, and $R(Y(t)) = (x(t+1), u(t))$. R is a partial shift operator, and $x(t)$ and $u(t)$ are quasi-differences [11]. Thus there exists an interval $I \subset \mathbb{N}$ such that for any complex number z and non-trivial solution $y(t)$ of (1.2),

$$\sum_{t \in I} R(y(t))^* \mathcal{W}(t) R(y(t)) > 0. \tag{1.7}$$

On the other hand the maximal difference operator generated by (1.2) is given by

$$D(L^*) = \big\{ y \in \ell^2([0,\infty)) : \ \text{there exists } \rho \in \ell^2([0,\infty)) \text{ such that } \\ \mathcal{J}\Delta Y(t) - \mathcal{P}(t)RY(t) = \mathcal{W}(t)\rho(t), \ \ t \in [0,\infty), \ \ L^*y = \mathcal{L}y \big\}.$$

Assume that for some natural number n, we restrict the domain of L^* using the boundary conditions such that $y(0) = y(t) = 0$, for all $t \geq n+1$, then we obtain a pre-minimal difference operator whose closure is a minimal difference operator. We denl denote this by L. Just like in the continuous case, one defines the deficiency indices and the selfadjoint operator extension of L whenever these indices are equal and L is not selfadjoint. For more details, see [11]. The deficiency indices and spectral results in this paper have been obtained through asymptotic integration for differential operators and asymptotic summation for difference operators. Asymptotic integration is based on a theorem of Levinson which states that if $Y'(x,z) = [\Lambda(x,z) + R(x,z)]Y(x,z)$ is a first order system of (1.1) such that $\Lambda = diag(\lambda_k(x,z))$ for $k = 1,2$ satisfies the z-uniform dichotomy condition and the elements of $R(x,z)$ are absolutely integrable, then the solutions of the system are of the form

$$y_k(x,z) = (e_k(x,z) + r_k(x,z)) \exp\left(\int_0^x \lambda_k(s,z)ds\right). \tag{1.8}$$

$e_k(x,z)$ is a normalised eigenvector while $r_k(x,z) = o(1)$ is the contribution to eigenvalue $\lambda_k(x,z)$ as a result of diagonalisation.

On the other hand, asymptotic summation is based on a theorem of Levinson-Benzaid-Lutz which states that if $Y(t+1,z) = [\Lambda(t,z) + R(t,z)]Y(t,z)$ is the first order system of (1.2) such that $\Lambda(t,z)$ satisfies the z-uniform dichotomy condition and elements of $R(t,z)$ are absolutely summable, then the solutions of (1.2) are of the form

$$y_k(t,z) = (e_k(t,z) + r_k(t,z)) \prod_0^{t-1} (\lambda_k(l,z)). \tag{1.9}$$

Our main results show that when $q_1(x)$ is unbounded and $\mid q_1(x) \mid^{-1}$ is not integrable, then the selfadjoint operator extension of the minimal differential operator generated by (1.1) has absolutely continuous spectrum, of multiplicity one, either contained on half line if $q_1(x) > 0$ or full real line if $q_1(x) < 0$. On the other hand, if $q_1(t)$ is unbounded and $\mid q_1(t) \mid^{-1}$ is not summable, then the selfadjoint operator extension of the minimal difference operator generated by (1.2) is pure discrete. These results, in addition to those in the cited references, settles in a general way, the problem of comparative analysis of spectral theory of second order difference operators and their continuous counterparts.

2. Results

Theorem 2.1. *Let T be the minimal differential operator generated by (1.1) on $\mathcal{L}^2[0,\infty)$ and assume that conditions (1.3), (1.4) and (1.6) are satisfied. Then*

(i) *Eigenvalues of (1.1) satisfy the uniform dichotomy condition.*

(ii) *If* $|q_1(x)|^{-1}$ *is integrable, then* $defT = (2,2)$ *and* $\sigma(H)$ *is discrete.*

(iii) *If* $|q_1(x)|^{-1}$ *is not integrable, then* $defT = (1,1)$*. Suppose* $q_1(x) > 0$ *then* $\sigma_{ac}(H) \subset [\bar{p}_0, \infty)$ *and if* $q_1(x) < 0$ *then* $\sigma_{ac}(H) = \mathbb{R}$ *with spectral multiplicity 1. Here* $\bar{p}_0 = lim\ sup\ p_0(x)$*.*

Proof. (i) Here, we apply asymptotic integration. This requires that (1.1) is converted into its first order form using quasiderivatives [9]. These are of the form:

$$y_1 = y(x) \qquad y_2 = p_1(x)y'(x) - iq_1(x)y(x).$$

This leads to a first order system of the form

$$Y' = AY,\ \ Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ \ A = \begin{bmatrix} \frac{iq_1}{p_1} & \frac{1}{p_1} \\ p_0 - \frac{q_1^2}{p_1} & i\frac{q_1}{p_1} \end{bmatrix}. \tag{2.1}$$

The coefficients are functions of x and p_0 should be interpreted as $p_0(x)-z$. Asymptotic integration in line with Levinson's theorem requires the eigenvalues of A. Computing the charateristic polynomial of A through $\det(A-\lambda I)$, multiplying the resultant polynomial by $-p_1$ and substituting λ with $-i\nu$, which is a unitary transformation and thus the spectrum is invariant, results into a Fourier polynomial of the form

$$\mathcal{P}_F(\nu, x, z) = p_1\nu^2 + 2q_1\nu + p_0. \tag{2.2}$$

There exists finitely many values of z where the roots of the polynomial (2.2) are repeated. The reader can refer to [1] to see how to handle the more generale case of $2n^{th}$ order operators and how to handle intervals where such polynomials have repeated roots. The remaining analysis is now restricted only to the interval where the two roots are distinct. Since we need to analyse the dichotomy condition, we will take $z = z_0 + i\eta$ where $z_0 = \text{Re}z$ and $\text{Im}z = \eta$, $\eta > 0$. One thus computes the ν roots of the polynomial and by backward substitution, obtains the eigenvalues λ which are analytic functions of x and z, approximately given by;

$$\lambda_1(x,z) = \frac{2iq_1}{p_1} - \frac{(p_0 - z_0)i}{2q_1} - \frac{\eta}{2q_1},\ \ \lambda_2(x,z) = \frac{(p_0 - z_0)i}{2q_1} + \frac{\eta}{2q_1}.$$

Here, $\text{Re}\lambda_1(x,z) = \frac{-\eta}{2q_1} + O(q_1^{-2})$, and $\text{Re}\lambda_2(x,z) = \frac{\eta}{2q_1} + O(q_1^{-2})$. In its simplest form, the z-uniform dichotomy condition requires that $\text{Re}(\lambda_1(x,z) - \lambda_2(x,z))$ is of constant sign modulo integrable terms. Even if $q_1(x) < 0$ or $q_1(x) > 0$, then either $Re\lambda_1(x,z) > 0$ or $Re\lambda_1(x,z) < 0$ respectively. A similar analysis is true for $Re\lambda_2(x,z)$. This implies that in each case of the sign of $q_1(x)$, one eigensolution will be bounded while the other is unbounded. This is the required dichotomy condition.

(ii) The first order system can now be diagonalised twice using eigenvectors. Approximately, the diagonalising matrix of A in (2.1) with only the leading terms is given by $M(x,z) = \begin{bmatrix} 1 & 1 \\ iq_1 & -iq_1 \end{bmatrix}$. Here, $\det M(x,z) = O(q_1(x))$. Using this matrix to diagonalise the system, by making a transformation of the form $Y(x,z) = M(x,z)W(x,z)$, we have

$$W'(x,z) = [\Lambda(x,z) + R(x,z)]W(x,z),$$

$$\Lambda(x,z) = diag(\lambda_1(x,z) + r_{11}(x,z), \lambda_2(x,z) + r_{22}(x,z)).$$

Here, $r_{11}(x,z) = O(q_1^{-1}(x))$, $r_{22}(x,z) = O(q_1^{-2}(x))$ are correction terms added to the diagonals as a result of diagonalisation. The remainder matrix $R(x,z)$ has $R_{jj}(x,z) = 0$, $j = 1,2$ while, $R_{jl} = O(f' \cdot q_1^{-1})$, $j,l = 1,2$, $j \neq l$. These terms are both $\mathcal{L}^2$ and $\mathcal{L}^1$ terms. A second diagonalisation is possible and for the details, see [2]. The deficiency indices can be read off from the asymptotics of the eigenvalue solution as $Imz \searrow 0$. The form of the solution is given by

$$y_j(x,z) = (e_j + r_j(x,z)).exp\left(\int_0^x \frac{\mp\eta}{|2q_1(s)|} ds\right), \quad j = 1,2.$$

Thus assume $|q_1(x)|^{-1}$ is integrable, then both the solutions are square integrable in the upper and lower half planes and hence results in $defT = (2,2)$. All the solutions are z-uniformly square integrable and hence discrete spectrum.

(iii) If $|q_1(x)|^{-1}$ is not integrable, then $y_1(x,z)$ is square integrable in the upper half plane if $q_1(x) > 0$ and fails to be square integrable in the lower half plane. $y_2(x,z)$ is square integrable in the lower half plane if $q_1(x) > 0$ but fails in the upper half plane. The situation is reversed if $q_1(x) < 0$. In each half plane with the appropriate sign of $q_1(x)$, $defT = (1,1)$. If $|q_1(x)|^{-1}$ is not integrable, then the correction term is given by $\frac{\mp\eta}{2|q_1(x)|}$ for $y_j(x,z)$, $j = 1,2$ solutions. Thus $y_1(x,z)$ is square integrable since $Re\lambda_1(x,z) = \frac{-\eta}{2|q_1(x)|}$, $\eta > 0$ but loses its square integrability as $\eta \longrightarrow 0^+$. In order to see this, note that

$$\|y_1\|^2 = c \cdot \lim_{x\to\infty} \exp\left(\int_0^x \frac{-\eta}{|q_1(s)|} ds\right),$$

for some positive constant c. This constant c is as result of the terms $e_1 + r_1(x,z)$ where e_1 is the normalized eigenvector and $r_1(x,z)$ is the correction term after diagonalisation and is bounded because of the assumptions in (1.3) and (1.4). Therefore, it is the exponential term that determines the boundedness of $\|y_1\|^2$. When $\eta > 0$ and as $x \to \infty$, the term $\|y_1\|^2$ decays to zero. But as $\eta \to 0$ from the right, the rate of decay of $\|y_1\|^2$ slowly decreases until it reaches the boundary point $\eta = 0$ where any small perturbation can easily make $\|y_1\|^2$ unbounded. Thus for $q_1(x) < 0$, it implies that $-\infty < z < \infty$, hence $\sigma_{ac}(H) = \mathbb{R}$ with spectral multiplicity 1. On the other hand, $y_2(x,z)$ is not integrable since $Re\lambda_2(x,z) = \frac{\eta}{2|q_1(x)|}$, $\eta > 0$. Thus for $q_1(x) > 0$, it implies that $\bar{p}_0 < z < \infty$, hence $\sigma_{ac}(H) \subset [\bar{p}_0, \infty)$, where $\bar{p}_0 = \limsup p_0(x)$. □

Theorem 2.2 *Let L be the minimal difference operator generated by (1.2) on $\ell^2[0,\infty)$ and assume that conditions (1.3), (1.5) and (1.7) are satisfied, then the eigenvalues of (1.2) satisfy the z-uniform dichotomy condition, the $def L = (1,1)$ and the spectrum is pure discrete.*

Proof. In this particular case, we apply asymptotics. This requires that (1.2) is converted into first order system. Here, we use quasi-differences as stated in [11]. We let $x(t) = y(t-1)$, $u(t) = p_1(t)(\Delta y(t-1)) - iq_1(t)y(t)$. Taking these as vector functions, we may assume, $Y(t) = \{x(t), u(t)\}^{tr}$. These lead to

$$\Delta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \frac{iq_1}{p_1} & \frac{1}{p_1} \\ p_0 - \frac{q_1^2}{p_1} & \frac{iq_1}{p_1} \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}. \tag{2.3}$$

The coefficients are functions of t with p_0 interpreted as $p_0(t) - z$. The form (2.3) is one of many ways of writing (1.2) in terms of its first order system and has been applied extensively in [11]. The form that is easily convertible to Levinson-Benzaid-Lutz form is given by,

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = \begin{bmatrix} S(t,z) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad S(t,z) = \begin{bmatrix} \frac{p_1}{p_1 - iq_1} & \frac{1}{p_1 - iq_1} \\ p_0 - \frac{q_1^2}{p_1} & 1 + \frac{iq_1}{p_1} \end{bmatrix} \tag{2.4}$$

For the eigenvalues of (1.2), we compute the characteristic polynomial of $S(t,z)$ given by $\mathcal{P}(t,\lambda,z) = \det(S(t,z) - \lambda I)$. Therefore,

$$\mathcal{P}(t,\lambda,z) = \lambda^2 - \lambda \left\{ 1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1} \right\} - \frac{p_0 - q_1^2}{p_1 - iq_1}.$$

By application of binomial expansion and approximating to $O(q_1^{-2}(t))$, we obtain λ-roots which are analytic functions of t and z:

$$\lambda_j(t,z) = \frac{1}{2}\left\{ \left(1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1} \right) \pm \frac{iq_1}{p_1} + O(q_1^{-2}) \right\}, \quad j = 1,2.$$

Explicitly, this implies that

$$\lambda_1(t,z) \approx \frac{iq_1}{p_1} + \frac{1}{2} + O(q_1^{-1}), \quad \lambda_2(t,z) \approx \frac{1}{2} + \frac{ip_1}{2q_1} + O(q_1^{-2}). \tag{2.5}$$

These two eigenvalues satisfy the z-uniform dichotomy condition. In its simplest form, dichotomy condition states that for any $\delta > 0$, however small, $|\frac{\lambda_1(t,z)}{\lambda_2(x,z)}|$ is either strictly greater than $1+\delta$ or strictly less than $1-\delta$. Since $|\lambda_1(t,z)| > 1$ for all $t \in \mathbb{N}$ because of (1.3) and $|\lambda_2(t,z)| < 1$, it follows that $|\frac{\lambda_1(t,z)}{\lambda_2(x,z)}| > 1+\delta$ which is the required uniform dichotomy condition.

The system can now be converted into Levinson's-Benzaid-Lutz form [4, 5] through diagonalisations. In this case, the diagonalising matrix, if the first component of the eigenvectors are normalised, is of the form $M(t,z) = \begin{bmatrix} 1 & 1 \\ \frac{q_1^2}{p_1} & -\frac{iq_1}{2} \end{bmatrix}$.

Even though the diagonalizing matrix is unbounded, its inverse is bounded. Here, the $(\det M(t,z))^{-1} = O(p_1(t)q_1^{-2}(t))$. The system is then transformed using

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = M(t,z)W(t,z).$$

After diagonalisation we have a first order of the form

$$W(t+1,z) = \Lambda_1(t,z) + R_1(t,z)]W(t,z),$$

$$\Lambda_1(t,z) = diag(\lambda_{1(t,z)} + \varrho_1(t,z), \lambda_2(t,z) + \varrho_2(t,z)).$$

The $\varrho_k(t,z)$ terms, $k = 1,2$, are obtained as a result of diagonalisations and are basically bounded and summable. The remainder matrix after the first diagonalisation, $R_1(t,z)$, has zeros in its main diagonal and the off diagonal terms are given by $(R_1)_{jl}(t,z) = O(q_1^{-1}(t)) \cdot \Delta f(t)$, $l,j = 1,2$, $j \neq l$, $f(t) = p_0(t)$, $p_1(t)$, $q_1(t)$. These are ℓ^2 and ℓ^1 terms by assumptions in (1.5). We now construct a matrix $\widetilde{S(t,z)}$ consisting of $\Lambda_1(t,z)$ and ℓ^2 terms from $R_1(t,z)$ then a second diagonalisation is possible using the eigenvectors of $\widetilde{S(t,z)}$. For more details, see [2]. After the second diagonalisation, the solutions will be given by the form (1.9). Thus the square summability of the eigensolutions are determined by

$$\lim_{t\to\infty} \langle y_k(t,z), y_k(t,z)\rangle \approx \lim_{t\to\infty} \Pi_{l=0}^{l=t-1} |\lambda_k(l,z)|^2; \quad k = 1,2.$$

This leads to $\lim_{t\to\infty} |y_1(t,z)|^2 = \infty$ and $\lim_{t\to\infty} |y_2(t,z)|^2 = 0$. This is because $|\lambda_1(t,z)| > 1$ and $|\lambda_2(t,z)| < 1$. A bounded solution implies that the solution is square summable and hence contribute to deficiency indices of L as shown by [11]. This will be true for $y_2(t,z)$ for all z both in the upper and lower halves of the complex plane. $\text{def}L = (1,1)$. The spectrum of all self-adjoint operator extension will consist of only eigenvalues and hence pure discrete. □

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