SOME ASPECTS IN THE STUDY OF INVARIANT SUBSPACES

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ABSTRACT

There is an outstanding problem in operator theory, the so-called "Invariant Subspace Problem" which has been open for more than half a century. The question is simple to state: does every bounded operator have a non-trivial invariant subspace? In spite of momentous efforts by functional analysist's, the problem continues to elude them even today. There have been significant achievements on occasions, sometimes after an interval of more than a decade in between, but its solution seems to be nowhere in sight. Our main objective in this thesis is to describe the treasure trove of the problem's heritage while we traverse the quest for a solution of the problem for some cases of operators. In chapter three we use a refinement of Lomonosov's theorem to show that; if an operator is bounded on a Banach space and there be a non-zero compact operator such that the commutator of the operator is of rank 1, then the operator has a non-trivial invariant subspace. We also show that if an operator on a Hilbert space is k-paranormal and has maximal vectors, then the operator has invariant subspaces. For the strong kind of invariance i.e. reduction we have proved that if a normal operator is either compact or its real part or imaginary part is a compact operator, then every invariant subspace of the operator is reducing. Finally in chapter three using invariant subspaces we obtain the structure of polynomially compact operators. For normal operators the spectral theorem yields many invariant subspaces. The step from normal to subnormal is, however, large: it is not known whether every subnormal operator is intransitive. In chapter four we consider a subnormal operator with a cyclic vector, in this case its extension space is simplified and we use the spectral decomposition of the normal extension of the operator to show that the subnormal operator has non trivial invariant subspaces. In this thesis we have thus solved the invariant subspace problem in the affirmative for the case of a commutator, a k-paranormal and a subnormal operator. Using invariant subspaces we have obtained the structure of polynomially compact operators on Banach spaces. At the same time we have provided an overview of the subject which even a specialist would relish. The results we have proved in this thesis give more insight into the invariant subspace problem and contribute immensely towards finding an affirmative answer to the problem. That existence theorem would be the first step towards a detailed structure theory for operators.

CHAPTER ONE

1.1 INTRODUCTION

The invariant subspace problem is the simple question: "Does every bounded operator T on a separable Hilbert space H over $\mathbb{C}$ have a non-trivial invariant subspace?" Here non-trivial subspace means a closed subspace of H different from $\{0\}$ and H. Invariant means that the operator T maps it to itself. The problem is easy to state, however, it is still open. The answer is 'no' in general for (separable) complex Banach spaces. For certain classes of bounded linear operators on complex Hilbert spaces, the problem has an affirmative answer. It seems unknown who first stated the problem. It apparently arose after Beurling [5] published his fundamental paper in Acta Mathematica in 1949 on invariant subspaces of simple shifts, or after von Neumann's unpublished result on compact operators which we shall discuss in the sequel.

Let H be any complex Hilbert space and T a bounded operator on H. An eigenvalue λ of T clearly yields an invariant subspace of T, namely the kernel of $T-\lambda I$. So if T has an eigenvalue, the problem is solved (the special case where T is multiplication by λ being trivial). However, not every bounded operator T on a complex Hilbert space has an eigenvalue. For example, the shift operator, T on ℓ^2, the Hilbert space of all square-summable sequences of complex numbers, defined by

$$Tx = (0, x_0, x_1, ...)$$

For each vector $x = (x_0, x_1, ...) \in \ell^2$, does not have any eigenvalue. However, if H is finite-dimensional, then of course every T on H has an eigenvalue, so the problem is solved for finite dimensional complex vector spaces.

Next, suppose H is infinite-dimensional but not separable. Let T be a bounded operator on H. Take a non-zero vector x and consider the closed subspace M generated by the vectors $\{x, Tx, T^2x, ...\}$. Then M is invariant under T and obviously $M \neq \{0\}$. Moreover, M does not coincide with H as this would contradict that H is non-separable. Thus every operator T on a non-separable infinite-dimensional complex Hilbert space H has a non-trivial invariant subspace. What remains to be examined is actually the invariant subspace problem; does every bounded operator T on an infinite dimensional separable complex Hilbert space H have a non-trivial invariant subspace?.

During the annual meeting of the American Mathematical Society in Toronto in 1976, the young Swedish mathematician Enflo [14] announced the existence of a Banach space and a bounded linear operator on it without any non-trivial invariant subspace. Enflo was visiting the University of California at Berkeley at that time. However, nothing appeared in print for several years and it was only in 1981 that he finally submitted a paper for publication in Acta Mathematica . Unfortunately the paper remained unrefereed with the referees for more than five years, though its manuscript had a world-wide circulation amongst mathematicians. This happened, as they say, because the paper was quite difficult and not well written. The paper was ultimately accepted in 1985 and it actually appeared in 1987 with only minor changes [14]. However, he had announced his construction of the counterexample earlier in the "Seminaire Maurey-Schwarz (1975-76)" and subsequently in the "Institute Mittag-Leffler Report 9 (1980)" (see [15] and [16]).

In the meantime, Read [35] following the ideas of Enflo [16] also constructed a counterexample and submitted it for publication in the Bulletin of the London Mathematical Society. A shorter version of this proof was published again by Read [35] in 1986. He also constructed in 1985 a bounded linear operator on the Banach space ℓ^1 without non-trivial invariant subspaces [36]. The temptation on the part of Read to have precedence over Enflo for solving the problem was considered professionally unethical by many mathematicians, because his work was essentially based on ideas of Enflo. For example, the French Mathematician Beauzamy [21] also sharpened the techniques of Enflo and produced a counterexample. He presented it at the Functional Analysis Seminar, University of Paris (VI-VII) in February, 1984. But he declined to publish his result in the Bulletin of the London Mathematical Society, although the Editors offered him the same facilities as they did to Read. Beauzamy's paper appeared later in June 1985 in Integral Equations and operator theory. The ℓ^1 example of [35] was further simplified by Davie [20] in (1988).

One should not get the impression that all counterexamples which have been produced so far are based directly or indirectly on the techniques developed by Enflo. As a matter of fact, a series of papers written by Read himself after his paper in 1984 makes a further significant contribution to the subject. For example, the counterexample that he constructed on ℓ^1 in 1985 is characteristically different from and simpler than Enflo's, and could be counted as a major achievement. Again, in yet another paper in 1988, Read constructed a bounded linear operator on ℓ^1 which has no invariant closed sets (let alone invariant subspaces) other than the trivial ones. Not only is this a stronger result, it also gives rise to a new situation; suppose that the invariant subspace problem is solved in the negative one day (as in the case of Banach spaces), one would ask a next question: "Does every bounded operator have a non-trivial invariant closed set?". Building on his earlier work, Read published in 1997 an example of a quasinilpotent bounded operator (i.e. $\lim \|T^n\|^{1/n} = 0$) on a Banach space without a non trivial invariant subspace. The same result is nicely described in [37].

Neumann [20] (unpublished) showed that every compact operator on a Hilbert space has a non trivial invariant subspace. The first proof of this result was published by Aronszajn and Smith [2] in 1954. The result was extended to polynomially compact operators by Bernstein and Robinson [6] in 1966 using techniques from non-standard analysis due to Robinson. Halmos [22] translated their proof into standard analysis. Interestingly, his paper appeared in the same issue of Pacific journal of Mathematics, just after theirs. In 1967, Arveson and Feldman [3] transformed the result in a still more general form by essentially chiseling the technique of Halmos; *if T is a quasinilpotent operator such that the uniformly closed algebra generated by T contains a non-zero compact operator, then T has a non-trivial invariant subspace.*

The result of Arveson and Feldman was, in a sense, the climax of the line of action initiated by Neumann. However, operator theorists were stunned in 1973 when the young Russian mathematician Lomonosov [30] obtained a more general result: *If a non-scalar bounded operator T on a Banach space commutes with a non-zero compact operator, then T has a non-trivial hyperinvariant subspace* (this means, a subspace which is invariant under every operator that commutes with T). This theorem was quite exciting for many reasons:

i. Lomonosov used a brand-new technique (namely, an ingenious use of Schauder's fixed point theorem) entirely different from the line of action followed hitherto by other mathematicians.

ii. His result was much stronger than what was known so far; every polynomially compact operator has a non-trivial invariant subspace.

iii. His theorem highlighted another, stronger, form of the 'invariant subspace problem" does every bounded linear operator on a Hilbert space have a non-trivial hyperinvariant subspace?".

iv. Many mathematicians tried to find alternative proofs of Lomonosov's theorem, say, by replacing the use of Schauder's fixed-point theorem by the Banach contraction principle, but the theorem stands as it was even today. M. Hilden, however, succeeded in proving its special case that every non-zero compact operator has a non-trivial hyperinvariant subspace without using any fixed point theorem. In fact, Hilden [31] assumed without any loss of generality a non-zero compact operator also to be quasinilpotent: if a non-zero compact operator is not quasinilpotent, then it must have a non-zero eigenvalue, and hence the eigenspace corresponding to this eigenvalue is a non-trivial hyperinvariant subspace. Hilden exploits the quasinilpotence of the compact operator to finish his proof.

v. Initially it was felt that Lomonosov's theorem might lead to a solution of the general 'invariant subspace problem' in the affirmative. However, seven years after his result, in 1980, Hadvin-Nordgren-Radjavi-Rosenthal [34] gave an example of an operator that does not commute with any non-zero compact operator.

A number of extensions and applications of Lomonosov's theorem have been obtained by several mathematicians. In chapter three we obtain a refinement of Lomonosov's theorem and use it to show that; if X is a Banach space and $T \in B(X)$ and there be a non-zero compact operator K such that $C = TK - KT$ is of rank 1, then T has a non-trivial invariant subspace. We also show if T is an operator on a Hilbert space H and T is k-paranormal and has maximal vectors, then T has invariant subspaces. For the strongest kind of invariance i.e reduction we have that; if $T \in B(H)$ is a hyponormal operator and M is an invariant subspace of T such that $S = TP$ (Where P is the orthogonal projector of H and M) is a normal operator, then M reduces T, using this we show that; if $T \in B(H)$ is normal and T is compact or $\operatorname{Re} T$ or $\operatorname{Im} T$ is a compact operator, then every

1.2 HERITAGES OF THE PROBLEM

For an operator T the lattice of all invariant subspaces of T ($\mathrm{Lat}\,T$) with set-inclusion as a partial order. For a general operator T, it is extremely difficult to describe $\mathrm{Lat}\,T$, particularly when we do not know whether there exists a bounded operator T for which $\mathrm{Lat}\,T$ is isomorphic to the lattice $\{0,1\}$ (this is the invariant subspace problem!). However, for certain special operators T, namely the shifts and the Volterra operators, the structure of $\mathrm{Lat}\,T$ is completely known. We now describe this, and discuss the role of shifts and their invariant subspaces in the structure theory of operators, as initiated by Rota [49].

Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H. The operator U on H such that $Ue_n = e_{n+1}, n = 0,1,2...$ is called the (forward) shift operator. A simple calculation shows that its adjoint S is the backward shift, given by $Se_0 = 0$ and $Se_n = e_{n-1}$ for $n \geq 1$. We shall be concerned with the following concrete representations of U and S. Let $L^2 = L^2(C,\mu)$ be the Hilbert space of all square-integrable functions defined on the unit circle C, where μ is the normalized Lesbesgue measure on C (i.e. $\mu(C)=1$).If for each integer n, $e_n = e_n(z) = z^n$, then $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis of L^2. The hardy space H^2 is the closed subspace of L^2 generated by the vectors. $\{e_0, e_1, e_2, ...\}$. We see that the multiplication by $e_1(z) = z$ on H^2 is U. As a second example let ℓ^2 be the Hilbert space of all square-summable complex sequences $x = (x_n)_{n=0}^{\infty}$. Then U and S on ℓ^2 appear as $Ux = (0, x_0, x_1, ...)$ and $Sx = (x_1, x_2, x_3, \ldots)$

1.2.1 Beurling's theorem and its ramifications:

In 1949, Beurling [5] characterized the invariant subspaces of the shift operator on the Hardy space H^2 on the unit circle C. His result is: *If M is an invariant subspace of the shift operator on the Hardy space H^2 on the unit circle C, then there exists an inner function ϕ on C (this means that ϕ is measurable and $|\phi(z)| = 1$ almost everywhere on C), such that $M = \phi H^2$. If both ϕ_1 and ϕ_2 are such functions, then ϕ_1 / ϕ_2 is equal to a* *constant function almost everywhere.* As Beurling's theorem showed an interplay between the theory of functions and the operator theory, it has naturally had numerous ramifications both in harmonic analysis and functional analysis. Mainly there have been three directions:

i. Replacing the Hardy space of scalar valued functions by the Hardy space of vector-valued functions;

ii. Extending Beurling's characterization to the Hardy space of scalar-valued functions on the torus;

iii. Viewing (i) and (ii) in the sense of de Branges [10], which puts Beurling's theorem as well as its vector-valued generalizations due to Halmos [23] and others in a more general setting.

1.2.2 Weighted shifts:

Shifts form an important class of operators. They have been rightly called the 'Building Blocks' of operator theory. Many important operators are, in a sense, 'made up' of shifts, for example, every pure isometry is a direct sum of shifts and every contraction with powers strongly tending to zero is a 'part' of a backward shift. More importantly, shifts serve as an unending source of counterexamples. Read uses a shift to construct his counterexample of a bounded operator on a Banach space without a non-trivial invariant subspace. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and let $w=\{w_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers. Consider the weighted forward shift T_w :

$$T_w e_n = w_{n+1} e_{n+1}, \quad n = 0,1,2,...$$

and the corresponding weighted backward shift S_w :

$$S_w e_0 = 0,$$
$$S_w e_n = \overline{w_n} e_{n-1}, \quad n = 1,2,3,...$$

A weighted forward shift is the adjoint of a weighted backward shift and vice versa. Note that a subspace M is invariant under an operator T if and only if its orthogonal complement $M^{\perp}$ is invariant under T^*. Hence determining LatS_w is equivalent to determining LatT_w. Let M_n denote the closed subspace spanned by $\{e_0, e_1, \cdots, e_n\}$. Then

$M_n \in LatS_w$ for all n. Under certain conditions on the weight sequence w, one can show that $\mathrm{LatS_w}$ consists of M_n's only: if a weight sequence $w = \{w_n\}_{n=1}^{\infty}$ is such that $\{|w_n|\}$ is monotonically decreasing and $\sum_{n=0}^{\infty} |w_n|^2 < \infty$, then every non-trivial invariant subspace in $\mathrm{Lat}\, S_w$ is some M_n. This result is due to Nikolskii [33]. The case $w_n = 2^{-n}$ was obtained in 1957 by Donoghue [11].

1.2.3 Volterra integral operators:

Consider the Volterra integral operator V defined on $L^2(0,1)$ by

$$(Vf)(x) = \int_0^x f(t)dt, \quad 0 \le x \le 1, \text{ for all } f \in L^2(0,1)$$

This operator is another one whose invariant subspaces have been characterized. For each $\alpha \in [0,1]$, let

$$M_\alpha = \{f \in L^2(0,1) : f = 0 \textit{ almost everywhere on} [0,\alpha]\}$$

Obviously $M_\alpha \in LatV$ for all $\alpha \in [0,1]$. In fact $LatV = \{M_\alpha : \alpha \in [0,1]\}$. This was proven by Dixmier [5] in case of the real space $L^2(0,1)$. Donoghue [11] and Brodskii [9] independently settled it in 1957for the complex space $L^2(0,1)$. These results have been extended to integral operators K on $L^2(0,1)$ defined by

$$(Kf)(x) = \int_0^x k(x,y)f(y)dy, \; 0 \le x \le 1, \text{ for all } f \in L^2(0,1),$$

where $K(x,y)$ is a square-integrable function on $[0,1]\times[0,1]$. The characterization of $\mathrm{Lat}\, K$ in this case may be used to obtain a functional-analytic proof of the famous classical Titchmarsh convolution theorem [28].

1.2.4 Rota's models of linear operators:

By a part of an operator T on a Hilbert space H, we mean the restriction $T|_M$ of T to an invariant subspace M of T. Let $\ell^2(0,1)$ denote the Hilbert space of all square-summable sequences $x = (x_0, x_1, \ldots, x_n, \ldots)$ in H. Take a bounded sequence $w = (w_n)$ of positive real numbers. The backward shift S_w on $\ell^2(H)$ is given by

$$S_w x = (w_1 x_1, w_2 x_2, \ldots, w_{n+1} x_{n+1}, \ldots)$$

Put $\beta_0 = 1$ and $\beta_n = w_1 w_2 \cdots w_n$, for $n \geq 1$. One has the following result: *Suppose T is a bounded operator on H and $\sum_{n=0}^{\infty} \beta_n^{-2} \|T^n\|^2 < \infty$. Then T is similar to a part of S_w on $\ell^2(H)$ in the following sense: define $A : H \to \ell^2(H)$ by*

$$Ax = \{\beta_0^{-1} x, \beta_1^{-1} Tx, \beta_2^{-1} T^2 x, \ldots\},$$

then the image M of A is closed and $S_w A = AT$. This implies M is an invariant subspace of S_w and T is similar to $S_w|_M$. If the spectral radius,

$$r(T) := \lim_{n \to \infty} \left(\|T^n\|^2\right)^{1/n}$$

of T is less than 1, then the conditions of the above result are satisfied for the constant sequence $w_n = 1$. This observation leads to the result Rota [39]: *If a bounded operator T on a Hilbert space H has spectral radius $r(T) < 1$, then T is similar to a part of the standard backward shift on $\ell^2(H)$. In particular, this holds for a strict contraction T, i.e. if $\|T\| < 1$.* Since any bounded operator can be 'scaled' so as to be a strict contraction, Rota's work yields a reformulation of the invariant subspace problem: Are the minimal non-zero invariant subspaces of backward shifts one-dimensional? More details on this work initiated by Rota may be found in [49].

CHAPTER TWO

LITERATURE REVIEW

In the field of mathematics known as functional analysis, one of the most prominent open problems is the invariant subspace problem, sometimes optimistically known as the invariant subspace conjecture. It is the question whether the following statement is true: *Given a complex Hilbert space H of dimension >1 and a bounded linear operator* $T: H \to H$*, then H has a non-trivial closed T-invariant subspace, i.e. there exists a closed linear subspace W of H which is different from* $\{0\}$ *and H such that* $T(W) \subseteq W$.

The statement is true for all finite-dimensional complex vector spaces of dimension at least 2: the eignevalues of a linear operator (matrix) are the zeros of its characteristic polynomial; this polynomial has zeros because of the fundamental theorem of algebra; a corresponding eigenvector will span an invariant subspace. The statement is true in the infinite-dimensional case if W is not required to be closed: pick any non-zero vector x in H and consider the subspace W of H spanned by $\{T^n(x): n \geq 0\}$. W is invariant. Moreover, W is a meager set in H and so by the Baire category theorem [4] must be distinct from H. While the general case of the conjecture is still open, several special cases have been settled:

i. The conjecture is true if the Hilbert space H is not separable (i.e. if it has an uncountable orthonormal basis). In fact, if x is a non-zero vector in H, the norm closure of the vector space generated by the infinite sequence $\{T^n(x): n \geq 0\}$ is separable and hence a proper subspace and also invariant.

ii. The spectral theorem shows that all normal operators admit invariant subspaces [20].

iii. Every compact operator has an invariant subspace, as proved by Aronszajn and Smith [2] in 1954. The theory of compact operators is in many ways similar to the theory of operators on a finite-dimensional space, so this result is not too surprising.

iv. Bernstein and Robinson [6] proved in 1966, using nonstandard analysis that if T^n is compact for some positive integer n, then T has an invariant subspace.

Halmos [26] subsequently provided a proof which did not rely on nonstandard methods.

v. Lomonosov [30] proved in 1973 that if T commutes with a non-zero compact operator then T has a hyperinvariant subspace. More generally he showed that if S commutes with a non-scalar operator T that commutes with a non-zero compact operator, then S has an hyperinvariant subspace.

In recent years, some mathematicians have attempted to construct counterexamples to the conjecture using the theory of random matrices. If one considers Banach spaces instead of Hilbert spaces, the conjecture becomes false; explicit examples of bounded operators without invariant subspaces have been exhibited by Enflo [16] (who in 1975 sketched out a construction, of which a revised version was produced in 1980 [15] and eventually published in 1987 [14]), however, the statement is true for certain classes of operators. In 1964, De Branges [10] published an alleged proof of the invariant subspace conjecture which was later found to be false.

Positive results are known for some special classes of operators. The cheapest way to get one is to invoke the spectral theorem and to conclude that normal operators always have non-trivial invariant subspaces. The earliest non-trivial result along these lines is the assertion that compact operators always have non-trivial invariant subspaces [2]. That result has been generalized ([6], [22], [30] and [31]), but the generalizations are still closely tied to compactness. Non-compact results are few; here is a sample; *if A is a contraction such that neither of the sequences $\{A^n\}$ and $\{A^{*n}\}$ tends strongly to 0, then A has a non-trivial invariant subspace* [32]. A bird's eye view of the subject is in [27], a more extensive bibliography is in [13] and a detailed treatment in [34].

It is helpful to approach the subject from a different direction; instead of searching for counter examples, study the structure of some non-counter examples. One way to do this is to fix attention on a particular operator and to characterize all its invariant subspaces; the first significant step in this direction is the work of Beurling (problem 157 in [19]). Nothing along these lines is easy. The second operator whose invariant subspaces have received detailed study is the Volterra integration operator ([9], [11], [28] and [40]). The results for it are easier to describe than for the shift, but harder to prove. If $(Vf)(x) = \int_0^\alpha f(y)dy$ for f in $L^2(0,1)$, and if, for each α in $[0,1]$, M_α is the subspace of those functions that vanish almost everywhere on $[0,\alpha]$ then M_α, is invariant under V; the principal result is that every invariant subspace of V is one of the M_α 's . An elegant way of obtaining these results is to reduce the study of the Volterra integration operator (as far as invariant subspaces are concerned) to that of the unilateral shift; this was done in [41].

The collection of all subspaces invariant under some particular operator is a lattice (closed under the formation of intersections and spans). One way to state the result about V is to say that its lattice of invariant subspaces is anti-isomorphic to the closed unit interval. ("Anti-" because as α grows M_α shrinks). The lattice of invariant subspaces of V^* is in an obvious way isomorphic to the closed unit interval. Is there an operator whose lattice of invariant subspaces is isomorphic to the positive integers? The question must be formulated with a little more care: every invariant subspace lattice has a largest element. The exact formulation is easy: is there an operator for which there is a one-to-one and order-preserving correspondence* $n \to M_n, n = 0,1,2,3,\cdots,\infty$ between the indicated integers (including ∞) and all invariant subspaces? The answer is yes. The first such operator was discovered by Donoghue [11]; a wider class of them is described in [33].

Suppose that $\{\alpha_n\}$ is a monotone sequence ($\alpha_n \geq \alpha_{n+1}, n = 0,1,2,\cdots$) of positive numbers ($\alpha_n > 0$) such that $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. The unilateral weighted shift with the weight sequence $\{\alpha_n\}$ will be called a monotone ℓ^2 shift. The span of the basis vectors $e_n, e_{n+1}, e_{n+2}, \cdots$ is invariant under such a shift, $n = 0,1,2,\cdots$. The orthogonal complement, i.e. the span M_n of $e_0,\cdots,e_{n-1}$, is invariant under the adjoint, $n = 1,2,3,\cdots$; the principal result is that every invariant subspace of that adjoint is one of these orthogonal complements.

2.1 BASIC PREREQUISITES

Definition 2.1.1

Let H be a Hilbert space. A map $\phi : H \times H \to \mathbb{C}$ is called a ***sesquilinear form*** on H if

i. $\phi(x+x',y)=\phi(x,y)+\phi(x',y)\forall x,x' \text{ and } y\in H$

ii. $\phi(\alpha x,y)=\alpha\phi(x,y)\forall x,y\in H \text{ and } \alpha\in\mathbb{C}$

iii. $\phi(x,y+y')=\phi(x,y)+\phi(x,y')\forall x,y,y'\in H$

iv. $\phi(x,\alpha y)=\overline{\alpha}\phi(x,y)\forall x,y\in H \text{ and } \alpha\in\mathbb{C}$.

A map $\hat{\phi}: H\to\mathbb{C}$ (H is a Hilbert space is called a ***quadratic form*** in H if there is a sesquilinear form $\phi : H\times H\to\mathbb{C}$ such that $\hat{\phi}(x)=\phi(x,x)\ \forall\ x\in H$

Lemma 2.1.2 (Polarization identity)

If $\hat{\phi}$ is the quadratic form associated with the sesquilinear form ϕ as above, then

$$\phi(x,y)=\frac{1}{4}\left[\hat{\phi}(x+y)-\hat{\phi}(x-y)+i\hat{\phi}(x+iy)-i\hat{\phi}(x-iy)\right] \qquad (2.1)$$

Proof:

Compute the right hand side of (2.1)

$$\hat{\phi}(x+y)=\phi(x+y,x+y)=\phi(x,x)+\phi(x,y)+\phi(y,x)+\phi(y,y).$$

$$i\hat{\phi}(x+iy)=i\phi(x+iy,x+iy)=i\left[\phi(x,x)+\phi(x,iy)+\phi(iy,x)+\phi(iy,iy)\right]$$

$$=i\left[\phi(x,x)-i\phi(x,y)+i\phi(y,x)-i^2\phi(y,y)\right]$$

$$=i\left[\phi(x,x)-i\phi(x,y)+i\phi(y,x)+\phi(y,y)\right].$$

$$-\hat{\phi}(x-y)=\phi(x-y,x-y)=-\phi(x,x)+\phi(x,y)+\phi(y,x)-\phi(y,y)$$

$$-i\hat{\phi}(x-iy)=-i\phi(x-iy,x-iy)=-i\left[\phi(x,x)+\phi(x,-iy)+\phi(-iy,x)+\phi(-iy,-iy)\right]$$

$$=-i\left[\phi(x,x)+i\phi(x,y)-i\phi(y,x)-i^2\phi(y,y)\right]$$

$$=-i\left[\phi(x,x)+i\phi(x,y)-i\phi(y,x)+\phi(y,y)\right].$$

Adding the four expressions, the right hand side of (2.1) reduces to $\phi(x,y)$

□

Lemma 2.1.3

If $\hat{\phi}, \hat{\psi}$ are the quadratic forms associated with the sesquilinear forms ϕ, ψ respectively then $\hat{\phi} = \hat{\psi} \Rightarrow \phi = \psi$

Proof:

Follows at once from Lemma 2.1.2 Indeed,

$$\phi(x,y) = \frac{1}{4}\left[\hat{\phi}(x+y) - \hat{\phi}(x-y) + i\hat{\phi}(x+iy) - i\hat{\phi}(x-iy)\right]$$

$$= \frac{1}{4}\left[\hat{\psi}(x+y) - \hat{\psi}(x-y) + i\hat{\psi}(x+iy) - i\hat{\psi}(x-iy)\right]$$

$$= \psi(x,y)$$

□

Proposition 2.1.4

Let X, Y be normed linear spaces over $\mathbb{C}$ and $B(X,Y)$ be the set of all bounded linear transformations (operators) on X into Y. If $T, S \in B(X,Y)$ and $\alpha \in \mathbb{C}$ define $T+S$, αT by

$$(T+S)(x) = T(x) + S(x) \; \forall x \in X$$

$$(\alpha T)(x) = \alpha Tx \quad \forall x \in X \text{ and } \forall \alpha \in \mathbb{C}$$

then $T+S$, αT are both in $B(X,Y)$ and

$$\|T+S\| \leq \|T\| + \|S\| \text{ and } \|\alpha T\| = |\alpha|\|T\|.$$

The function $\|\cdot\| : T \to \|T\|$ is a norm in the linear space $B(X,Y)$. Thus $(\mathrm{B}(X,Y), \|\cdot\|)$ is a normed linear space over $\mathbb{C}$.

For a proof see [4].

Proposition 2.1.5

If Y is a Banach Space; then $(\mathrm{B}(X,Y), \|\cdot\|)$ is also a Banach Space.

Proof:

Let (T_n) be Cauchy in the normed linear space $(\mathrm{B}(X,Y), \|\cdot\|)$. We must show that $(T_n) \xrightarrow{\|\cdot\|} T$ for a $T \in B(X,Y)$. For any $x \in H$ we have (for all m, n $\in \mathbb{N}$)

$$\|T_n(x)-T_m(x)\| = \|(T_n-T_m)x\| \le \|T_n-T_m\|\|x\|$$

Since (T_n) is Cauchy with respect to $\|.\|$, $\|T_n-T_m\|\to 0$ as both $m,n\to\infty$. Thus for each fixed $x\in X$

$$\|T_n(x)-T_m(x)\|\to 0 \text{ as both } m,n\to\infty$$

Thus for each fixed $x\in X$

$$\|T_n(x)-T_m(x)\|\to\infty \text{ as both } m,n\to\infty$$

i.e $(T_n(x))_{n=1}^{\infty}$ is strongly Cauchy in the Banach space Y (for each $x\in X$). By strong completeness of Y, $s-\lim_{n\to\infty} T_n(x)$ exists in Y, call this limit $T(x)$ i.e.

$$T(x) = s-\lim_{n\to\infty} T_n(x) \quad \forall x\in X.$$

Clearly

i. T is linear on X into Y.

ii. $T\in B(X,Y)$

iii. $T_n \xrightarrow{\|\cdot\|} T$ i.e $\lim_{n\to\infty}\|T_n-T\|=0$.

□

Suppose $Y=X$. In this case consider $T_1,T_2\in B(X)$ and the composition T_2T_1 defined by

$$T_2T_1(x)=T_2(T_1(x))\,\forall x\in X.$$

Clearly

i. $T_2T_1 : X\to X$ is linear

ii. $T_2T_1\in B(X)(=B(X,X))$.(see [13]).

It can be verified that $\forall T_1,T_2,T_3\in B(X)$

$$T_1(T_2+T_3)=T_1T_2+T_1T_3$$

$$T_1(\alpha T_2)=\alpha(T_1T_2)=(\alpha T_1)(T_2)$$

see [20]. In general T_1T_2 need not be the same as T_2T_1. Thus $B(X)$ is what we call an ***algebra*** (which need not be commutative). $(B(X),+,\cdot,\|\cdot\|)$ is a normed algebra over $\mathbb{C}$.

Proposition 2.1.6 (Banach Steinhauss Theorem)

Let X, Y be Banach spaces and $\{T_\alpha : \alpha \in \Lambda\}$ be a family of elements of $B(X,Y)$ such that at each $x \in X$, $\{T_\alpha(x) : \alpha \in \Lambda\}$ is bounded; i.e. there exists a positive real β_x such that $\|T_\alpha(x)\| \le \beta_x \ \forall \alpha \in \Lambda$ then $\{\|T_\alpha\| : \alpha \in \Lambda\}$ is bounded. (Uniform boundedness theorem). See [4]

Proposition 2.1.7 (Projection theorem)

Let M be a closed linear subspace of a Hilbert space H then $H = M \oplus M^\perp$, where $M^\perp = \{y \in H : < y, x >= 0 \ \forall x \in H\}$. See [4]

Proposition 2.1.8 (Riesz Representation Theorem for Hilbert spaces)

Let H be a Hilbert space and $f \in H^*$ i.e f is a bounded linear functional on H. There exists a unique element $y_f \in H$ such that

$$f(x) = \langle x, y_f \rangle \quad \forall x \in H$$

moreover

$$\|f\| = \|y_f\| \text{ (see [4]).}$$

Let H, K be Hilbert spaces (over $\mathbb{C}$) and T be a bounded linear operator i.e $T \in B(H,K)$. Define a map $f_y : H \to \mathbb{C}$ by (for any fixed $y \in K$)

$$f_y(x) = \langle Tx, y \rangle.$$

Clearly, f_y is a linear functional on H and f_y is bounded (see [4]). Thus f_y is a bounded linear functional on H. Hence by proposition 2.1.8

$$f_y(x) = \langle x, y^* \rangle \quad \forall x \in H$$

for a unique y^* in H and we have $\|f_y\| = \|y^*\|$. Thus

$$f_y(x) = \langle Tx, y \rangle = \langle x, y^* \rangle$$

Consider the map $T^* : K \to H$ defined by $T^*(y) = y^*$. It follows that

i. T^* is linear.

ii. T^* is bounded and $\|T^*\| \le \|T\|$.

Definition 2.1.9

If $T \in B(H,K)$, then there exists a unique $T^* \in B(K,H)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ and \ \|T^*\| \le \|T\|.$$

T^* is called the (Hilbert) ***adjoint*** of T. Thus any $T \in B(H,K)$ is associated with a unique $T^* \in B(K,H)$ (called the adjoint of T) and $\|T^*\| \le \|T\|$. We apply the above arguments to $T^* \in B(K,H)$. $T \in B(H)$ is said to be ***self-adjoint*** if $T = T^*$. Clearly

$$\langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, T^{**}x\rangle} = \langle T^{**}x, y\rangle$$

where $T^{**} \in B(H,K)$ is the adjoint of T^* i.e. $T^{**} = (T^*)^*$ and $\|T^{**}\| \le \|T^*\|$. Also

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle T^{**}x, y\rangle \quad \forall x \in H \ and \ \ y \in K$$

thus

$$\langle Tx - T^{**}x, y\rangle = 0 \ \forall x \in H \ and \ y \in K$$

i.e.

$$Tx - T^{**}x = 0 \ \ \forall x \in H \ \text{ i.e. } T^{**} = T \Rightarrow \ \|T^{**}\| = \|T\|.$$

Since

$$\|T\| = \|T^{**}\| \le \|T^*\| \le \|T\|$$

equality holds throughout, thus

$$\|T^{**}\| = \|T\|.$$

H is of interest in operator theory to deal with the situation when $K = H$. Thus for every $T \in B(H)$ there is a unique $T^* \in B(H)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ \forall x, y \in H$$

and also $\|T^*\| = \|T\|$.

Definition 2.1.10

An operator $T \in B(H)$ is said to be ***invertible*** if there exists an operator $S \in B(H)$ such that $ST = TS = I_H$. Where I_H is the identity map on H.

Proposition 2.1.11

Let H be a Hilbert space and $S,T \in B(H)$. The following hold:

i. $0^* = 0$; where 0 is the zero operator $0x = \overline{0}\ \forall x \in H$

ii. $I^* = I$ where I is the identity operator on H

iii. $(S+T)^* = S^* + T^*$

iv. $(\alpha T)^* = \overline{\alpha} T^*\ \ \forall \alpha \in \mathbb{C}$

v. $(ST)^* = T^* S^*$

vi. T is invertible if and only if T^* is invertible and then $(T^*)^{-1} = (T^{-1})^{\alpha}$.

See [4].

Lemma: 2.1.12

Let $T \in B(H)$. Then $\|T^*T\| = \|T\|^2$

Proof:

Clearly

$$\|T^*T\| \le \|T^*\|\|T\| = \|T\|T\|\| = \|T\|^2$$

i.e

$$\|T^*T\| \le \|T\|^2 \qquad (2.2).$$

On the other hand

$$0 \le \|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \le \|T^*Tx\|\|x\| \le \|T^*T\|\|x\|^2 \quad (\text{since } T^*T \in B(H)).$$

Hence

$$\|Tx\| \le \|T^*T\|^{\frac{1}{2}}\|x\|\ \forall x \in H$$

i.e.

$$\|T\| \le \|T^*T\|^{\frac{1}{2}} \quad i.e. \|T\|^2 \le \|T^*T\| \qquad (2.3).$$

(2.2) and (2.3) imply

$$\|T^*T\| \le \|T\|^2 \le \|T^*T\| \text{ i.e } \|T^*T\| = \|T\|^2$$

□

Remarks:

1. $\left(T^{*}T\right)^{*}=T^{*}\left(T^{*}\right)^{*}=T^{*}T^{**}=T^{*}T$

2. $\left\|TT^{*}\right\|=\left\|\left(T^{*}\right)^{*}T^{*}\right\|=\left\|T^{*}\right\|^{2}=\left\|T\right\|^{2}$

Definition 2.1.13

$T\in B(H)$ is called a ***contraction*** *if* $\|T\|\leq 1$

Definition 2.1.14

An operator T in H (with domain as the linear subspace D_T) is said to be ***bounded from below*** if there is a positive constant β such that $\|Tx\|\geq\beta\|x\| \quad \forall x\in D_T$

Remark:

T is bounded from below $\Rightarrow T$ is one-to-one, i.e. T is injective for let $Tx=\bar{0}$ so $\|Tx\|=0$, therefore $0\geq\beta\|x\|$ *and* $\beta>0$, thus $\|x\|=0\Leftrightarrow x=\bar{0}$. Thus $Tx=\bar{0}\Rightarrow x=\bar{0}$ and T is linear, therefore T is one-to- one.

Proposition 2.1.15

Let $T\in B(H)$ and be bounded from below. Then range of T is closed.

Proof:

Clearly $\Re_T\subseteq\overline{\Re_T}$. We must show that $\overline{\Re_T}\subseteq\Re_T$, whence $\overline{\Re_T}=\Re_T$. Let $y\in\overline{\Re_T}$. Then there exists a sequence (y_n) of elements of $\Re_T$ such that $y_n\xrightarrow{s}y$. Thus (y_n) is strongly Cauchy in H. Since T is bounded from below, T is 1-1, and hence for each y_n unique $x_n\in H$ such that $Tx_n=y_n$. Thus (Tx_n) is strongly Cauchy. Now

$$\|Tx_m-Tx_n\|=\|T(x_m-x_n)\|\geq\beta\|x_m-x_n\|$$

since (Tx_n) is strongly Cauchy. Therefore $\beta\|x_m-x_n\|\to 0,\ as\ m,n\to\infty$. But β is a positive constant, i.e. $\|x_m-x_n\|\to 0,\ as\ m,n\to\infty$. Therefore (x_m) is strongly Cauchy in H and H is strongly complete, hence $x_n\xrightarrow{s}x$ for a unique $x\in H$. Since T is bounded, so $T:H\to H$ is continuous, and hence

$$Tx_n\xrightarrow{s}Tx \text{ i.e. } y_n\xrightarrow{s}Tx \text{ but } y_n\xrightarrow{s}y.$$

Therefore by uniqueness of the strong limit, $Tx = y$ i.e. $y \in \Re_T$ thus $y \in \overline{\Re_T} \Rightarrow y \in \Re_T$ i.e. $\overline{\Re_T} \subseteq \Re_T$ hence $\Re_T$ is closed.

□

Proposition 2.1.16

$T \in B(H)$ is invertible if and only if T is bounded from below and $\Re_T$ is dense in H.

Proof:

Let T be bounded from below and $\overline{\Re_T} = H$. Then T is one-to-one and by proposition 2.1.15, $\Re_T$ is closed, i.e $\overline{\Re_T} = \Re_T$, hence the condition $\overline{\Re_T} = H \Rightarrow \Re_T = H$, i.e. T is onto. Hence for each $y \in H$, there is an $x \in H$ such that $Tx = y$ and only one such x, exists, since T is 1-1. So for each $y \in H$ there exists unique $x \in H$ such that $Tx = y$. Call the set inverse of T by T^{-1}, so $T^{-1}y = x$. We know that T^{-1} is linear. The condition "T is bounded from below" $\Rightarrow$ there exists a positive constant β such that $\|Tx\| \geq \beta \|x\|$ *for all* x, writing y in place of Tx and $T^{-1}y$ in place of x we get for all $y \in H$

$$\|y\| \geq \beta \|T^{-1}y\| \quad \forall y \in H \text{ i.e. } \|T^{-1}y\| \leq \frac{1}{\beta}\|y\| \quad \forall y \in H$$

i.e. $T^{-1} \in B(H)$. In other words, T is invertible. Conversely let T be invertible, i.e. there exists $S \in B(H)$ such that $ST = I = TS$ i.e. T is 1-1 and onto i.e. $\Re_T = H$, i.e. $\Re_T$ is dense in H since T is onto for each $y \in H$ there is a unique (for T is one-to-one) $x \in H$ such that $Tx = y$, so $Sy = x$. Since S is bounded we have $\|Sy\| \leq \|S\|\|y\| \quad \forall y \in H$. Replacing Sy by x and y by Tx, we obtain

$$\|x\| \leq \|S\|\|Tx\| \quad \forall x \in H$$

i.e.

$$\|Tx\| \geq \frac{1}{\|S\|}\|x\| \quad \forall x \in H \text{ (and } \|S\| > 0\text{)}$$

which implies T is bounded from below.

□

Remark:

If $T: X \to X$ is one-to-one and onto, the map T^{-1} exists, but need not be bounded (when X is a normed linear space). In operator theory we have a conclusion on the other hand;

Proposition 2.1.17 (Banach's inverse Theorem)

Let X be a Banach space, $T \in B(X)$ which is one-to-one and onto. Then the set inverse $T^{-1} \in B(X)$, i.e. T is invertible and its inverse is bounded (see [4]).

2.2 PROJECTORS, INVARIANCE AND REDUCING PROPERTY

Definition 2.2.1

Let X be a linear space over $\mathbb{R}$ *or* $\mathbb{C}$ and D be a linear subspace of X and $T: D \to X$ be a linear transformation. A number $\lambda \in \mathbb{R}$ *or* $\mathbb{C}$ is called an ***eigenvalue*** of T if there is a non-zero $x \in D$ such that $Tx = \lambda x$, x is then called an ***eigenvector*** for T corresponding to the eigenvalue λ. The null space $\eta_{T-\lambda I}$ of the linear transformation $T - \lambda I$ is called the ***eigenspace*** corresponding to the eigenvalue λ and represented by $\eta_T(\lambda)$.

Proposition 2.2.2

Let M be a closed linear subspace of a Hilbert space $H\left(so\ H = M \oplus M^{\perp}\right)$ (by proposition 2.1.7). Each $x \in M$ can be written uniquely as $x = x' + x''$ where $x' \in M\ and\ x'' \in M^{\perp}$. Define a function $P: H \to H$ by $Px = x'$(with x', x'' as above) for all $x \in H$. Then

i. $P \in B(H)$

ii. $\|P\| \leq 1$. Infact $\|P\| = 1$ if $\{\bar{0}\} \subset M$ and $\|P\| = 0$ if $M = \{\bar{0}\}$

iii. 0 and 1 are the only possible eigenvalues of P.

iv. P is Idempotent i.e. $P^2 = P$

v. $P^* = P$

P is called the ***orthogonal projection*** on H onto M. Conversely if an operator $P \in B(H)$ satisfies the conditions $P^2 = P$ and $P^* = P$, then P is the orthogonal projection on H onto the subspace $\Re_P = M = \{x \in H : Px = x\}$ (see [4]).

Proposition 2.2.3

If $T \in B(H)$ is a contraction and idempotent, then T is an orthogonal projection on H.

Proof:

So $\|T\| \leq 1$ and $T^2 = T$. We will show that T is self-adjoint and then $T = T^*$ and $T^2 = T$ together imply that T is an orthogonal projection on H. Now $\forall x \in H$

$$0 \leq \|T^*Tx - Tx\|^2 = \langle T^*Tx - Tx, T^*Tx - Tx \rangle$$

$$= \|T^*Tx\|^2 - \langle T^*Tx, Tx \rangle - \langle Tx, T^*Tx \rangle + \|Tx\|^2$$

$$= \|T^*Tx\|^2 - \langle Tx, T^2x \rangle - \langle T^2x, Tx \rangle + \|Tx\|^2$$

$$= \|T^*Tx\|^2 - \langle Tx, Tx \rangle - \langle Tx, Tx \rangle + \|Tx\|^2 \left(\text{since } T^2 = T\right)$$

$$= \|T^*Tx\|^2 - \|Tx\|^2 - \|Tx\|^2 + \|Tx\|^2$$

$$= \|T^*Tx\|^2 - \|Tx\|^2$$

$$\leq \|Tx\|^2 - \|Tx\|^2 = 0$$

(since $\|T^*Tx\|^2 \leq \|T^*\| \|Tx\| = \|T\| \|Tx\| \leq \|Tx\|$ for $\|T^*\| = \|T\|$ and $\|T\| \leq 1$). Therefore

$$\|T^*Tx - Tx\| = 0 \quad \forall x \in H.$$

Hence

$$T^*Tx = Tx \quad \forall x \in H,$$

i.e.

$$T^*T = T \tag{2.4}$$

Taking adjoints of both sides we get

$$\left(T^*T\right)^* = T^*\left(T^*\right)^* = T^*T = T^* \tag{2.5}$$

So (2.4) and (2.5) imply

$$T = T^*$$

i.e. T is self-adjoint. Thus T is self-adjoint and idempotent. Hence T is an orthogonal projection.

□

Proposition 2.2.4

If P is an orthogonal projection on H, so is $I-P$

Proof:

We shall show that $I-P$ is idempotent and self-adjoint, then it follows that $I-P$ is an orthogonal projection. $(I-P)^* = I^* - P^* = I - P$ (for P is an orthogonal projection hence $P^2 = P$ and $P = P^*$, also $I^* = I$). Thus $I-P$ is self-adjoint. Now

$$\begin{aligned}(I-P)^2 &= (I-P)(I-P)\\ &= I - IP - PI + P^2\\ &= I - P - P + P\\ &= I - P.\end{aligned}$$

Thus $I-P$ is idempotent. Therefore $I-P$ is an orthogonal projection onto $\Re_{I-P}$. If range of $P = M$ we can see that $\Re_{I-P} = M^{\perp}$. For

$$\begin{aligned}\Re_{I-P} &= \{(I-P)x : x \in H\}\\ &= \{x - Px : x \in H\}\end{aligned}$$

Now $Px \in M$ and each $x \in H$ has unique decomposition in $M \oplus M^{\perp}$

$$x = Px + (x - Px)$$

(where $Px \in M$ and $(x-Px) \in M^{\perp}$) So

$$\Re_{I-P} = M^{\perp}$$

□

Proposition 2.2.5

Let X be a Banach space and (x_n) be a sequence of elements of X. If $\sum_{n\in\mathbb{N}} x_n$ is absolutely convergent, then $\sum x_n$ converges to an element of X in the norm of X.

Moreover, if $\sum x_n$ converges to x, we have $\sum_{n\in\mathbb{N}} \|x_n\| \le \|x\|$ (see [13])

Proposition 2.2.6

Let X be a Bananch space and $T \in B(X)$ with $\|T\| < 1$ then $I-T$ is invertible.

Proof:

Consider the operator

$$I + T + T^2 + T^3 + \cdots$$

$$\|T^2\| = \|TT\| \le \|T\|^2, \cdots, \|T^n\| \le \|T\|^n \ \forall n \in \mathbb{N}$$

Now

$$1 + \|T\| + \|T^2\| + \|T^3\| + \cdots \le 1 + \|T\| + \|T\|^2 + \|T\|^3 + \cdots$$

The right hand side is a geometric series of common ratio $\|T\| < 1$ and hence is convergent. Therefore $I + T + T^2 + T^3 + \cdots$ is convergent to a limit $S \in B(X)$ by proposition 2.2.5. Let

$$S_n = I + T + T^2 + T^3 + \cdots + T^{n-1} \ \forall n \in \mathbb{N}$$

thus

$$S_n(I - T) = (I + T + T^2 + T^3 + \cdots + T^{n-1})(I - T)$$

$$= (I + T + T^2 + T^3 + \cdots + T^{n-1}) - (T + T^2 + \cdots + T^n)$$

$$= I - T^n.$$

Similarly

$$(I - T)S_n = I - T^n.$$

Thus

$$(I - T)S_n = S_n(I - T) \quad \forall n \in \mathbb{N}.$$

As $n \to \infty$, $S_n \to S$, and in the limit we have

$$(I - T)S = S(I - T), S \in B(X).$$

Thus, $I - T$ is invertible and

$$(I - T)^{-1} = S = I + T + T^2 + T^3 + \cdots$$

□

Proposition 2.2.7

If $T \in B(H)$ and $\|I - T\| < 1$ then T is invertible.

This follows from proposition 2.2.6, where we replace T by $I - T$.

Definition 2.2.8

Let H be a Hilbert space, $T \in B(H)$ and M be a closed linear subspace of H. We say that M is ***invariant*** with respect to T or T***-invariant*** if $x \in M \Rightarrow Tx \in M$. If T is defined on D_T (subspace of H) then M is said to be T***-invariant*** if $Tx \in M$ for all $x \in M \cap D_T$. If $M = \{\bar{0}\}$ or $M = H$, then M is always T-invariant. For $x \in H \Rightarrow Tx \in H$ and $x = \bar{0} = T\bar{0} = \bar{0} \in \{\bar{0}\}$. $\{\bar{0}\}$ and H are called the ***trivial invariant subspaces*** for T. η_T is T-invariant for if $x \in \eta_T, Tx = \bar{0} \in \eta_T$. Hence it is of interest to consider whether an operator $T \in B(H)$ has non-trivial invariant subspaces.

Proposition 2.2.9

Let H be a Hilbert space and $T \in B(H)$. Then a closed linear subspace M of H is T-invariant if and only if $M^\perp$ is T^*-invariant

Proof:

Let M be T-invariant. To show that $M^\perp$ is T^*-invariant. Let $x \in M^\perp$ i.e.

$$\langle x, Ty \rangle = 0 \ \forall y \in M$$

(M is T-invariant $\Leftrightarrow Ty \in M$ for all $y \in M$). But $\langle x, Ty \rangle = \langle T^* x, y \rangle$ so $\langle T^* x, y \rangle = 0 \quad \forall y \in M$ i.e. $T^* x \perp M$, i.e. $T^* x \in M^\perp$. Thus $x \in M^\perp \Rightarrow T^* x \in M^\perp$ thus $M^\perp$ is T^*-invariant. By the same result it follows that; $M^\perp$ is T^*-invariant $\Rightarrow (M^\perp)^\perp$ is $(T^*)^*$-invariant (since $T \in B(H)$ and $M^\perp$ is a closed linear subspace). But $(M^\perp)^\perp = M$ since H is a Hilbert space and $(T^*)^* = T$ for $T \in B(H)$. So $M^\perp$ is T^*-invariant $\Rightarrow M$ is T-invariant. Thus for $T \in B(H)$, if M is a closed linear subspace of H then

$$M \text{ is } T\text{-invariant} \Leftrightarrow M^\perp \text{ is } T^*\text{-invariant}$$

□

Definition 2.2.10

Let $T \in B(H)$ and M be a closed linear subspace of H. We say that M ***reduces*** T if M and $M^\perp$ are both invariant under T i.e. $Tx \in M \quad \forall x \in M$ and $Ty \in M^\perp \ \forall y \in M^\perp$. M

is then called a ***reducing subspace*** for T. It is clear that H and $\{\bar{0}\}$ are both reducing subspaces for any $T \in B(H)$. These H and $\{\bar{0}\}$ are called the ***trivial reducing subspaces*** (for any $T \in B(H)$). Hence it is of interest to consider whether an operator $T \in B(H)$ has non-trivial reducing subspaces. T is said to be ***irreducible*** if it has no non-trivial reducing subspaces $\Leftrightarrow$ closed linear subspaces.

Remark:

M reduces $T \Leftrightarrow$ 'M is invariant under T' and '$M^{\perp}$ is invariant under T'

$\Leftrightarrow$ 'M is invariant under T' and 'M is invariant under T^*'

$\Leftrightarrow$ '$M^{\perp}$ is invariant under T^*' and 'M is invariant under T^*'

$\Leftrightarrow$ 'M is invariant under T and T^*' and '$M^{\perp}$ is invariant under T and T^*'.

Both the concepts "invariance" and "reduction" as described above are geometric in their expressions involving a linear subspace M and its image. This geometric concept can be translated into a purely algebraic concept involving operators with perfect equivalence as seen in the next result.

Proposition 2.2.11

Let $T \in B(H)$ and M be a subspace of H. Let P be the orthogonal projection on H onto M. Then

1. M is invariant under T if and only if $PTP = TP$.
2. M reduces T if and only if $T \leftrightarrow P$ and $P \leftrightarrow T^*$.

Proof:

1. Let M be invariant under T. Let $x \in H$, then $Px \in M$, since M is invariant under T, so $T(Px) \in M$ i.e. $TPx \in M$,

$$P(TPx) = TPx \ (\text{since } Py = y \ \forall y \in M)$$

and this is valid $\forall x \in H$, hence $PTP = TP$. Conversely let $PTP = TP$, to show that M is invariant under T. Let $x \in M$, to show that $Tx \in M$

$$PTP = TP \Rightarrow PTPx = TPx$$

(but $Px = x, \forall x \in M$) $\Rightarrow PTx = Tx$ i.e. $Tx \in M$.

2. M reduces $T \Leftrightarrow M$ is invariant under T and T^*

$$\Leftrightarrow PTP = TP \text{ (by part (1))}$$

$$\Leftrightarrow (PTP)^* = (TP)^* \Leftrightarrow P^*T^*P^* = P^*T^*$$

$$\Leftrightarrow PTP = PT \; (\because P^* = P \; and \; T^* = T)$$

$$\Leftrightarrow TP = PT \Leftrightarrow (TP)^* = (PT)^* \; i.e. \; PT^* = T^*P \; i.e \; P \leftrightarrow T^*$$

□

Remark1:

Thus the statement "M reduces T" can be given an equivalent version (algebraic or operator theory) as "$T \leftrightarrow P$" (where P is orthogonal projector on H onto M). Let H be a Hilbert space and $T \in B(H)$. We say that an orthogonal projector P ***reduces*** T if $P \leftrightarrow T$. (this is equivalent to saying that M reduces T, where $M = \Re_P$).

Remark2:

Let $T \in B(H)$ have a proper reducing subspace M. Now $H = M \oplus M^{\perp}$ (by proposition 2.1.7). Since T maps M into M and $M^{\perp}$ into $M^{\perp}$ hence we can split T into two bounded linear operators; $T|_M, T|_{M^{\perp}}$ and study these instead of T. Also $T = T|_M \oplus T|_{M^{\perp}}$. Note:

$$T|_M \in B(M, M) \text{ and } T|_M : M \to M, \; T|_{M^{\perp}} \in B(M^{\perp}, M^{\perp}) \text{ and } T|_{M^{\perp}} : M^{\perp} \to M^{\perp}$$

It is possible that even $T|_M, T|_{M^{\perp}}$, have themselves reducing subspaces and so on; so that these operators can be further split.

Proposition 2.2.12

Let $T \in B(H)$ be self-adjoint and M be a closed linear subspace of H. Then M reduces T if and only if it is T-invariant.

Proof:

M reduces $T \Leftrightarrow M, M^{\perp}$ are both T-invariant,

$M^{\perp}$ is T-invariant $\Leftrightarrow (M^{\perp})^{\perp}$ is T^*-invariant

$\Leftrightarrow M$ is T-invariant ($T^* = T$ since T is self adjoint).

2.3 THE OPERATOR e^T AND NORMAL OPERATORS IN $B(H)$

Definition 2.3.1

Let X be a Banach space and $T \in B(X)$. The expression e^T stands for the infinite series

$$I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots + \frac{1}{n!}T^n + \cdots,$$

where I is the identity operator on X. It is clear that $e^T \in B(X)$. Indeed we first have $\|T^n\| \le \|T\|^n \quad \forall n \in \mathbb{N}$. (for $\|T^2\| \le \|T\|\|T\| = \|T\|^2$ e.t.c use induction on n). Now the expression

$$1 + \|T\| + \frac{1}{2!}\|T\|^2 + \frac{1}{3!}\|T\|^3 + \cdots + \frac{1}{n!}\|T\|^n + \cdots$$

converges in $\mathbb{R}$ and its sum is $e^{\|T\|} \in \mathbb{R}$. Thus $\sum_{n=0}^{\infty} \frac{T^n}{n!} \left(T^0 = I\right)$ is absolutely convergent in $B(X)$. For

$$\left\|\sum_{n=0}^{\infty} \frac{T^n}{n!}\right\| \le \sum_{n=0}^{\infty} \left\|\frac{T^n}{n!}\right\| \le \sum_{n=0}^{\infty} \frac{1}{n!}\|T\|^n < \infty$$

and hence by proposition 2.2.5 $\sum_{n=0}^{\infty} \frac{1}{n!}T^n$ converges in the norm of $B(X)$ to an element of $B(X)$ which is denoted by e^T.

Lemma 2.3.2

$T \in B(X) \Rightarrow \|e^T\| \le e^{\|T\|}$ (see [4]).

Lemma 2.3.3

$e^0 = I$ for $0 \in B(X)$

Proof:

For $0 \in B(X)$ put $T = 0$ in

$$e^T = I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots + \frac{1}{n!}T^n + \cdots,$$

and we get $e^0 = I$.

□

Proposition 2.3.4

Let $T \in B(X)$. Then

1. e^T is invertible
2. $e^T e^{-T} = \mathrm{I}$
3. If $S \in B(X)$ and $S \leftrightarrow T$ then $e^{S+T} = e^S e^T$

Proof:

We prove (3) for subsequently (1) and (2) follow at once. By definition

$$e^T = I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots + \frac{1}{n!}T^n + \cdots$$

$$e^S = I + S + \frac{1}{2!}S^2 + \frac{1}{3!}S^3 + \cdots + \frac{1}{n!}S^n + \cdots$$

$$e^{S+T} = I + (S+T) + \frac{1}{2!}(S+T)^2 + \frac{1}{3!}(S+T)^3 + \cdots + \frac{1}{n!}(S+T)^n + \cdots.$$

Let

$$a_n = I + T + \frac{1}{2!}T^2 + \cdots + \frac{1}{n!}T^n$$

$$b_n = I + S + \frac{1}{2!}S^2 + \cdots + \frac{1}{n!}S^n$$

$$c_n = I + (S+T) + \frac{1}{2!}(S+T)^2 + \cdots + \frac{1}{n!}(S+T)^n$$

and

$$\tilde{a}_n = 1 + \|T\| + \frac{1}{2!}\|T\|^2 + \cdots + \frac{1}{n!}\|T\|^n,$$

$$\tilde{b}_n = 1 + \|S\| + \frac{1}{2!}\|S\|^2 + \cdots + \frac{1}{n!}\|S\|^n$$

$$\tilde{c}_n = 1 + (\|S\| + \|T\|) + \frac{1}{2!}(\|S\| + \|T\|)^2 + \cdots + \frac{1}{n!}(\|S\| + \|T\|)^n \quad \forall n \in \mathbb{N}.$$

By our earlier discussion, we note that

$$a \to e^{\mathrm{T}}, b_n \to e^S \;\; c_n \to e^{S+\mathrm{T}} \text{ in } B(X)$$

$$\tilde{a}_n \to e^{\|T\|},\; \tilde{b}_n \to e^{\|S\|},\; \tilde{a}_n + \tilde{b}_n \to e^{\|S\|+\|T\|} \text{ in } \mathbb{R}.$$

Now

$$a_n b_n - c_n = \left(I + T + \frac{1}{2!}T^2 + ... + \frac{1}{n!}T^n \right)\left(I + S + \frac{1}{2!}S^2 + ... + \frac{1}{n!}S^n \right)$$

$$-\left(I + (S+T) + \frac{1}{2!}(S+T)^2 + ... + \frac{1}{n!}(S+T)^n \right).$$

In expanding $(S+T)^2$, $(S+T)^3$ e.t.c we use $S \leftrightarrow T$; so that we have

$$(S+T)^2 = (S+T)(S+T) = S^2 + ST + TS + T^2 = S^2 + 2ST + T^2$$

(since S commutes with T so $ST = TS$).

$$(S+T)^3 = S^3 + 3S^2T + 3ST^2 + T^3 \text{ e.t.c.}$$

Hence

$$a_n b_n - c_n = \sum_{i=1}^{n} a_{ik} T^i S^k ,$$

where (a_{ik}) is an nxn matrix with positive entries on the main diagonal and below, and 0's elsewhere (see [13]). Therefore

$$\|a_n b_n - c_n\| = \left\| \sum_{i=1}^{n} a_{ik} T^i S^k \right\| \le \sum_{i=1}^{n} a_{ik} \|T^i\| \|S^k\| \le \sum_{i=1}^{n} a_{ik} \|T\|^i \|S\|^k = \tilde{a}_n \tilde{b}_n - \tilde{c}_n .$$

As $n \to \infty$, $\tilde{a}_n \tilde{b}_n - \tilde{c}_n \to e^{\|T\|} e^{\|S\|} - e^{\|T\| + \|S\|} = e^{\|T\|} e^{\|S\|} - e^{\|T\|} e^{\|S\|} = 0$. So

$$\|a_n b_n - c_n\| \le \tilde{a}_n \tilde{b}_n - \tilde{c}_n \to 0 \quad as \; n \to \infty .$$

But

$$a_n \to e^T , b_n \to e^S , c_n \to e^{T+S} , \; \|a_n b_n - c_n\| \to \|e^T e^S - e^{T+S}\| = 0 \quad as \; n \to \infty ,$$

therefore $e^T e^S = e^{T+S}$ when $T \leftrightarrow S$. Let $T \in B(X)$ so $-T \in B(X)$. $e^T, e^{-T} \in B(X)$ and $T \leftrightarrow -T$ therefore

$$e^T e^{-T} = e^{T-T} = e^0 = I$$

i.e e^T is invertible and its inverse is $\left(e^T\right)^{-1} = e^{-T}$. Consider a Hilbert space H. Each $T \in B(H)$ has a unique adjoint T^* for $T \in B(H)$, $e^T \in B(H)$.

□

Lemma 2.3.5

$\left(e^{T}\right)^{*}=e^{T^{*}}$

Proof:

Now

$$e^{T}=I+T+\frac{T^{2}}{2!}+\frac{T^{3}}{3!}+\cdots+\frac{T^{n}}{n!}+\cdots$$

$$e^{T^{*}}=I+T^{*}+\frac{\left(T^{*}\right)^{2}}{2!}+\frac{\left(T^{*}\right)^{3}}{3!}+\cdots+\frac{\left(T^{*}\right)^{n}}{n!}+\cdots.$$

Let

$$A_{n}=I+T+\frac{T^{2}}{2!}+\frac{T^{3}}{3!}+\cdots+\frac{T^{n}}{n!}+\cdots$$

$$A_{n}^{*}=I+T^{*}+\frac{\left(T^{*}\right)^{2}}{2!}+\frac{\left(T^{*}\right)^{3}}{3!}+\cdots+\frac{\left(T^{*}\right)^{n}}{n!}+\cdots,\ \forall n\in\mathbb{N}$$

then

$$A_{n}\to e^{T},A_{n}^{*}\to e^{T^{*}}\ in\ B(H)\left(A_{n}\in B(H)\right).$$

If A_{n} is a sequence of elements of $B(H)$ which converges to $A\in B(H)$ then $\left(A_{n}^{*}\right)$ converges to A^{*} in $B(H)$. Indeed $\|A_{n}-A\|\to 0\ as\ n\to\infty$. We must show that $\|A_{n}^{*}-A^{*}\|\to 0$. Clearly $A_{n}-A\in B(H)$ so $(A_{n}-A)^{*}$ exists in $B(H)$ i.e. $A_{n}^{*}-A^{*}$ exists in $B(H)$. Moreover

$$\|A_{n}-A\|=\left\|(A_{n}-A)^{*}\right\|=\|A_{n}^{*}-A^{*}\|.$$

Therefore

$$\lim_{n\to\infty}\|A_{n}-A\|=0\Rightarrow\lim_{n\to\infty}\|A_{n}^{*}-A^{*}\|=0\ \text{ i.e. } A_{n}^{*}-A^{*}\text{ is in } B(H).$$

By the result proved it follows that since $A_{n}\to e^{T}$ so $(A_{n})^{*}\to\left(e^{T}\right)^{*}$ but $A_{n}^{*}\to e^{T^{*}}$. By the uniqueness of the limit, it follows that $\left(e^{T}\right)^{*}=e^{T^{*}}$.

□

Proposition 2.3.6

Let H be a complex Hilbert space and $T \in B(H)$. If $\langle Tx, x \rangle = 0 \; \forall x \in H$, then $T = 0$.

Proof:

Remark: if we have $\langle Tx, y \rangle = 0 \quad \forall x, y \in H$ the conclusion $T = 0$ is immediate for

$$\phi(x, y) = \langle Tx, y \rangle :$$

where ϕ is sesquilinear form and $\hat{\phi}$ is the associated quadratic form

$$\hat{\phi}(x) = \langle Tx, x \rangle .$$

Now by lemma 2.1.2

$$\phi(x, y) = \frac{1}{4}\left[\hat{\phi}(x + y) - \hat{\phi}(x - y) + i\hat{\phi}(x + iy) - i\hat{\phi}(x - iy) \right]$$

i.e.

$$\langle Tx, y \rangle = \frac{1}{4}\left[\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle + i\langle T(x + iy), x + iy \rangle - i\langle T(x - iy), x - iy \rangle \right]$$

Now by hypothesis $\langle Tz, z \rangle = 0 \quad \forall z \in H$. Therefore

$$\langle Tx, y \rangle = 0 \quad \forall x, y \in H \; \therefore \; T = 0 .$$

The above result is not true in real Hilbert spaces (see [4])

□

Definition 2.3.7

An operator $T \in B(H)$ is called ***isometric*** if $\|Tx\| = \|x\| \quad \forall x \in H$

Lemma 2.3.8

For a $T \in B(H)$ the following conditions are equivalent

i. T is isometric

ii. $T^*T = I$

Proof:

$i) \Rightarrow ii)$; By (i)

$\|Tx\|^2 = \|x\|^2 \; \forall x \in H$ i.e. $\langle Tx, Tx \rangle = \langle x, x \rangle \quad \forall x \in H$ i.e.

$\langle T^*Tx, x \rangle = \langle Ix, x \rangle \; \forall x \in H$.

$T^*T = I$ by proposition 2.3.6 (H is a complex Hilbert space). Indeed

$$\langle T^*Tx, x\rangle - \langle Ix, x\rangle = 0 \; \forall x \in H$$

$$\Rightarrow \langle T^*Tx - Ix, x\rangle = 0 \; \forall x \in H$$

$$\Rightarrow \langle (T^*T - I)x, x\rangle = 0 \quad \forall x \in H$$

in fact the previous results implies.

$(ii) \Rightarrow (i)$;

Now for all x

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle Ix, x\rangle \left(\because T^*T = I\right)$$

$$= \langle x, x\rangle = \|x\|^2$$

i.e. $\|Tx\| = \|x\|$ i.e T is isometric.

□

Lemma 2.3.9

If H is a complex Hilbert space: $S, T \in B(H)$ and $\langle Sx, x\rangle = \langle Tx, x\rangle \; \forall x \in H$ then $S = T$

Proof:

$$\langle Sx, x\rangle = \langle Tx, x\rangle \Rightarrow \langle Sx - Tx, x\rangle = 0 \; i.e \langle (S-T)x, x\rangle = 0$$

therefore $S - T = 0$ i.e $S = T$

□

Definition 2.3.10

An operator $T \in B(H)$ is called ***unitary*** if $TT^* = T^*T = I$. Thus $T \leftrightarrow T^*$ and $T^*T = I$. Clearly T is invertible and $T^{-1} = T^*$. $T \in B(H)$ is called a ***normal operator*** if T commutes with T^* i.e. $TT^* = T^*T$.

Lemma 2.3.11

For a $T \in B(H)$ the following conditions are equivalent

i. T is normal

ii. $\|Tx\| = \|T^*x\| \; \forall x \in H$

Proof:

$(i) \Rightarrow (ii)$

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle$$

$$= \langle TT^*x, x\rangle \text{ (for } T \text{ is normal by (i) } TT^* = T^*T\text{)}$$

$$= \langle T^*x, T^*x\rangle$$

$$= \|T^*x\|^2 \quad \forall x \in H .$$

Therefore $\|Tx\| = \|T^*x\|$ which is (ii), by a previous result $T^*T = TT^*$ $i.e. T \leftrightarrow T^* \Rightarrow T$ is normal.

$(ii) \Rightarrow (i)$

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 = \|T^*x\|^2$$

$$= \langle T^*x, T^*x\rangle = \langle (T^*)^* T^*x, x\rangle$$

$$= \langle TT^*x, x\rangle \ \forall x \in H$$

therefore by lemma 2.3.9, $T^*T = TT^*$ i.e. $T \leftrightarrow T^* \Rightarrow T$ is normal

□

A self adjoint operator T is normal, for $T = T^*$ so since $T \leftrightarrow T, T \leftrightarrow T^* = T$. A unitary operator $U \in B(H)$ is normal, for $UU^* = I = U^*U \Rightarrow U \leftrightarrow U^*$.

A special observation!

Let $T \in B(H)$ be normal. We can write

$$T = A + iB = \frac{(T + T^*)}{2} + i\frac{(T - T^*)}{2i} \quad \forall T \in B(H).$$

Now $A, B \in B(H)$ and

$$A^* = \left(\frac{1}{2}(T + T^*)\right)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T^* + T) = A$$

$$B^* = \left(\frac{1}{2i}(T - T^*)\right)^* = -\frac{1}{2i}(T^* - T^{**}) = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = B .$$

Thus both A, B are self-adjoint. But T is normal so $T \leftrightarrow T^*$ i.e. $T^*T = T^*T$. Since $T = A + iB$, so $T^* = (A+iB)^* = A^* - iB^* = A - iB$. Therefore

$$TT^* = (A+iB)(A-iB)$$

$$= A^2 - iAB + iBA + B^2$$

$$T^*T = (A-iB)(A+iB)$$

$$= A^2 + iAB - iBA + B^2$$

$$T^*T = TT^* \Rightarrow A^2 - iAB + iBA + B^2 = A^2 + iAB - iBA + B^2$$

$$\Rightarrow 2iAB - 2iBA = 0$$

$$\Rightarrow 2iAB = 2iBA$$

$$\Rightarrow AB = BA \quad \text{i.e. } A \leftrightarrow B.$$

Thus every normal $T \in B(H)$ can be expressed as $A + iB$, where A, B are both self-adjoint and $A \leftrightarrow B$. The uniqueness of this decomposition follows since if $T = A + iB$ is such a decomposition then $T^* = (A+iB)^* = A^* - iB^* = A - iB$. Solving the operator equations

$$T = A + iB,\ T^* = A - iB$$

We get unique solutions

$$A = \frac{T + T^*}{2},\ B = \frac{1}{2i}\left(T - T^*\right)$$

Proposition 2.3.12 (Fuglede's theorem)

If $T \in B(H)$ is normal and T commutes with $S \in B(H)$ then the adjoint of T (i.e. T^*) also commutes with S i.e. $TS = ST \Rightarrow T^*S = ST^*$.

Proof:

Since $T \leftrightarrow S, T^n \leftrightarrow S$ for all $n \in \mathbb{N}$. Indeed the result is obviously valid when $n = 1$. Suppose (induction hypothesis) $T^{n-1} \leftrightarrow S$ then

$$T^nS = (TT^{n-1})S = T(T^{n-1}S)$$

$$= T(ST^{n-1}) = (TS)T^{n-1}$$

$$= (ST)T^{n-1} = ST^n.$$

Therefore $T^n \leftrightarrow S \quad \forall n = 0,1,2,\cdots$. For any $z \in \mathbb{C}$, consider the operator $e^{i\bar{z}T} \in B(H)$.

Note

$$e^{i\bar{z}T} = \sum_{n=0}^{\infty} \frac{i^n \bar{z}^n T^n}{n!}.$$

Since $S \leftrightarrow T^n \quad \forall n = 0,1,2,\cdots$ we obtain

$$Se^{i\bar{z}T} = e^{i\bar{z}T} S. \tag{2.6}$$

(for if $A_n = \sum_{m=0}^{n} \frac{i^m \bar{z}^m T^m}{m!}$, then $S \leftrightarrow A_n \quad \forall n \in \mathbb{N}$. Now $A_n \xrightarrow{\|\|} e^{i\bar{z}T}$ $as\ n \to \infty$. As $S \leftrightarrow A_n$ for each $n \in \mathbb{N}$, $S \leftrightarrow \lim_{n\to\infty} A_n = e^{i\bar{z}T}$). Therefore $S = e^{-i\bar{z}T} Se^{i\bar{z}T}$ (from (2.6)). Now consider the operator $e^{-izT^*} Se^{izT^*}$

$$e^{-izT^*} Se^{izT^*} = e^{-izT^*}\left(e^{-i\bar{z}T} Se^{i\bar{z}T}\right)e^{izT^*} \text{ (from (2.6))}$$

$$= e^{-i\left(zT^* + \bar{z}T\right)} Se^{i\left(\bar{z}T + zT^*\right)}$$

(since $T \leftrightarrow T^*$)

$$\left(e^{i\left(\bar{z}T + zT^*\right)}\right)^* \left(e^{i\left(\bar{z}T + zT^*\right)}\right) = \left(e^{-i\left(zT^* + \bar{z}T^*\right)}\right)\left(e^{i\left(\bar{z}T + zT^*\right)}\right)$$

$$= e^0 = I = \left(e^{i\left(\bar{z}T + zT^*\right)}\right)\left(e^{-i\left(\bar{z}T + zT^*\right)}\right)$$

$$= \left(e^{i\left(\bar{z}T + zT^*\right)}\right)\left(e^{-i\left(\bar{z}T + zT^*\right)}\right)^*.$$

Therefore the operator $e^{i\left(\bar{z}T + zT^*\right)}$ is unitary and its norm is 1. Thus

$$e^{-i\left(zT^* + \bar{z}T\right)} Se^{i\left(zT^* + \bar{z}T\right)} \text{ is } U^* SU$$

where U is the unitary operator $e^{i\left(\bar{z}T + zT^*\right)}$. It follows that

$$\left\|e^{-izT^*} Se^{izT^*}\right\| = \left\|USU^*\right\| \le \|U\|\|S\|\left\|U^*\right\| = \|S\|.$$

Hence for any $x, y \in H$

$$\left|\left\langle e^{-izT^*} Se^{izT^*} x, y\right\rangle\right| \le \left\|e^{-izT^*} Se^{izT^*}\right\|\|x\|\|y\| \le \|S\|\|x\|\|y\|.$$

Now $\left\langle e^{-izT^*} S e^{izT^*} x, y \right\rangle$ is analytic in z and bounded in the entire complex plane i.e is a bounded entire function and hence by Liouville's theorem [7], the function must be constant on $\mathbb{C}$. Setting, in particular $z = 0$ we get

$$e^{-izT^*} S e^{izT^*} = S \text{ i.e } S e^{izT^*} = e^{izT^*} S.$$

Equating coefficients of like powers of z on both sides we get $ST^* = T^*S$.

Remark:

Let $S \in B(H)$ and $T \in B(H)$. If $T \leftrightarrow S$ it does not follow that $T^* \leftrightarrow S$ (it is obvious however that $T^* \leftrightarrow S^*$). For

$$TS = ST \Rightarrow (TS)^* = (ST)^* \text{ i.e. } S^*T^* = T^*S^* \text{ i.e. } S^* \leftrightarrow T^*.$$

However if $T \leftrightarrow S \Rightarrow T^* \leftrightarrow S$, then since $T \leftrightarrow T$, so $T^* \leftrightarrow T$ i.e T is normal. The Fuglede's theorem is a converse of this; i.e. T is normal and $T \leftrightarrow S \Rightarrow T^* \leftrightarrow S$.

2.4 SPECTRUM, OPERATIONS $\wedge, \vee, \perp$ AND BASIC DEFINITIONS

Definition 2.4.1

A number $\lambda \in \mathbb{C}$ is said to belong to the ***resolvent set*** $\rho(T)$ of an operator T if $(T - \lambda I)$ is invertible. The complement of $\rho(T)$ in $\mathbb{C}$ is called the ***spectrum*** of T and we represent it by $\sigma(T)$. Therefore

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \textit{ is not invertible}\}$$

(in the operator sense). By proposition 2.1.16, $T \in B(H)$ is invertible iff T is bounded from below and $\Re_T$ is dense in H. So $(T - \lambda I)$ is invertible if and only if $(T - \lambda I)$ is bounded from below and $\Re_{T-\lambda I}$ is dense in H. The contrapositive statement is $(T - \lambda I)$ is not invertible if and only if '$(T - \lambda I)$ is not bounded from below or $\Re_{T-\lambda I}$ is not dense in H'. Therefore $\lambda \in \sigma(T) \Rightarrow$ $(T - \lambda I)$ is not bounded from below or $\overline{\Re}_{T-\lambda I} \neq H$. Let

$$\pi\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \textit{ is not bounded from below}\}$$

$$\Gamma\sigma(T) = \{\lambda \in \mathbb{C} : \Re_{T-\lambda I} \textit{ is not dense in } H\}$$

$\pi\sigma(T)$ is called the ***approximate point spectrum*** of T whereas $\Gamma\sigma(T)$ is called the ***compression spectrum*** of T. So

$$\sigma(T) = \pi\sigma(T) \cup \Gamma\sigma(T).$$

Note: the sets $\pi\sigma(T)$ and $\Gamma\sigma(T)$ may overlap. It is clear that if we define a $\lambda \in \mathbb{C}$ to be an approximate eigenvalue of T if there exists a sequence (x_n) of elements of H such that $\|x_n\| = 1 \quad \forall n \in \mathbb{N}$ and

$$\|(T - \lambda I)x_n\| \to 0 \, as \;\; n \to \infty$$

then $\pi\sigma(T)$ is the set of all approximate eigenvalues of T. For $(T - \lambda I)$ is bounded from below $\Leftrightarrow$ there exists a real number ε such that

$$\|(T - \lambda I)x\| \geq \varepsilon \|x\| \; \forall x \in H$$

i.e.

$$\left\|(T - \lambda I)\left(\frac{x}{\|x\|}\right)\right\| \geq \varepsilon \;\; \forall x \in H$$

such that $x \neq 0$. Now $\left\|\frac{x}{\|x\|}\right\| = 1 \, \forall x \in H$. Thus

$$\|(T - \lambda I)y\| \geq \varepsilon \;\text{ for all }\; y \in H \;\text{ satisfying }\; \|y\| = 1.$$

Hence there exists a sequence (x_n) such that $\|x_n\| = 1$ and

$$\|(T - \lambda I)x_n\| \to 0 \, as \;\; n \to \infty$$

i.e. λ is not an approximate eigenvalue, The converse can be similarly seen. If λ is an eigenvalue of T, it is clear that $\lambda \in \pi\sigma(T)$(see [13]). Let

$$\mathrm{P}\sigma(T) = \{\lambda : \lambda \, is \, an \, eigenvalue \, of \, T\},$$

thus $P\sigma(T) \subset \pi\sigma(T)$. $P\sigma(T)$ is called ***point spectrum*** of T. $\lambda \in P\sigma(T) \Rightarrow$ there exists an $x \in H$ such that $x \neq \bar{0}$ and

$$(T - \lambda I)x = \bar{0} \text{ i.e. } \|(T - \lambda I)x\| = 0$$

if we take $x_n = \dfrac{x}{\|x\|}$ $\forall n \in \mathbb{N}$ we note that $\|x_n\| = 1$ $\forall n \in \mathbb{N}$ and

$$\|(T-\lambda I)x_n\| = 0, \ i.e. (T-\lambda I)x_n \to 0 \ as\, n \to \infty$$

i.e. $\lambda \in \pi\sigma(T)$. There are five disjoint sets, some of which may be void or not void, depending on T. The usual tradition in spectral theory is to have the following disjoint divisions of $\sigma(T)$

$$\Gamma\sigma(T) - P\sigma(T) = \text{Residual spectrum of } T\,(denoted\ by\ R\sigma(T))$$

$$\pi\sigma(T) - (P\sigma(T) \cup \Gamma\sigma(T)) = \text{Continuous spectrum of } T \text{ (denoted by } C\sigma(T))$$

Proposition 2.4.2 (spectral mapping theorem)

Let X be a Banach space and p be a polynomial with complex coefficients. Let $T \in B(X)$, then

$$\sigma(p(T)) = p(\sigma(T))$$

where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.

For a proof see [13].

Definition 2.4.3

Le X be a normed linear space and $T \in B(X)$. The number sup $\{|\lambda| : \lambda \in \sigma(T)\}$ (note: $\sigma(T) \neq \phi$ i.e. $\sigma(T)$ is not empty) is called the ***spectral radius*** of T and represented by $r(T)$. It follows that $r(T) \leq \|T\|$. Indeed

$$\lambda I - T = \lambda\left(I - \frac{T}{\lambda}\right) \ if\ \lambda \neq 0. \tag{2.7}$$

Then $\lambda I - T$ is invertible if and only if $\left(I - \dfrac{T}{\lambda}\right)$ is invertible (by proposition 2.2.6, since X is Banach space, we know that $I - T$ is invertible if $\|T\| < 1$). Looking now at (2.7), we conclude that $\left(I - \dfrac{T}{\lambda}\right)$ is invertible if $\left\|\dfrac{T}{\lambda}\right\| < 1$, i.e. if $|\lambda| > \|T\|$. Hence if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$. Consequently

$$\sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\| \text{ i.e. } r(T) \leq \|T\|.$$

The next result is proved for bounded linear operators in the Hilbert space context.

Proposition 2.4.4

Let H be a complex Hilbert space and $T \in B(H)$. Then $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists and equals $r(T)$.

Proof:

It follows from the proposition 2.4.2, that

$$\{r(T)\}^n = r(T^n) \text{ for all } n \in \mathbb{N}.$$

To see this, consider the polynomial $p(t) = t^n$ and use proposition 2.4.2, namely;

$$\sigma(p(T)) = p(\sigma(T))$$

for $T \in B(H)$ i.e. $\sigma(T^n) = (\sigma(T))^n$ where $(\sigma(T))^n = \{\lambda^n : \lambda \in \sigma(T)\}$. Therefore

$$\begin{aligned} r(T^n) &= \sup\{|\mu| : \mu \in \sigma(T^n)\} \\ &= \sup\{|\lambda^n| : \lambda \in \sigma(T)\} \; for \; \sigma(T^n) = \{\sigma(T)\}^n \\ &= \sup\{|\lambda|^n : \lambda \in \sigma(T)\} \\ &= \{\sup\{|\lambda| : \lambda \in \sigma(T)\}\}^n \\ &= \{r(T)\}^n \end{aligned}$$

Thus (note $r(T) \geq 0$)

$$r(T) = \{r(T^n)\}^{\frac{1}{n}} \quad \forall n \in \mathbb{N}.$$

We have already seen that for any $A \in B(H)$, $r(A) \leq \|A\|$. Since $T^n \in B(H)$, so $r(T^n) \leq \|T^n\|$ and hence we get

$$r(T) \leq \|T^n\|^{1/n}.$$

Letting $n \to \infty$ the last line implies that

$$r(T) \leq \lim_{\inf} \|T^n\|^{\frac{1}{n}} \tag{2.8}$$

(we are just considering the sequence of reals $\left(\left\|T^n\right\|^{\frac{1}{n}}\right)_{n=1}^{\infty}$ on the right hand side and hence it is relevant to invoke the limit infimum of this sequence without consideration of the convergence of the latter!). Let $\lambda \neq 0$ and $\frac{1}{\lambda} \in \rho(T)$. Now

$$\frac{1}{\lambda} I - T = \frac{1}{\lambda}(I - \lambda T).$$

If $\|\lambda T\| < 1$ then $I - \lambda T$ is invertible and consequently $\frac{1}{\lambda}$ would be in $\rho(T)$ as asserted. In such a situation the mapping, given any $x, y \in H$,

$$\lambda \to \lambda\left\langle (I - \lambda T)^{-1} x, y \right\rangle$$

would be analytic for all λ satisfying $\frac{1}{|\lambda|} > r(T)$. But

$$\lambda\left\langle (I - \lambda T)^{-1} x, y \right\rangle = \lambda\left\langle \sum_{n=0}^{\infty} (\lambda T)^n x, y \right\rangle = \lambda \sum_{n=0}^{\infty} \left\langle (\lambda T)^n x, y \right\rangle \tag{2.9}$$

for $(I - \lambda T)^{-1} = \sum_{n=0}^{\infty} (\lambda T)^n$ (Neumann's series) (see [13]). Therefore

$$(I - \lambda T)^{-1} x = \sum_{n=0}^{\infty} (\lambda T)^n x$$

(strong convergence in H on the right hand side). The convergence of the right hand side of (2.9) implies that the sequence of scalars $\left(\left\langle (\lambda T)^n x, y \right\rangle\right)_{n=1}^{\infty}$ (for any $x, y \in H$ fixed) must be bounded. Considering the functionals represented by the elements $(\lambda T)^n x \in H$, we note that this sequence of functionals is bounded pointwise and hence as a consequence of the uniform boundedness principle [4], it follows that there is a real $M > 0$ such that $|\lambda|^n \left\|T^n\right\| \leq M \ \forall n \in \mathbb{N}$ for λ satisfying $|\lambda| \leq \frac{1}{r(T)}$ (see [13]). Hence $|\lambda| \left\|T^n\right\|^{\frac{1}{n}} \leq M^{\frac{1}{n}} \ \forall n \in \mathbb{N}$ and for all λ satisfying $|\lambda| \leq \frac{1}{r(T)}$. Let $n \to \infty$, we know that $M^{\frac{1}{n}} \to 1$ (from calculus). Hence

$$|\lambda| \limsup \|T^n\|^{\frac{1}{n}} \leq 1 .$$

Since this is true for all $|\lambda| \leq \frac{1}{r(T)}$ we obtain $\frac{1}{r(T)} \limsup \|T^n\|^{\frac{1}{n}} \leq 1$ i.e.

$$\limsup \|T^n\|^{\frac{1}{n}} \leq r(T) \qquad (2.10).$$

From (2.8) and (2.10) we get

$$r(T) \leq \liminf \|T^n\|^{\frac{1}{n}} \leq \limsup \|T^n\|^{\frac{1}{n}} \leq r(T) .$$

Hence

$$\lim\sup \|T^n\|^{1/n} = \limsup \|T^n\|^{1/n} = r(T) .$$

i.e. $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists and equals $r(T)$.

□

Proposition 2.4.5

Let $T \in B(H)$ be normal, then T^n is normal for all $n \in \mathbb{N}$

Proof:

T is normal $\Rightarrow T^2$ is normal. T is normal $\Rightarrow TT^* = T^*T$

$$(T^2)^* = (TT)^* = T^*T^* = (T^*)^2 .$$

So

$$\begin{aligned}(T^2)(T^2)^* &= (TT)(T^*T^*) = T(TT^*)T^* = (TT^*)(TT^*) \\ &= (T^*T)(T^*T) = T^*(TT^*)T = T^*(T^*T)T \\ &= (T^*T^*)(TT) = (T^*)^2(T^2) \\ &= (T^2)^*(T^2)\end{aligned}$$

i.e. $T^2 \leftrightarrow (T^2)^*$, i.e. T^2 is normal. By induction if follows that $T \in B(H)$ is normal $\Rightarrow T^n$ is normal for each $n \in \mathbb{N}$.

□

We are now in a position to prove the result

Proposition 2.4.6

If H is a Hilbert space and $T \in B(H)$ is normal then $\|T^n\| = \|T\|^n$ and $r(T) = \|T\|$.

Proof:

Since $T \in B(H)$ so is T^n $\forall n \in \mathbb{N}$. If $S, T \in B(H)$ we know $ST \in B(H)$ and $\|ST\| \le \|S\|\|T\|$ (see [13]). Putting $S = T$ we get $\|T^2\| \le \|T\|^2$. By induction on n we get

$$\|T^n\| \le \|T\|^n \qquad (2.11).$$

Hence we need to show that if $T \in B(H)$ is normal then $\|T\|^n \le \|T^n\|$. Since T is normal, we have

$$\|Tx\| = \|T^* x\| \quad \forall x \in H,$$

putting Tx in place of x we get

$$\|T(Tx)\| = \|T^*(Tx)\| \; \forall x \in H \text{ i.e. } \|T^2 x\| = \|T^* Tx\| \; \forall x \in H.$$

Taking the supremum of both sides over all $x \in H$ satisfying $\|x\| \le 1$ we obtain $\|T^2\| = \|T^* T\|$. But $\|T^* T\| = \|T\|^2$ (lemma 2.1.12). Now

$$\|T^2\| = \|T\|^2 \text{ if T is normal} \qquad (2.12).$$

Likewise since T^2 is normal, replacing T by T^2 in (2.12) we get $\|T^4\| = \|T^2\|^2 = \left(\|T\|^2\right)^2 = \|T\|^4$. In general it follows (by induction) that $\|T^m\| = \|T\|^m$ for all $m = 2^k$ where $k \in \mathbb{N}$. Consider any $n \in \mathbb{N}$. We can always write $n = 2^m - r$ for some $r \in \mathbb{N}$, therefore $n + r = 2^m$. Then $\|T^{n+r}\| = \|T\|^{n+r}$ i.e.

$$\|T\|^{n+r} = \|T^{n+r}\| = \|T^n T^r\| \le \|T^n\|\|T^r\| \le \|T^n\|\|T\|^r$$

i.e $\|T\|^n \|T\|^r \le \|T^n\|\|T\|^r$, canceling $\|T\|^r$ from both sides (of course $\|T\| \ne 0$) we get

$$\|T\|^n \le \|T^n\|, \qquad (2.13)$$

which is the required reverse inequality. Thus from (2.11) and (2.13) if $T \in B(H)$ is normal, then $\|T^n\| = \|T\|^n$ $\forall n \in \mathbb{N}$.

Now

$$r(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\|T\|^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\|T\|=\|T\|$$

□

Definition 2.4.7

Let X be a normed linear space and S be a non-void subset of X. The intersection of all the closed linear subspaces of X containing S is denoted by $\vee S$.

Remark:

The family of all closed linear subspaces of X containing S is clearly non-void. For X is a closed linear subspace belonging to this family. The intersection of any family of closed subsets of X is closed and also the intersection of any family of linear subspaces of X containing S is again a linear subspace containing S. Hence the intersection of all closed linear subspaces of X containing S is a closed linear subspace of X. Thus $\vee S$ is closed and contains S. $\vee S$ is the "smallest" closed linear subspace of X containing S.

Proposition 2.4.8

$\vee S=\overline{[S]}$, where $[S]$ is the linear span of S and the bar indicates closure (strong) in the normed linear space X.

Proof:

$[S]$ is the linear subspace spanned by S and, obviously, $\overline{[S]}\supseteq S$ (see [4]). Hence $\overline{[S]}$ is a closed linear subspace of X containing S. Hence $\overline{[S]}\supseteq\vee S$, for $\vee S$ =intersection of all closed linear subspaces of X containing S. It remains to establish the reverse inclusion i.e. $\overline{[S]}\subseteq\vee S$. Let $x\in\overline{[S]}$. Hence there is a sequence $(x_n)\in[S]$ such that $x_n\xrightarrow{s}x$. Each x_n is a linear combination of elements of S. Hence each x_n belongs to any linear subspace of X containing S. Hence x belongs to the intersection of all closed linear subspaces of X containing S i.e. $x\in\vee S$. Thus $\overline{[S]}\subseteq\vee S$.Hence $\vee S=\overline{[S]}$.

□

Notation:

Suppose $\{M_\alpha : \alpha \in \Lambda\}$ is a family of linear subspaces of a normed linear space X. Then $\overline{\left[\bigcup_{\alpha\in\Lambda} M_\alpha\right]} = \vee\left\{\bigcup_{\alpha\in\Lambda} M_\alpha\right\}$ is denoted by the symbol $\vee_{\alpha\in\Lambda} M_\alpha$. If cardinality $\Lambda = n$, say $\Lambda = \{1,2,\cdots,n\}$ then

$$\vee_{\alpha\in\Lambda} M_\alpha = \vee_{i=1}^{n} M_i = \vee\{M_1 \cup M_2 \cup \cdots \cup M_n\}$$

which may be written as $M_1 \vee M_2 \vee \cdots \vee M_n$ also. Thus $\vee_{\alpha\in\Lambda} M_\alpha$ is the "smallest" closed linear subspace of X containing each one of the M_α's. Likewise, if each M_α is a linear space of X, then $\bigcap_{\alpha\in} M_\alpha$ is the "largest" linear subspace of X contained in all of the M_α's. We use the symbol $\wedge_{\alpha\in\Lambda} M_\alpha$ to represent $\bigcap_{\alpha\in\Lambda} M_\alpha$. In practice, we are interested in closed linear subspaces of X only and in the set Γ of all closed linear subspaces of X, we have two closed operations $\vee, \wedge$ which ensure that if $M_\alpha \in \Gamma \ \forall \alpha \in \Lambda$, then

$$\vee_{\alpha\in\Lambda} M_\alpha, \ \wedge_{\alpha\in\Lambda} M_\alpha \in \Gamma.$$

So Γ is a closed lattice with respect to the (lattice) operations $\vee, \wedge$. In particular if $X = H$, a Hilbert space, then we have an additional operation $\perp$ which gives the orthogonal complement. We know that if M is a closed linear subspace of H, then $M^\perp$ is also a closed linear subspace of H (see [4]). Thus $M \in \Gamma \Rightarrow M^\perp \in \Gamma$. So in the lattice of all closed linear subspaces of H we have three operations $\vee, \wedge$ and $\perp$ with respect to which Γ is closed. Moreover, we have

Proposition 2.4.9

Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of closed linear subspaces of a Hilbert space H. Then

$$\left(\vee_{\alpha\in\Lambda} M_\alpha\right)^\perp = \wedge_{\alpha\in\Lambda} M_\alpha^\perp$$

and

$$\left(\wedge_{\alpha\in\Lambda} M_\alpha\right)^\perp = \vee_{\alpha\in\Lambda} M_\alpha^\perp$$

(De Morgan "type" relations). For a proof see [4].

Definition 2.4.10

We take the standard base $\{e_n : n = 0,1,2,\cdots\}$ in H where $e_n = (0,0,0,\cdots,0,1,0,\cdots)$ where n is in $(n+1)^{th}$ place i.e.

$$e_0 = (1,0,0,\cdots)$$

$$e_1 = (0,1,0,0,\cdots) \text{ e.t.c.}$$

Let

$$x = (x_0, x_1, x_2, \cdots) \in \ell^2 \quad i.e \sum_{n=0}^{\infty} |x_n|^2 < \infty .$$

Define $U : \ell^2 \to \ell^2$ by

$$Ux = (0, x_0, x_1, x_2, \cdots) \; \forall x \in \ell^2 .$$

Thus

$$Ue_0 = e_1, Ue_1 = e_2, \cdots, Ue_n = e_{n+1} \; \forall n = 0,1,2,\cdots .$$

U is easily seen to be linear [38] and is called the ***unilateral right shift operator*** on ℓ^2. It is bounded indeed

$$\|Ux\| = \|(0, x_0, x_1, x_2, \cdots)\| = \sqrt{\sum_{n=0}^{\infty} |x_n|^2} = \|x\| \; \forall x \in \ell^2 .$$

So U is an isometry. Now $\|U\| = 1$. Since $\sigma(U)$ is such that $|\lambda| \le \|U\| \; \forall \lambda \in \sigma(U)$, it follows that $|\lambda| \le 1 \; \forall \lambda \in \sigma(U)$ i.e. the spectrum of U is contained in the closed unit disc of $\mathbb{C}$. Since $\|U^*\| = \|U\|$ (see [18]), it follows that $\sigma(U^*)$ is also contained in the closed unit disc of $\mathbb{C}$. Let $x = (x_0, x_1, x_2, \cdots)$, $y = (y_0, y_1, y_2, \cdots)$ both be in ℓ^2. Then

$$\langle Ux, y \rangle = \langle (0, x_0, x_1, x_2, \cdots), (y_0, y_1, y_2, \cdots) \rangle$$

$$= x_0 \overline{y_1} + x_1 \overline{y_2} + \cdots$$

$$= \langle (x_0, x_1, x_2, \cdots), (y_1, y_2, y_3, \cdots) \rangle = \langle x, (y_1, y_2, \cdots) \rangle .$$

But $U \in B(H)$ so

$$\langle Ux, y \rangle = \langle x, U^* y \rangle$$

(by definition 2.1.9). So

$$\langle x, U^*y\rangle = \langle x, (y_1, y_2, \cdots)\rangle \quad \forall x \in H$$

therefore

$$U^*y = (y_1, y_2, \cdots) \text{ i.e. } U^*(y_0, y_1, y_2, \cdots) = (y_1, y_2, \cdots).$$

Thus U^* is a contraction, viz $\|U^*\| \leq 1$. U^* is called the ***left-shift operator*** on ℓ^2. In fact $\|U\| = 1$. For if $x = (x_0, x_1, \cdots) \in \ell^2$ and $x \neq \bar{0}$ then $Ux = (0, x_0, x_1, \cdots)$ so $\|Ux\| = \|x\| \quad \forall x \in H$. Therefore $\|U\| = 1 = \|U^*\|$ i.e. U is an isometry.

Definition 2.4.11

Let $\{e_n : n = 0, \pm 1, \pm 2, \cdots\}$ be an orthonormal basis for ℓ^2. The order in orthonormal basis is $\{\cdots, e_{-3}, e_{-2}, e_{-1}, (e_0), e_1, e_2, e_3, \cdots\}$. The bracket () indicates the central position(occupied by e_0). If $x \in \ell^2$, we represent it by the Fourier series $x = \sum_{i=-\infty}^{\infty} x_i e_i$ or coordinates as $\{\cdots, x_{-3}, x_{-2}, x_{-1}, (x_0), x_1, x_2, x_3, \cdots\}$. The ***bilateral shift*** S is defined by

$$Sx = (\cdots, x_{-3}, x_{-2}, (x_{-1}), x_0, x_1, x_2, x_3, \cdots)$$

where the shift is to the right by one place so that x_{-1} has moved to the central position. Clearly $\|Sx\| = \|x\| \quad \forall x \in \ell^2$. Therefore S is one-to-one and onto. S is unitary. It can easily be verified that $S^*x = (\cdots, x_{-3}, x_{-2}, x_{-1}, x_0, (x_1), x_2, x_3, \cdots)$ (see [17]) where the coordinates are shifted to the left by one place.

Definition 2.4.12

Let $\{e_n : n = 0, \pm 1, \pm 2, \cdots\}$ $(or\ \{e_n : n = 0, 1, 2, \cdots\})$ be an orthonormal basis for ℓ^2. The operator D defined by $De_n = \alpha_n e_n \ \forall n \in \mathbb{N}$ is called a ***diagonal operator*** with diagonal $\{\alpha_n\}$.

Definition 2.4.13

An operator $A = SD$, where S is a shift operator (unilateral or bilateral) and D is a diagonal operator is called a ***weighted shift*** with weights $\{\alpha_n\}$, where $\{\alpha_n\}$ is the diagonal of D. Thus if $\{e_n\}$ is an orthonormal basis for H, then

$$SDe_n = S(\alpha_n e_n) = \alpha_n Se_n = \alpha_n e_{n+1} \ \forall n \in \mathbb{N}.$$

Thus if S is the unilateral shift operator then

$$\begin{aligned} SD(x_0e_0 + x_1e_1 + x_2e_2 + \cdots) &= S(x_0\alpha_0e_0 + x_1\alpha_1e_1 + x_2\alpha_2e_2 + \cdots) \\ &= x_0\alpha_0Se_0 + x_1\alpha_1Se_1 + x_2\alpha_2Se_2 + \cdots \\ &= x_0\alpha_0e_1 + x_1\alpha_1e_2 + x_2\alpha_2e_3 + \cdots \\ &= (0, x_0\alpha_0, x_1\alpha_1, x_2\alpha_2, \cdots)(e_0, e_1, e_2, e_3 \cdots) \end{aligned}$$

Definition 2.4.14

$A \in B(H)$ is said to be ***nilpotent*** if $A^k = 0$ for a positive integer k and the smallest positive integer k for which $A^k = 0$ is called the ***index of nilpotence***. If A is nilpotent $\Rightarrow \sigma(A) = \{0\}$. For $A^k = 0$ so $r(A^k) = 0 \Rightarrow \sigma(A^k) = \{0\}$, by proposition 2.4.2

$$\sigma(A^k) = \{\sigma(A)\}^k = \{\lambda^k : \lambda \in \sigma(A)\} = \{0\},$$

therefore $\lambda = 0$. A^k has a non-trivial kernel so $P\sigma(A) = \{0\}$, i.e. non-void. $A \in B(H)$ is said to be ***quasi-nilpotent*** if $\lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} = 0$. A is nilpotent $\Rightarrow A$ is quasinilpotent. If A is quasinilpotent, $r(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} = 0$ i.e. $\sigma(A) = \{0\}$. If A is nilpotent, $P\sigma(A) = \{0\}$ i.e. is non-void, in the case of a quasinilpotent operator, this in not so (see [11]).

Definition 2.4.15

Let (X, χ, μ) be a measure space and $f : X \to \mathbb{C}$ be a function. We say that f is ***essentially bounded*** if $|f|$ is bounded $\mu.a.e$ on X. In this case there exists a positive $a \in \mathbb{R}$ such that

$$\mu(\{x \in X : |f(x)| > a\}) = 0,$$

then

$$\inf\{a : \mu(\{x \in X : |f(x)| > a\})\} = 0$$

is called the ***essential supremum*** of the function f and represented by the symbol $\|f\|_\infty$.

Clearly for each $k \in \mathbb{N}$

$$\mu\left(\left\{x \in X : |f(x)| > \|f\|_\infty + \frac{1}{k}\right\}\right) = 0 \qquad (2.14)$$

and

$$\mu\{x\in X:|f(x)|>\|f\|_{\infty}\}=\mu\left\{x\in X:|f(x)|>\|f\|_{\infty}+\frac{1}{k}\right\}.$$

Therefore by countable subadditivity of μ

$$\mu(\{x\in X:|f(x)|>\|f\|_{\infty}\})\leq\sum_{k=1}^{\infty}\mu\left(\left\{x\in X:|f(x)|>\|f\|_{\infty}+\frac{1}{k}\right\}\right)=0$$

(by (2.14)) i.e.

$$\mu(\{x\in X:|f(x)|>\|f\|_{\infty}\})=0$$

therefore

$$|f(x)|\leq\|f\|_{\infty}\ \mu.a.e\ \text{on } X$$

i.e. f is bounded $\mu.a.e$ on X. Clearly if f is bounded on X, then f is essentially bounded. The set of all essentially bounded χ-measurable functions ((X,χ,μ) is a measure space) is represented by $L^{\infty}(X,\chi,\mu)$ or L^{∞} or $L^{\infty}(\mu)$. An operator A on $L^2(\mu)$ given by $Af=\phi.f$ where $f\in L^2$ and $\phi:X\to\mathbb{C}$ is called a ***multiplication operator*** on L^2 corresponding to the ***multiplier*** ϕ. A is called the ***operator induced by*** ϕ. Incase $X=\mathbb{N}$ and μ is the counting measure (i.e. the measure of a subset of $\mathbb{N}$ is the number of elements in that subset) then the operator A reduces to a diagonal operator. The subset Σ of $\mathbb{C}$ defined by

$$\Sigma=\{\lambda\in\mathbb{C}:\mu(\phi^{-1}(N(\lambda;\varepsilon)))>0\ \textit{for all real}\ \varepsilon>0\}$$

is called the ***essential range*** of an essentially bounded function $\phi:X\to\mathbb{C}$.

Definition 2.3.11

Suppose A, B are operators on Hilbert spaces H, K. Consider

$$H\oplus K=\{(x,y):x\in H\ and\ y\in K\}$$

(H is identified with $H\oplus\{\overline{0}_K\}$, K is identified with $\{\overline{0}_H\}\oplus K$.). The operator $A\oplus B$ called the ***direct sum*** of A and B is defined by

$$(A\oplus B)(x,y)=(Ax,By)\in H\oplus K$$

2.5 CLASSES OF NON-NORMAL OPERATORS

Definition 2.5.1

An operator $T \in B(H)$ is said to be ***quasinormal*** if $T^*T \leftrightarrow T$, i.e. $\left(T^*T\right)T = T\left(T^*T\right)$. It is obvious that if a bounded linear operator is normal, then it is also quasinormal i.e. normality $\Rightarrow$ quasinomality. For T is normal $\Leftrightarrow T^*T = TT^*$ therefore

$$\left(T^*T\right)T = \left(TT^*\right)T = T\left(T^*T\right)$$

i.e. $T^*T \leftrightarrow T$, i.e. T is quasinormal. The converse is not true. For instance, consider an isometry $U \in B(H)$ which is not unitary, i.e. not normal. Then $U^*U = I$ (but $UU^* \neq I$).

Clearly $U^*U \leftrightarrow U$ i.e.

$$I \leftrightarrow U \Rightarrow IU = UI \text{ i.e. } \left(U^*U\right)U = U\left(U^*U\right)$$

Thus U is quasinormal but not normal. An illustration of such a U is the simple unilateral shift.

Proposition 2.5.2

An operator $T \in B(H)$ with polar decomposition $T = UP$ (where U is an isometry from $\overline{\Re}_P$ onto $\overline{\Re}_T$ and $P = \sqrt{T^*T} \geq 0$) is quasinormal if and only if $UP = PU$

Proof:

First let $UP = PU$ i.e. $U \leftrightarrow P$. Now

$$T^*T = (UP)^*(UP) = P^*U^*UP = P^*P = P^2$$

(for $U^*U = I$ and $P^* = P$ i.e. P self adjoint). Clearly then $U \leftrightarrow P^2$. Indeed

$$UP^2 = (UP)P = (PU)P = P(UP) = P(PU) = P^2U$$

i.e. $U \leftrightarrow P^2$. Now $P \leftrightarrow P^2$ obviously. Thus

$$\left(T^*T\right)T = P^2T = P^2(UP) = \left(P^2U\right)P = \left(UP^2\right)P = UP^3$$

and

$$T\left(T^*T\right) = (UP)P^2 = UP^3.$$

Thus $\left(T^*T\right)T = T\left(T^*T\right) \Rightarrow T^*T \leftrightarrow T$ i.e T is quasi-normal. Conversely let T be quasinormal with polar decomposition $T = UP$. We must show that $UP = PU$. So

$T \leftrightarrow T^*T$. Now $T^*T \geq 0$ and $P = \sqrt{T^*T}$. Any operator which commutes with T^*T commutes with its square-root, therefore $T \leftrightarrow \sqrt{T^*T} = P$ i.e. $T \leftrightarrow P$ i.e. $UP \leftrightarrow P$ i.e. $(UP)P = P(UP)$ i.e. $UP^2 = PUP$ therefore $UP^2 - PUP = 0$ i.e. $(UP - PU)P = 0$ i.e. $UP - PU$ annihilates the range of P i.e. $\Re_P$ i.e. $UP - PU$ annihilates $\overline{\Re_P}$ (by continuity of $UP - PU$ on H). Consider $\overline{\Re}_P^{\perp}$, with $\eta_P = \eta_U$ also

$$\left.\begin{array}{l} UPx = \bar{0} \ \ \forall x \in \eta_P \\ and \\ PUx = \bar{0} \ \ \ \forall x \in \eta_P \end{array}\right\} \ and\ \eta_U = \eta_P.$$

Thus

$$(UP - PU)x = \bar{0} \ \ \forall x \in \overline{\Re}_P^{\perp}.$$

Since $H = \overline{\Re_P} \oplus (\eta_U \ or\ \eta_P)$ it follows that

$$(UP - PU)x = \bar{0} \ \ \forall x \in H,$$

therefore $UP - PU = 0$ i.e. $UP = PU$.

□

Definition 2.5.3

Let H and K be Hilbert spaces and $T : H \to K$ a linear map. We say that T is a ***partial isometry*** if there is a closed subspace M of H such that $T|_M$ is an isometry and $Tx = \bar{0}\ \forall x \in M^{\perp}$ (thus $\|Tx\| = \|x\|\ \forall x \in M$). M is then called the ***initial space*** of the partial isometry T, whereas $T(M)$ is called the ***final space*** of T.

Definition 2.5.4

An operator $T \in B(H)$ (where H is a Hilbert space) is said to be ***subnormal*** if there is a Hilbert space K of which H is a closed subspace and a normal operator $\mathrm{N} \in \mathrm{B}(\mathrm{K})$ such that

i. H is invariant under N i.e. $Nx \in H \quad \forall x \in H$ and

ii. $N|_H = T$.

In other words, T is said to be subnormal if it has a normal extension.

Remark:

Every normal operator $T \in B(H)$ is obviously subnormal. There are operators which are subnormal but not normal. For instance consider the unilateral shift U ; which is not normal. But U has a normal extension which is the bilateral shift (note that the bilateral shift is unitary and consequently normal and if A is the bilateral shift and $M = \bigvee_{n=0}^{\infty} e_n$ then $A|_M = U$, the simple unilateral shift)

Proposition 2.5.5

If $T \in B(H)$ is quasinormal then T is subnormal, thus normality $\Rightarrow$quasinormality$\Rightarrow$subnormality

Proof:

First of all for a $T \in B(H)$, $\eta_T = \eta_{T^*T}$. Indeed, let $x \in \eta_T$. Then $Tx = \overline{0}$. Hence $T^*(Tx) = \overline{0}$ i.e. $x \in \eta_{T^*T}$ i.e

$$\eta_T \subseteq \eta_{T^*T} \tag{2.15}$$

Conversely let $x \in \eta_{T^*T}$ i.e. $T^*Tx = \overline{0}$ then $\langle T^*Tx, x\rangle = 0$ i.e.

$$\langle Tx, Tx\rangle = 0 \Rightarrow \|Tx\|^2 = 0$$

i.e. $Tx = 0$ i.e. $x \in \eta_T$, therefore

$$\eta_{T^*T} \subseteq \eta_T \tag{2.16}$$

(2.15) and (2.16) imply

$$\eta_T = \eta_{T^*T}.$$

If T is quasinormal, then $(T^*T)T = T(T^*T)$. Taking adjoints of both sides we get

$$T^*T^*T = T^*TT^* \text{ i.e. } T^*(T^*T) = (T^*T)T^*$$

i.e. $T^*T \leftrightarrow T^*$, therefore η_{T^*T} is invariant under T^* (if $A \leftrightarrow B$ then η_A is invariant under B. $x \in \eta_A \Rightarrow Ax = \overline{0}$, $B(Ax) = \overline{0}$ i.e. $BAx = \overline{0}$ but $BA = AB$ therefore $A(Bx) = \overline{0}$ $\Rightarrow Bx \in \eta_A$. Thus $x \in \eta_A$ and $A \leftrightarrow B \Rightarrow Bx \in \eta_A$ i.e. η_A is invariant under B). Since $T^*T \leftrightarrow T$ so η_{T^*T} is invariant under T. Thus η_{T^*T} is invariant under T and T^*.

Therefore η_{T^*T} is a reducing subspace for T. Since $\eta_{T^*T} = \eta_T$ (for any $T \in B(H)$, we have; η_T is a reducing subspace for T (when T is quasinormal) i.e. $\eta_T, \eta_T^{\perp}$ (note $\eta_T \oplus \eta_T^{\perp} = H$) are both invariant under T. We can therefore decompose T into two parts

$$T = T' \oplus T'' \text{ where } T' : \eta_T \to \eta_T \text{ and } T'' : \eta_T^{\perp} \to \eta_T^{\perp}.$$

Replace T' by 0 and then T'' acts on the Hilbert space $\eta_T^{\perp}$ and T'' has the trivial null space, i.e. $\eta_{T''} = \{\bar{0}\}$ and since $T = 0 \oplus T''$ and T is quasinormal it follows that T'' is quasinormal. We can therefore without loss of generality consider a quasinormal $T \in B(H)$ with the trivial kernel. Let T have the polar decomposition UP; where U is a partial isometry and $P \geq 0$. Let E be the orthogonal projection UU^*. Then

$$(I-E)U = 0 = U^*(I-E)$$

(for $(I-E)U = U - EU = U - UU^*U = 0$ and $U^*(I-E) = U^* - U^*E = U^* - U^*UU^* = 0$ for U, U^* are partial isometries, U^*U is projection onto initial space of U, UU^* is projection onto final space of U). Let $V, Q \in B(H \oplus H)$ defined by

$$V = \begin{pmatrix} U & (I-E) \\ 0 & U^* \end{pmatrix} \quad Q = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

The following facts are now verifiable

i) Q is positive, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in H \oplus H$

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Px \\ Py \end{pmatrix}$$

i.e.

$$\left\langle \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} Px \\ Py \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

$$= \langle Px, x \rangle + \langle Py, y \rangle$$

$$\geq 0 \ \forall \begin{pmatrix} x \\ y \end{pmatrix} \in H \oplus H$$

i.e. $Q \geq 0$.

ii) V is unitary, for

$$V^* = \begin{pmatrix} U^* & 0 \\ (I-E)^* & (U^*)^* \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix}$$

$$V^*V = \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix} \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix}$$

$$= \begin{pmatrix} U^*U & U^*(I-E) \\ (I-E)U & (I-E)^2 + UU^* \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & (I-E)+UU^* \end{pmatrix} \text{ But } E = UU^*$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$VV^* = \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix}$$

$$= \begin{pmatrix} UU^* + (I-E)^2 & (I-E)U \\ U^*(I-E) & U^*U \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

(Note since $\eta_T = \{0\}$ (see above) the partial isometry U is isometric on $\eta_T^\perp = H$, i.e. U is an isometry hence $U^*U = I$) therefore

$$V^*V = VV^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$\Rightarrow V$ is unitary. Also $V \leftrightarrow Q$ indeed

$$VQ = \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

$$= \begin{pmatrix} UP & (I-E)P \\ 0 & U^*P \end{pmatrix}$$

and

$$QV = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix}$$

$$= \begin{pmatrix} PU & P(I-E) \\ 0 & PU^* \end{pmatrix}.$$

Now $UP = PU$ and $U^*P = PU^*$ and

$$P(I-E) = P - PE = P - PUU^* = P - (PU)U^*$$

$$= P - (UP)U^* = P - U(PU^*) = P - U(U^*P)$$

$$= P - (UU^*)P = P - EP$$

$$= (I-E)P.$$

The product

$$VQ = \begin{pmatrix} UP & (I-E)P \\ 0 & U^*P \end{pmatrix}$$

is the required normal extension of the subnormal operator $T = UP$.

□

Remark 1:

If $T \in B(H)$ is normal, then it has the polar decomposition $T = UP$ with U unitary and P positive and $UP = PU$. Since T is normal $\|T^*x\| = \|Tx\| \quad \forall x \in H$. Thus since $P = \sqrt{T^*T}$ we have already seen that $\|Tx\| = \|Px\|$ therefore $\|Tx\| = \|Px\| = \|T^*x\| \quad \forall x \in H$, therefore

$$\eta_T = \eta_P = \eta_{T^*}.$$

In the decomposition $T = UP$ (for a $T \in B(H)$), U was an isometry from $\overline{\Re}_P$ to $\overline{\Re}_T$. Since $T \in B(H)$ is now taken to be normal, we have

$$\overline{\Re}_P = \eta_{P^*}^{\perp} = \eta_P^{\perp} (\because P \geq 0)$$

$$= \eta_{T^*}^{\perp} \left(\because \eta_P = \eta_{T^*} \; when\, T\; is\; normal\right)$$

$$= \overline{\Re}_{(T^*)^*}^{\perp} = \overline{\Re}_T$$

Thus the isometry $U:\overline{\Re_P}\to\overline{\Re_T}$ is simply an isometry $U:\overline{\Re_T}\to\overline{\Re_T}$. We can define an extension of U to H by letting say $Ux=x \quad \forall x\in\overline{\Re_T}^{\perp}$. Thus $\|Ux\|=\|x\| \quad \forall x\in\overline{\Re_T}^{\perp}$. Since $\|Ux\|=\|x\| \quad \forall x\in\overline{\Re_T}$. It follows that U is an isometry on H. It is also clear that

$$\Re_U=\overline{\Re_T}\oplus\overline{\Re_T}^{\perp}=H.$$

Remark 2:

From the expression

$$VQ=\begin{pmatrix} UP & (I-E)P \\ 0 & U^*P \end{pmatrix}$$

It is evident that $VQ|_H=UP=T:H\to H$ (see [8], [18] and [44]).

Remark 3:

If $T\in B(H)$ is subnormal, what can we say about T^*? Since T is subnormal, there is a Hilbert space $K\supset H$ and a normal operator $N\in B(K)$ and such that $N|_H=T$. Therefore $\forall x,y\in H$

$$\langle T^*x,y\rangle=\langle x,Ty\rangle=\langle x,Ny\rangle \quad (\text{since } N|_H=T)$$

$$=\langle N^*x,y\rangle=\langle N^*x,Py\rangle$$

where P is the orthogonal projection on K onto its subspace H (since $y\in H=\Re_p$, it follows that $Py=y$).

$$\langle T^*x,y\rangle=\langle P^*N^*x,y\rangle=\langle PN^*x,y\rangle \text{ (for } P^*=P).$$

Thus the operator $PN^*\in B(K)$ is invariant under H and T^* is the restriction of PN^* to H. Thus $T^*x=PN^*x \quad \forall x\in H$. Also

$$\|T^*x\|^2=\|PN^*x\|^2\le\|N^*x\|^2$$

($\|P\|\le 1$ for P is an orthogonal projection (by proposition 2.2.2)). Therefore since N is normal $\|N^*x\|=\|Nx\|$. Thus

$$\|T^*x\|^2\le\|Nx\|^2 \text{ i.e. } \|T^*x\|^2\le\|Tx\|^2 \text{ (since } N|_H=T)$$

that is

$$\langle T^*x, T^*x\rangle \le \langle Tx, Tx\rangle$$

i.e.

$$\langle TT^*x, x\rangle \le \langle T^*Tx, x\rangle \qquad (2.17).$$

Since TT^*, T^*T are positive operators for any $T \in B(H)$ equation (2.17) implies $TT^* \le T^*T$. Thus T is subnormal implies $TT^* \le T^*T$.

Definition 2.5.6

An operator $T \in B(H)$ is said to be ***hyponormal*** if $TT^* \le T^*T$. Thus from our discussions we conclude that; every subnormal operator is hypornormal. So T is normal $\Rightarrow T$ is quasinormal $\Rightarrow T$ is subnormal $\Rightarrow T$ is hypornormal.

There is another way of looking at a subnormal operator which extends T and acts on a Hilbert space $K \supset H$ so, $T \subset N$ (i.e. N extends T). Hence we may write.

$$N = \begin{pmatrix} T & R \\ 0 & S \end{pmatrix}$$

where the matrix elements T and 0 guarantee for

i) N extends T or $N|_H = T$ and

ii) N is invariant under H (i.e. maps H into H)

Since N is normal, $N \leftrightarrow N^*$. Now

$$NN^* = \begin{pmatrix} T & R \\ 0 & S \end{pmatrix}\begin{pmatrix} T^* & 0 \\ R^* & S^* \end{pmatrix} = \begin{pmatrix} TT^* + RR^* & RS^* \\ SR^* & SS^* \end{pmatrix}$$

$$N^*N = \begin{pmatrix} T^* & 0 \\ R^* & S^* \end{pmatrix}\begin{pmatrix} T & R \\ 0 & S \end{pmatrix} = \begin{pmatrix} T^*T & T^*R \\ R^*T & R^*R + S^*S \end{pmatrix}$$

$$N^*N - NN^* = \begin{pmatrix} T^*T & T^*R \\ R^*T & R^*R + S^*S \end{pmatrix} - \begin{pmatrix} TT^* + RR^* & RS^* \\ SR^* & SS^* \end{pmatrix}$$

$$= \begin{pmatrix} T^*T - (TT^* + RR^*) & T^*R - RS^* \\ R^*T - SR^* & (R^*R + S^*S) - SS^* \end{pmatrix}.$$

Since $N^*N - NN^* = 0$; we must necessarily have $T^*T - TT^* - RR^* = 0$ i.e. $T^*T - TT^* = RR^*$ and $RR^* \geq 0$. Thus $T^*T - TT^* \geq 0$. Therefore

$$\langle (T^*T - TT^*)x, x\rangle = \langle RR^*x, x\rangle \geq 0 \quad \forall x \in H$$

i.e. $\langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle$ which shows that $TT^* \leq T^*T$.

Example:

The simple unilateral shift U on ℓ_+^2 is subnormal and its normal extension is the bilateral shift W. Is U^* subnormal?, hyponormal? Now U is an isometry on ℓ_+^2 so $U^*U = I$. If U^* were subnormal, then U^* would be hyponormal. In the latter case

$$U^*(U^*)^* \leq (U^*)^* U^* \text{ i.e. } U^*U \leq UU^*.$$

But $U^*U = I$ (for a simple unilateral shift) and UU^* is an orthogonal projection on H. We cannot have $I \leq UU^*$ therefore U^* is not subnormal and not hyponormal. Incase U^* is subnormal (and therefore hyponormal) we must have $UU^* = I$ i.e. $U^*U = I$ and $UU^* = I$ i.e. U is unitary. There are hyponormal operators which are not subnormal; consider the unilateral weighted shift U with weights (α_n) where α_n is a sequence of non-zero complex numbers $(n = 0, 1, 2, \cdots)$ the operator U is not subnormal but is hyponormal (see [8] and [44]).

Definition 2.5.7

If $T \in B(H)$ is such that T^* is hyponormal then T is called a ***semi-normal*** operator

Proposition 2.5.8

If $T \in B(H)$ is quasinormal (subnormal, hyponormal) and idempotent then T is an orthogonal projection.

For a proof see [8], [26] or [47].

Proposition 2.5.9

A partially isometric operator is normal if and only if it can be written as a direct sum of a unitary operator and a zero operator. It is subnormal if and only if it can be written as a sum of an isometric operator and a zero operator.

Proof:

Let V be a partial isometry, M its initial space, so $V(M)$ is final space of V. Then $V^*V =$ projection on M, $VV^* =$ projection on final space $V|_M$ is an isometry.

$$V = V|_M \oplus V|_{M^\perp}, \quad V|_{M^\perp} = 0.$$

By the Wold-decomposition theorem (see [47] and [38]) an isometry can be written as a pure isometry plus a zero-operator and the pure isometry being a unilateral shift is subnormal, for it has a normal extension(bilateral shift). If $V|_M$ is unitary, then $V|_M$ is normal, so $V|_M \oplus V|_{M^\perp}$ is also normal. Conversely; let V be a partial isometry which is hyponormal (subnormal operators are hyponormal) then $VV^* \leq V^*V$ i.e. $P_{V(M)} \leq P_M$ (where P is the orthogonal projection on H onto M e.t.c).Therefore $V(M) \subseteq M$ i.e. M is invariant under V. Clearly $M^\perp$ is also invariant under V (for $x \in M^\perp \Rightarrow Vx = \overline{0} \in M^\perp$) therefore M reduces V. Thus $V|_M$ is an isometry and $V|_{M^\perp}$ is 0. If V is normal then $VV^* = V^*V$, hence $V(M) = M$, therefore $V|_M : M \to V(M) = M$ being an isometry is unitary.

□

Proposition 2.5.10

If $T \in B(H)$ is hyponormal then $\|T^n\| = \|T\|^n \quad \forall n \in \mathbb{N}$.

(For a proof see [44]).

Corollary 2.5.11

If a hyponormal operator is quasi-nilpotent, it must be the zero operator.

Proof:

If $T \in B(H)$ is hyponormal then $\|T^n\| = \|T\|^n \; \forall n \in \mathbb{N}$ (proposition 2.5.10) i.e.

$$\|T^n\|^{1/n} = \|T\| \quad \forall n \in \mathbb{N}.$$

Now T is quasinilpotent $\Rightarrow \lim_{n\to\infty} \|T^n\|^{1/n} = 0$ i.e. T is the zero operator.

□

Definition 2.5.12

An operator $T \in B(H)$ is said to be ***(N,1)-normal*** or ***paranormal*** if all $x \in H$ with $\|x\| = 1$, we have $\|Tx\|^2 \le \|T^2 x\|$. $T \in B(H)$ is said to be ***(N,k)-normal*** or ***k-paranormal*** if $\|Tx\|^k \le \|T^k x\|$ for all $x \in H$ satisfying $\|x\| = 1$

2.6 HYPERINVARIANT SUBSPACES

Definition 2.6.1

A subspace M of a Hilbert space H is said to be ***hyperinvariant*** for an operator T on H if $SM \subset M$ (for all operators S that commute with T). This concept is related to invariant subspaces for algebras of operators: the hyperinvariant subspaces of T are the invariant subspaces of the (strongly closed) operator algebra.

$$\{T\}' = \{S / ST = TS\}$$

Of course a scalar operator $T = cI$ has only the trivial hyperinvariant subspaces $\{0\}$ and H. when H is finite dimensional the converse assertion is true: a nonscalar operator has nontrivial hyperinvariant subspaces. More generally, a well known result of Burnside [30] asserts that the only algebra of $n \times n$ matrices without nontrivial invariant subspaces is the algebra of all $n \times n$ matrices. Lomonosov proved that every non-scalar (compact) operator has a nontrivial hyperinvariant subspace. Is $B(H)$ the only strongly closed operator algebra without nontrivial invariant subspaces? An affirmative answer for the latter problem (which seems quite unlikely) would imply an affirmative answer for the former, and hence for the invariant subspace problem.

Consider the case of a normal operator N. Because the spectral projections for N commute with any operator $\{N\}'$; the spectral subspaces are hyperinvariant. On the other hand, since $A \in \{N\}'$ implies $A^* \in \{N\}'$ (proposition 2.3.12), the projection P on a hyperinvariant subspace commutes with any operator in $\{N\}'$. It is well known that when H is separable, such projections must be spectral. Thus in general it may be useful to think of the hyperinvariant subspaces as assuming the role of the spectral subspaces. The following result, due to Rosenthal and Stampfli [38], shows that certain invariant subspaces must be hyperinvariant solely by virtue of their position in the lattice of all invariant subspaces.

Proposition 2.6.2

Let δ be a countable family of invariant subspaces for an operator T, with the property that for any invariant subspaces $M \in \delta$ and $N \notin \delta$, either $M \subset N$ or $N \subset M$. Then δ consists of hyperinvariant subspaces.

Proof:

Observe first that for any operator S and $|\lambda| > \|S\|$, the operators S and $(S-\lambda I)^{-1}$ have the same invariant subspaces. For the series $\sum \lambda^{-n-1} S^n$ converges to $(S-\lambda I)^{-1}$ in the operator norm, and so $SM \subset M$ implies

$$(S-\lambda I)^{-1} M \subset M .$$

On the other hand, if $(S-\lambda I)^{-1} M \subset M$ then $M \subset (S-\lambda I) M$. If the inclusion were proper there would be a unit vector $x \in M$ orthogonal to $(S-\lambda I)x$, so that

$$0 = \langle (S-\lambda I)x, x \rangle \geq |\lambda| \langle Sx, x \rangle \geq |\lambda| - \|S\| > 0 ,$$

a contradiction. Thus $M = (S-\lambda I)M$ and $SM \subset M$. Now suppose that S commutes with T and let $M \in \delta$. Then the subspaces $(S-\lambda I)M, |\lambda| > \|S\|$, are invariant for T. If $(S-\lambda I)M \notin \delta$ for some $|\lambda| > \|S\|$, then by hypothesis

$$M \subset (S-\lambda I)M \text{ or } (S-\lambda I)M \subset M ,$$

so that $(S-\lambda I)^{-1} M \subset M$ or $(S-\lambda I)M \subset M$, and in either case $SM \subset M$. If $(S-\lambda I)M \in \delta$ for all $|\lambda| > \|S\|$, then $(S-\lambda_1 I)M = (S-\lambda_2 I)M$ for some $\lambda_1 \neq \lambda_2$ since δ is countable, and therefore

$$M = (S-\lambda_2 I)^{-1}(S-\lambda_1 I)M = \left(I + (\lambda_2 - \lambda_1)(S-\lambda_2 I)^{-1}\right)M ,$$

so that again $SM \subset M$. Hence M is hyperinvariant.

□

Corollary 2.6.3

If the invariant subspaces of T are countable in number, then every invariant subspace is hyperinvariant. (See [17] or [38]).

Corollary 2.6.4

If M is an invariant subspace of T that is comparable with every other invariant subspace of T, then M is hyperinvariant. (See [3] or [17]).

Corollary 2.6.5

If M is an invariant subspace of T from a chain, then every invariant subspace is hyperinvariant. (See [2] or [17]).

Definition 2.6.6

An operator such that the invariant subspaces form a chain (i.e. for any invariant subspace M and N, either $M \subset N$ or $N \subset M$) is called ***unicellular***. When H is finite-dimensional, the unicellular operators are those of the form $\lambda I + N$, where N is cyclic and nilpotent, (in other words, cyclic with one point spectrum). For arbitrary H, it is not known whether the spectrum of a unicellular operator must reduce to a point [38].

Proposition 2.6.7

Let M be a subspace of ℓ_+^2. If there exists $y \in \ell_+^2$ such that $|x_n| \le y_n \|x\|$ for all $x \in M$ and $n \ge 0$, then M is finite dimensional. (See [27] and [29]).

Proposition 2.6.8 (Nikolskii)

Let S be the weighted shift operator with weights $\{\lambda_n\}$, and assume that $\{|\lambda_n|\}$ is non-increasing, $\lambda_n \neq 0$ for all $n \ge 0$, and $\sum |\lambda_n|^p < \infty$ for some $p \in (0, \infty)$. Then M is a nonzero invariant subspace of S if and only if $M = M_n$ for some $n \ge 0$.

Proof:

Let n be the least integer for which there is $x \in M$ with $x_n \neq 0$. Then $M \subset M_n$, and it will be shown that equality holds. There is no loss of generality in assuming that $n = 0$, in which case it must be shown that $M = \ell_+^2$. Fix an integer $N \ge 1$ such that $p \le 2N$; then $\gamma = \{|\lambda_n|^N\} \in \ell_+^2$ (see [33]). If $x \in M$ with $x_0 \neq 0$, then because $x, Sx, \cdots, S^N x$ are linearly independent elements of M, it is clear that a suitable linear combination is a vector

$y \in M$ with $y_0 = 1$ and $y_1 = \cdots = y_N = 0$. Now let z be orthogonal to M, so that $\langle z, S^n y \rangle = 0$ for all $n \geq 0$. Then

$$\sum_{k=0}^{\infty} \lambda_k \cdots \lambda_{k+n-1} y_k \overline{z}_{n+k} = 0$$

$$\overline{z} = -(\lambda_0 \cdots \lambda_{n-1})^{-1} \sum_{k=N+1}^{\infty} \lambda_k \cdots \lambda_{k+n-1} y_k z_{n+k},$$

for all $n \geq 1$. But if $n, k \geq N+1$,

$$\left| \frac{\lambda_k \cdots \lambda_{k+n-1}}{\lambda_0 \cdots \lambda_{n-1}} \right| \leq \left| \frac{\lambda_{N+1} \cdots \lambda_{N+n}}{\lambda_0 \cdots \lambda_{n-1}} \right| = \left| \frac{\lambda_n \cdots \lambda_{N+n}}{\lambda_0 \cdots \lambda_N} \right| \leq \frac{|\lambda_n|^N}{|\lambda_0 \cdots \lambda_N|}$$

since $\{|\lambda_n|\}$ is non-increasing, and therefore $|z_n| \leq B|\lambda_n|^N \|z\|$ for all $n \geq N+1$, where $B = |\lambda_0 ... \lambda_N|^{-1} \|y\|$. Hence $\dim M^{\perp} < \infty$ (by proposition 2.6.7). If $M^{\perp} \neq \{0\}$ there must be an eigenvector $e \in M^{\perp}$ for S^*. But S^* is quasinilpotent (because S is (see [17] and [33])), so $S^* e = 0$ and therefore $e_n = 0$ for $n \geq 1$. Since $\langle e, y \rangle = 0$ and $y_0 = 1$ it follows that $e = 0$, a contradiction. Thus $M^{\perp} = \{0\}$ and $M = \ell^2_+$.

□

The rest of this section will be devoted to showing that operators that are close (in a suitable sense) to being unitary have nontrivial hyperinvariant subspaces.

Definition 2.6.9

Operators A and B are ***quasi-similar*** if there are one-to-one operators P and Q each having dense range such that $AP = PB$ and $QA = BQ$. Similar operators have isomorphic lattices of invariant subspaces. Although this does not seem to be true for quasi-similarity, we have:

Proposition 2.6.10

If A and B are quasi-similar and A has a nontrivial hyperinvariant subspace, then so does B.

Proof:

There are one-to-one operators P and Q with dense range such that $AP = PB$ and $QA = BQ$. Suppose M is hyperinvariant for A, and consider the subspace

$$N = \{SQx : x \in M \text{ and } SB = BS\}.$$

It is clear that N is hyperinvariant for B and that $N \neq \{0\}$ whenever $M \neq \{0\}$. If $SB = BS$ then $(PSQ)A = PSBQ = PBSQ = A(PSQ)$, and therefore $PSQM \subset M$ because M is hyperinvariant for A. Hence $PN \subset M$, from which it follows that $N \neq H$ whenever $M \neq H$.

□

Proposition 2.6.11

Let T be a contraction such that $\|T^n x\|$ does not $\to 0$ and $\|T^{*n} x\|$ does not $\to 0$ for all $x \neq 0$. Then T is quasi-similar to a unitary operator. (See [21], [28] or [32]).

Proposition 2.6.12

Let T be a contraction, and suppose that there are vectors x_0 and y_0 such that $\|T^n x_0\|$ and $\|T^* y_0\|$ do not approach 0. Then either T has a nontrivial hyperinvariant subspace or $T = cI$.

Proof:

The subspaces $\{x \mid T^n x \to 0\}$ and $\{y \mid T^{*n} y \to 0\}^{\perp}$ are hyperinvariant for T. If both are proper then T is quasi-similar to a unitary operator V (by proposition 2.6.11). If V is scalar so is T; otherwise T has a nontrivial hyperinvariant subspace (by proposition 2.6.10).

□

Proposition 2.6.13

Let A be an operator such that $\operatorname{Re} A$ is of finite rank and $\operatorname{Re} A \leq 0$. Then A has a nontrivial invariant subspace.

Proof:

According to the Cayley transform $T = (A + \mathrm{I})(A - I)^{-1}$ is an everywhere defined contraction [28]. In addition, because

$$I - T^*T = -4(A^* - I)^{-1}(\operatorname{Re} A)(A - I)^{-1},$$

$I - T^*T$ is of finite rank. In the same way so is $I - TT^*$. If $T^n \to 0$ strongly, then T is unitarily equivalent to a part of a backward shift of finite multiplicity, therefore T has a nontrivial invariant subspace (see [17] or [27]). If $T^{*n} \to 0$ strongly, then in the same way T^* has a nontrivial invariant subspace, and hence so does T. If neither of these is the case, a nontrivial invariant subspace exists by proposition 2.6.12, thus in all cases T has a nontrivial invariant subspace. The proof is completed by showing $TM \subset M$ implies $AM \subset M$. Since $T = I + 2(A - I)^{-1}$, $TM \subset M$ implies

$$(A-I)^{-1} M \subset M.$$

If the inclusion were proper, there would exist $x \neq 0$ such that

$$\langle (A-I)^{-1} x, x \rangle = 0,$$

so that for $y = (A-I)^{-1} x$,

$$0 = \langle y, (A-I)y \rangle = \langle y, Ay \rangle - \|y\|^2 \leq -\|y\|^2,$$

and consequently $y = 0$ and $x = 0$, a contradiction. Hence

$$(A-I)^{-1} M = M, \ (A-I)M = M, \text{ and } AM \subset M.$$

□

Remark:

This result is valid if $\operatorname{Re} A$ is compact and its sequence of eigenvalues lies in ℓ^p_+ for some $p \in [1, \infty]$ (see [21] and [27]). The question is open if $\operatorname{Re} A$ is merely compact.

2.7 COMPACT OPERATORS AND INVARIANT SUBSPACES

Definition 2.7.1

Let (X, τ) be a topological space. A subset T of X is said to be ***relatively compact*** if it's closure $\overline{T}$ is compact. We say that T is ***sequentially compact*** if every sequence (x_n) of elements of T has a convergent subsequence with limit in T.

Definition 2.7.2

Let X, Y be normed linear spaces over $\mathbb{R}$ or $\mathbb{C}$. A linear transformation $T: X \to Y$ is said to be ***compact*** (equivalently ***completely continous***) if for every bounded subset M of X, $T(M)$ is relatively compact, i.e $\overline{T(M)}$ is compact.

Proposition 2.7.3

Let H, K be Hilbert spaces and (x_n) be a weakly convergent sequence in H. Let $T: H \to K$ be compact. Then (Tx_n) is strongly convergent. (See [4], [13] or [20]).

Proposition 2.7.4

Let $T: H \to K$ be compact if $x_n \xrightarrow{w} x$ then $Tx_n \xrightarrow{s} Tx$

Proof:

Since $T \in B(H, K)$ we have $x_n \xrightarrow{w} x \Rightarrow Tx_n \xrightarrow{w} Tx$ (see [4]). Now since T is compact, (x_n) is weakly convergent $\Rightarrow (Tx_n)$ is strongly convergent (by proposition 2.7.3), say to $y \in K$. Thus $Tx_n \xrightarrow{s} y$. But strong convergence $\Rightarrow$ weak convergence. So $Tx_n \xrightarrow{s} y \Rightarrow Tx_n \xrightarrow{w} y$. By uniqueness of weak limit (of a weakly convergent sequence) we have $Tx_n \xrightarrow{w} y, Tx_n \xrightarrow{w} Tx \Rightarrow Tx = y$. Therefore $Tx_n \xrightarrow{s} y = Tx$ i.e. $Tx_n \xrightarrow{s} Tx$.

□

Proposition 2.7.5

Let H, K be a Hilbert spaces and $T \in B(H, K)$. The following statements are equivalent

i. T is compact

ii. T^*T is compact

iii. T^* is compact

Proof:

$(i) \Rightarrow (ii)$;

$T \in B(H, K) \Rightarrow T^* \in B(K, H)$ (definition 2.1.9). T is compact and T^*T is meaningful. T is compact, T^* is bounded $\Rightarrow T^*T$ is compact (see [4] or [12]).

$(ii) \Rightarrow (i)$;

Let (x_n) be a weakly convergent sequence in H such that $x_n \xrightarrow{w} x$. Since T^*T is compact. $T^*Tx_n \xrightarrow{s} T^*Tx$ *i.e.* T^*T is strongly Cauchy in H (by proposition 2.7.4).

Since (x_n) is weakly convergent, (x_n) is bounded, hence there exists $c > 0$ such that $\|x_n\| \le c \ \forall n \in \mathbb{N}$. Now for all $m, n \in \mathbb{N}$

$$\|Tx_n - Tx_m\|^2 = \langle T(x_n - x_m), T(x_n - x_m)\rangle$$
$$= \langle T^*T(x_n - x_m), (x_n - x_m)\rangle \quad (\because T^*T \ge 0)$$
$$\le \|T^*T(x_n - x_m)\| \|x_n - x_m\|.$$

Since T^*T is strongly Cauchy in H,

$$\|T^*T(x_n - x_m)\| \to 0 \text{ as both } m, n \to \infty.$$

Since $\|x_k\| \le c \ \ \forall k \in \mathbb{N}$, so

$$\|x_n - x_m\| \le \|x_n\| + \|x_m\| \le 2c.$$

Therefore $\|Tx_n - Tx_m\| \to 0$ as both $m, n \to \infty$, i.e. (Tx_n) is strongly Cauchy in K. But K is strongly complete therefore (Tx_n) converges strongly in K. So T maps weakly convergent sequences to strongly convergent sequences, therefore T is compact

$(i) \Leftrightarrow (iii)$)

Since T is compact and $T^* \in B(K, H)$, so $TT^* \in B(K)$ is compact. i.e. $(T^*)^* T^*$ is compact. From $(i) \Leftrightarrow (ii)$ we conclude, T^* is compact. Now $T^* \in B(H)$ and $(T^*)^* = T$. So T^* is compact $\Rightarrow (T^*)^*$ is compact $\Rightarrow T$ is compact.

□

Proposition 2.7.6

Let $T \in B(H)$ and M be a closed linear subspace of H which reduces T. (i.e. $M, M^\perp$ are invariant under T, equivalently, M is invariant under T and T^*). Then $T|_M \in B(M)$ and $(T|_M)^* = T^*|_M$.

Proof:

Since T is bounded and $(T|_M): M \to M$, we have $\|Tx\| \le k\|x\| \ \ \forall x \in H$, and thus

$$\|Tx\| \le k\|x\| \quad \forall x \in M \ \text{ i.e. } \|T|_M x\| \le k\|x\|$$

i.e. $T|_M \in B(M)$. Therefore $(T|_M)^*$ exists and also belongs to $B(M)$. Let $x, y \in M$

$$\langle T|_M x, y\rangle = \langle x, (T|_M)^* y\rangle \qquad (2.18).$$

But also

$$\langle T|_M x, y\rangle = \langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, (T^*|_M) y\rangle \quad \forall y \in M \qquad (2.19).$$

Therefore $T^*|_M y = T^* y$. From (2.18) and (2.19), we get

$$\langle x, (T|_M)^* y\rangle = \langle x, T^*|_M y\rangle \quad \forall x, y \in M,$$

therefore $(T|_M)^* = T^*|_M$.

□

Proposition 2.7.7

Let $T \in B(H)$ be normal and M be a closed linear subspace of H. Then $T|_M$ is normal if and only if M reduces T.

Proof:

Let M reduce T. Then M is invariant under T and T^* i.e. $T: M \to M$, $T^*: M \to M$, $T: M^\perp \to M^\perp, T^*: M^\perp \to M^\perp$. Since T is normal, $T^*T = TT^*$. Therefore $(T^*T)|_M = (TT^*)|_M$. But

$$(T^*T)|_M = (T^*|_M)(T|_M)$$

(for $T: M \to M$, $T^*: M \to M$). Likewise

$$(TT^*)|_M = (T|_M)(T^*|_M).$$

Therefore

$$(T|_M)(T^*|_M) = (T^*|_M)(T|_M).$$

By the proposition 2.7.6 $T^*|_M = (T|_M)^*$. Thus we obtain

$$(T|_M)(T|_M)^* = (T|_M)^*(T|_M)$$

i.e. $T|_M$ is normal. Conversely, let $T|_M$ be normal we must show that M reduces T.

Since $T|_M$ is normal, so $T|_M \leftrightarrow (T|_M)^* = T^*|_M$ therefore

$$\left(T|_M\right)\left(T^*|_M\right)=\left(T^*|_M\right)\left(T|_M\right) \text{ i.e.} \left(T|_M\right)\left(T|_M\right)^*=\left(T|_M\right)^*\left(T|_M\right),$$

therefore $T|_M$, $T^*|_M \in B(M)$ we have to show that M is invariant under T and T^*.

Note: $T|_M \in B(M)$, therefore $T|_M$ maps M into M i.e. M. is invariant under T. Now

$$\left(T|_M\right)^*=\left(T^*|_M\right)\in B(M)$$

i.e. $T^*|_M$ maps M into M i.e. M is invariant under T^*. Thus M reduces T.

□

Proposition 2.7.8

Let $T \in B(H)$ be compact and normal. Then if $x \in H$ and $x \perp \eta_{\lambda I-T}$ for all $\lambda \in \mathbb{C}$ then $x=\overline{0}$. In other words $\bigcap_{\lambda\in\mathbb{C}} \eta_{\lambda I-T}^{\perp}=\{\overline{0}\}$.

Proof:

Let $L=\bigcup_{\lambda\in\mathbb{C}} \eta_{\lambda I-T}$, then

$$L^{\perp}=[L]^{\perp}=\overline{[L]}^{\perp}=\overline{\left[\bigcup_{\lambda\in\mathbb{C}}\eta_{\lambda I-T}\right]}^{\perp}=\left(\bigvee_{\lambda\in\mathbb{C}}\eta_{\lambda I-T}\right)^{\perp}$$

$$=\bigwedge_{\lambda\in\mathbb{C}}\eta_{\lambda I-T}^{\perp}=\bigcap_{\lambda\in\mathbb{C}}\eta_{\lambda I-T}^{\perp}$$

(standard Hilbert space results for any non-void $L \subset H$). Hence we need to show that $L^{\perp}=\{\overline{0}\}$. Since T is normal, both T, T^* commute with $T-\lambda I\ (\lambda\in\mathbb{C})$ (see [47]). We shall show for all $\lambda \in \mathbb{C}$.

$$T\left(\eta_{\lambda I-T}\right)\subseteq\eta_{\lambda I-T} \tag{2.20}$$

and

$$T^*\left(\eta_{\lambda I-T}\right)\subseteq\eta_{\lambda I-T} \tag{2.21}$$

i.e. $\eta_{\lambda I-T}$ reduces T. To see (2.20), let $y\in T\left(\eta_{\lambda I-T}\right)$ i.e. $y=Tx$ for some $x\in\eta_{\lambda I-T}$ i.e.

$$(\lambda I-T)x=\overline{0}$$

i.e. $Tx=\lambda x$, therefore $y=\lambda x$. But $x\in\eta_{\lambda I-T}$ therefore $\lambda x\in\eta_{\lambda I-T}$ i.e. $y\in\eta_{\lambda I-T}$, this proves (2.20). To see (2.21) let $y\in T^*\left(\eta_{\lambda I-T}\right)$ i.e. $y=T^*z$ for some $z\in\eta_{\lambda I-T}$ i.e. $(\lambda I-T)z=\overline{0}$ i.e. $Tz=\lambda z$. Now

$$(\lambda I - T)y = (\lambda I - T)T^* z = \lambda T^* z - TT^* z = \lambda T^* z - T^* Tz \ (T \text{ is normal})$$

$$= \lambda T^* z - T^*(Tz) = \lambda T^* z - T^*(\lambda z)$$

$$= \lambda T^* z - \lambda T^* z = \bar{0}$$

i.e. $y \in \eta_{\lambda I - T}$. Thus $T^*(\eta_{\lambda I - T}) \subseteq \eta_{\lambda I - T}$ which establishes (2.21)? Hence, it follows that

$$T(L) \subseteq L \text{ and } T^*(L) \subseteq L \qquad (2.22).$$

Indeed

$$T(L) = T\left(\bigcup_{\lambda \in \mathbb{C}} \eta_{\lambda I - T}\right) = \bigcup_{\lambda \in \mathbb{C}} T(\eta_{\lambda I - T}) \subseteq \bigcup_{\lambda \in \mathbb{C}} \eta_{\lambda I - T} = L,$$

by (2.20). Likewise

$$T^*(L) = T^*\left(\bigcup_{\lambda \in \mathbb{C}} \eta_{\lambda I - T}\right) = \bigcup_{\lambda \in \mathbb{C}} T^*(\eta_{\lambda I - T}) \subseteq \bigcup_{\lambda \in \mathbb{C}} \eta_{\lambda I - T} = L,$$

by (2.21). Results (2.22) imply

$$T^*(L^\perp) \subseteq L^\perp \text{ and } T^{**}(L^\perp) \subseteq L^\perp \qquad (2.23).$$

Now $L^\perp$ is a closed linear subspace of H. Therefore (2.23) implies

$$T^*(L^\perp) \subseteq L^\perp \text{ and } T(L^\perp) \subseteq L^\perp \text{ (since } T^{**} = T) \qquad (2.24).$$

$(2.24) \Rightarrow L^\perp$ reduces T. Suppose $L^\perp$ is not zero. There is a non-zero $x \in L^\perp$. Thus $L^\perp$ is not the zero subspace. Put $S = T\big|_{L^\perp}$, since T is normal and $L^\perp$ is a closed linear subspace of H which reduces T, so $T\big|_{L^\perp}$ i.e. S is normal (by proposition 2.7.7). Note $S : L^\perp \to L^\perp$. Let (x_n) be any bounded sequence of elements of $L^\perp$ i.e. $\|x_n\| \le k$ for some k>0 and for all $n \in \mathbb{N}$. Since T is compact, there exists a subsequence (x_{nk}) of (x_n) such that (Tx_{nk}) converges strongly in H say to y. Since $x_{nk} \in L^\perp$ and $T\big|_{L^\perp} = S$, we get that $Sx_{nk} \xrightarrow{s} y \in H$ i.e. S is compact (by proposition 2.7.3). Since $L^\perp$ is closed and $Sx_{nk} \in L^\perp$ (for $S : L^\perp \to L^\perp$) and $Sx_{nk} \xrightarrow{s} y$ it follows that $y \in L^\perp$. Since $S : L^\perp \to L^\perp$ is compact and normal, S has a non-zero eigenvalue λ (see lemma 2.7.9 for this assertion). Let x be an eigenvector corresponding to λ; i.e. $Sx = \lambda x$ for an $x \in L^\perp$ and nonzero. $Sx = \lambda x \Rightarrow Tx = \lambda x$ for $T\big|_{L^\perp} = S$. At the same time, from $Tx = \lambda x$ we have

$(\lambda I-T)x=\bar{0}$; i.e. $x\in\eta_{\lambda I-T}$. But $L=\bigcup_{\lambda\in\mathbb{C}}\eta_{\lambda I-T}$, so $x\in L$. Thus $x\in L\cap L^{\perp}\subseteq\{\bar{0}\}$, which is a contradiction. Hence the assumption that $L^{\perp}\neq\{\bar{0}\}$ is untenable. Therefore $L^{\perp}=\{\bar{0}\}$ i.e.

$$x\perp\eta_{\lambda I-T}\ \forall\lambda\in\mathbb{C}\Rightarrow x=\bar{0}.$$

□

We used the

Lemma 2.7.9

If $T\in B(H)$ is compact and normal then its point spectrum is non-void. It has a non-zero eigenvalue.

Proof:

Since T is bounded, there is a $\lambda\in\sigma(T)$ such that $|\lambda|=r(T)$ (see [4]). Since T is normal, we have $r(T)=\|T\|$ (proposition 2.4.6). If $T\in B(H)$ is normal, then there exists $\lambda\in\sigma(T)$ such that $|\lambda|=\|T\|$. Assuming $T\neq 0$, we have $\|T\|\neq 0$, so $\lambda\neq 0$. Since $T\in B(H)$ is compact, we have $\sigma(T)-\{\bar{0}\}=P\sigma(T)-\{\bar{0}\}$ (see [13]). Now

$$\lambda\neq 0 \text{ and } \lambda\in\sigma(T)\Rightarrow\lambda\in\sigma(T)-\{\bar{0}\}\Rightarrow\lambda\in P\sigma(T)-\{\bar{0}\}.$$

Therefore $\lambda\in P\sigma(T)$. Thus there is a non-zero eigenvalue for $T\in B(H)$ which is compact and normal.

□

Incidentally this result also shows that the spectrum $\sigma(T)$ of a compact normal T is non-void.

Proposition 2.7.10

Dim $H=\infty$. Let $T\in B(H)$ be compact and normal and T have a finite point spectrum $\{\lambda_1,\cdots,\lambda_n\}$. Then T is of finite rank. Moreover

1) If P_j is the orthogonal projector on H onto $\eta_{\lambda_j I-T}$ $(j=1,\cdots,n)$ then $I=P_1+\cdots+P_n$

2) $P_i\perp P_j$ if $i\neq j$

3) T has the spectral decomposition $T = \sum_{i=1}^{n} \lambda_i P_i$

Remark:

This result may be extended to proposition 2.7.8 also. Dim $H = \infty$, $T \in B(H)$ is normal and of finite rank i.e. $T \in B(H)$ is normal and compact. If $T \in B(H)$ is normal and of finite rank, then $\sigma(T)$ consists of only $P\sigma(T)$ and cardinality of $P\sigma(T)$ is finite [4]. Thus $T \in B(H)$ is normal, compact and has a finite number of eigenvalues. Hence we can apply the proposition in context and have (1), (2) and (3).

Proof:

1) Let $\{\lambda_1, \ldots, \lambda_n\}$ be the finite point spectrum of T. Let

$$M = \left[\bigcup_{i=1}^{n} \eta_{\lambda_i I - T} \right]$$

Since T is normal, eigenvectors corresponding to distinct eigenvalues are mutually orthogonal (see [4] or [47]). Hence the eigenspaces $\eta_{\lambda_i I - T}$ are mutually orthogonal i.e. $i \neq j \Rightarrow \eta_{\lambda_i I - T} \perp \eta_{\lambda_j I - T}$. So we can write

$$M = \eta_{\lambda_1 I - T} \oplus \eta_{\lambda_2 I - T} \oplus \cdots \oplus \eta_{\lambda_n I - T}.$$

Since each of these null spaces is closed (if $T \in B(H)$, η_T is a closed subspace [4]) and hence M is a closed linear subspace of H. We shall show that $M = H$. In order to see this, we should show that $M^{\perp} = \{\overline{0}\}$. Let

$$L = \bigcup_{i=1}^{n} \eta_{\lambda_i I - T}.$$

Then $L^{\perp} = [L]^{\perp} = M^{\perp}$. Hence we need to show that $L^{\perp} = \{\overline{0}\}$. If $\lambda \in \mathbb{C}$ and $\lambda \notin P\sigma(T)$ (i.e λ is none of the λ_i's, $i = 1, \cdots, n$) then $\lambda I - T$ is one-to-one, i.e. $\eta_{\lambda I - T} = \{\overline{0}\}$, therefore $L = \bigcup_{\lambda \in \mathbb{C}} \eta_{\lambda I - T}$. Now by proposition 2.7.8, we get $L^{\perp} = \{\overline{0}\}$. Hence $M^{\perp} = \{\overline{0}\}$, i.e. $M = H$. Thus we have

$$H = \eta_{\lambda_1 I - T} \oplus \eta_{\lambda_2 I - T} \oplus \cdots \oplus \eta_{\lambda_n I - T}.$$

Hence if $x \in H$, we have unique $x_i \in \eta_{\lambda_i I-T}$ $(i=1,\cdots,n)$ such that $x = x_i + x_2 + ... + x_n$.

Now P_i is the orthogonal projector on H onto $\eta_{\lambda_i I-T}$. So $P_i x = x_i$ i.e.

$$x = P_1 x + P_2 x + \cdots + P_n x = (P_1 + P_2 + \cdots + P_n) x .$$

Therefore

$$P_1 + P_2 + \cdots + P_n = I .$$

2) Also if $i \neq j$ we saw $\eta_{\lambda_i I-T} \perp \eta_{\lambda_j I-T}$ and P_i, P_j are orthogonal projectors onto $\eta_{\lambda_i I-T}$, $\eta_{\lambda_j I-T}$ respectively. Therefore

$$P_i \perp P_j \quad \forall i \neq j .$$

3) Since $x = x_1 + \cdots + x_n$ where $x_i \in \eta_{\lambda_i I-T}$ and T is bounded

$$Tx = Tx_1 + \cdots + Tx_n = \lambda_1 x_1 + ... + \lambda_n x_n \left(\because x_i \in \eta_{\lambda_i I-T}, (i=1,\cdots,n) \text{ and } Tx_i = \lambda_i x_i\right)$$

$$= \lambda_1 P_1 x + \cdots + \lambda_n P_n x \text{ (since } Tx_i = \lambda_i x_i \text{)}$$

$$= (\lambda_1 P_1 + \cdots + \lambda_n P_n) x$$

i.e.

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n = \sum_{i=1}^{n} \lambda_i P_i$$

(spectral decomposition for T). If one of the eigenvalues is 0, delete it. Assume that after its deletion (if 0 was an eigenvalue) we had

$$T = \lambda_i P_i + \cdots + \lambda_n P_n \text{ so } Tx = \sum_{i=1}^{n} \lambda_i P_i x \quad \forall x \in H .$$

Now $P_i x \in \eta_{\lambda_i I-T}$ therefore $\lambda_i P_i x \in \eta_{\lambda_i I-T}$ therefore

$$Tx \in \eta_{\lambda_1 I-T} \oplus \cdots \oplus \eta_{\lambda_n I-T} \tag{2.25}$$

since T is compact, for each $\lambda \neq 0, \eta_{\lambda I-T}$ is finite dimensional [4] therefore $\eta_{\lambda_i I-T}$ is finite dimensional for each $i = 1,\cdots,n$. Hence (2.25) shows that

$$\eta_{\lambda_1 I-T} \oplus \cdots \oplus \eta_{\lambda_n I-T}$$

is finite dimensional i.e. $\Re_T$ is finite dimensional. Therefore T is of finite rank.

□

Proposition 2.7.11

Let H be a pre Hilbert space and $\{M_\alpha : \alpha \in \Lambda\}$ be a family of mutually orthogonal linear subspaces of H. Then

$$\overline{\left[\underset{\alpha\in\Lambda}{U} M_\alpha\right]} = \underset{\alpha\in\Lambda}{\vee} M_\alpha = \overline{\sum_{\alpha\in\Lambda} \oplus M_\alpha} \left(= \overline{M},\ where\ M = \sum \oplus M_\alpha \right).$$

(Meaning of M or $\sum\limits_{\alpha\in\Lambda} M_\alpha : M$ consists of all x which are sums of summable families

$\{x_\alpha : x_\alpha \in M_\alpha, \alpha \in \Lambda\}$ *i.e.* $M = \left\{ x \in H : x = \sum\limits_{\alpha\in\Lambda} x_\alpha\ where\ x_\alpha \in M_\alpha \right\}$) (see [1] or [4]).

Remark:

The bar above M in the term $\overline{M}$ can be deleted when H is a Hilbert space and $\{\mathrm{M}_\alpha : \alpha \in \Lambda\}$ is an orthogonal family of closed linear subspaces.

Proposition 2.7.12

Let H be a Hilbert space and $\{M_\alpha : \alpha \in \Lambda\}$ be an orthogonal family of closed linear subspaces and $M = \sum\limits_{\alpha\in\Lambda} \oplus M_\alpha$. Then

$$\underset{\alpha\in\Lambda}{\vee} M_\alpha = \overline{\left[\underset{\alpha\in\Lambda}{\cup} M_\alpha\right]} = \sum_{\alpha\in\Lambda} \oplus M_\alpha = M \quad \text{(see [4] or [12]).}$$

Remark:

Each x has a unique decomposition in M. For if $x = \sum x_\alpha = \sum x'_\alpha$, where $x_\alpha, x'_\alpha \in \mathrm{M}_\alpha\ \forall \alpha \in \Lambda$. We must show that $x_\alpha = x'_\alpha\ \forall \alpha \in \Lambda$. To see this, it is enough to show that $\bar{0}$ has the decomposition $\bar{0} = \sum\limits_{\alpha\in\Lambda} y_\alpha$ with $y_\alpha \in M_\alpha$, then $y_\alpha = \bar{0}$

$\left(\bar{0} = x - x = \sum\limits_{\alpha\in\Lambda} x_\alpha - \sum\limits_{\alpha\in\Lambda} x'_\alpha = (x_\alpha - x'_\alpha) \right)$.

Proposition 2.7.13

Let H be a Hilbert space and $\{M_\alpha : \alpha \in \Lambda\}$ be a family of mutually orthogonal closed linear subspaces of H. Then

$$x \perp M_\alpha \text{ for each } \alpha \in \Lambda \Rightarrow x = \bar{0}$$

if and only if $\underset{\alpha\in\Lambda}{\vee} M_\alpha = H$.

Proof:

By projection theorem

$$H = \left(\underset{\alpha \in \Lambda}{\vee} M_\alpha \right) \oplus \left(\underset{\alpha \in \Lambda}{\vee} M_\alpha \right)^\perp \qquad (2.26).$$

Suppose the condition is satisfied, i.e. $x \perp M_\alpha \ \forall \alpha \in \Lambda \Rightarrow x = \bar{0}$

$$x \perp M_\alpha \ \forall \alpha \in \Lambda \Rightarrow x \perp \overline{\left[\underset{\alpha \in \Lambda}{\cup} M_\alpha \right]}, \ i.e. \ x \in \overline{\left[\underset{\alpha \in \Lambda}{\cup} M_\alpha \right]}^\perp .$$

Hence

$$\overline{\left[\underset{\alpha \in \Lambda}{\cup} M_\alpha \right]}^\perp = \left(\underset{\alpha \in \Lambda}{\vee} M_\alpha \right)^\perp = \{\bar{0}\}.$$

Substituting in (2.26) we get $H = \underset{\alpha \in \Lambda}{\vee} M_\alpha$. Conversely let

$$H = \underset{\alpha \in \Lambda}{\vee} M_\alpha \left(= \sum_{\alpha \in \Lambda} \oplus M_\alpha \right).$$

Let $x \perp M_\alpha$ for each $\alpha \in \Lambda$. Then

$$x \perp \overline{\left[\underset{\alpha \in \Lambda}{\cup} M_\alpha \right]} = \underset{\alpha \in \Lambda}{\vee} M_\alpha = H .$$

Therefore $x = \bar{0}$.

□

Proposition 2.7.14

Let H be a Hilbert space and $\{P_\alpha : \alpha \in \Lambda\}$ be an orthogonal family of orthogonal projectors (i.e. $P_\alpha \perp P_\beta$ whenever $\alpha, \beta \in \Lambda$ and $\alpha \neq \beta$). Then $P = \sum_{\alpha \in \Lambda} P_\alpha$ is an orthogonal projector on the subspace $\underset{\alpha \in \Lambda}{\vee} M_\alpha \left(= \sum_{\alpha \in \Lambda} \oplus M_\alpha \right)$, where M_α is the range of P_α for each $\alpha \in \Lambda$.

Proof:

Consider the orthogonal projector P on $\sum_\alpha \oplus M_\alpha$. If $x \in H$ then $x = y + z$ where $y \in M$ and $z \in M^\perp$. Since $y \in M$, we can write $y = \sum_{\alpha \in \Lambda} y_\alpha$ where $y_\alpha \in M_\alpha$. Therefore

$$x = \sum_{\alpha \in \Lambda} y_\alpha + z, \ y_\alpha \in M_\alpha \ \forall \alpha \in \Lambda \ \ and \ \ z \in M^\perp .$$

For any $\beta \in \Lambda$

$$P_\beta x = P_\beta\left(\sum_{\alpha\in\Lambda} y_\alpha\right) + P_\beta z$$

$$= \sum_{\alpha\in\Lambda} P_\beta y_\alpha + P_\beta z \qquad (2.27).$$

Now since P_β is the orthogonal projector onto M_β and $y_\beta \in M_\beta$, we get $P_\beta y_\beta = y_\beta$. But if $\alpha \neq \beta$ then $P_\beta y_\alpha = \overline{0}$ (for $y_\beta \in M_\beta$ and $M_\alpha \perp M_\beta =$ range of P_β) therefore

$$\sum_{\alpha\in\Lambda} P_\beta y_\alpha = P_\beta y_\beta \qquad (2.28).$$

We have $M_\beta \subseteq \sum_{\alpha\in\Lambda} \oplus M_\alpha = M$. Therefore $M_\beta^\perp \supseteq M^\perp$. Since $z \in M^\perp, so\, z \in M_\beta^\perp$.

Therefore

$$P_\beta z = \overline{0} \qquad (2.29).$$

Substituting in (2.27) these results (2.28) and (2.29), we get

$$P_\beta x = P_\beta y_\beta + \overline{0} = P_\beta y_\beta = y_\beta .$$

From $x = y + z$, we have $Px = Py + Pz, y \in M, z \in M^\perp$. But P is orthogonal projector onto M; so $Pz = \overline{0}$. Therefore $Px = Py = y = \sum_{\alpha\in\Lambda} y_\alpha = \sum_{\alpha\in\Lambda} P_\alpha x$. Therefore

$$Px = \left(\sum_{\alpha\in\Lambda} P_\alpha\right)x \;\; \forall x \in H \text{ i.e. } P = \sum_{\alpha\in\Lambda} P_\alpha .$$

□

Proposition 2.7.15 (Spectral decomposition theorem for compact normal operators)

Let $T \in B(H)$ be compact and normal. Then

i. $$H = \eta_T \oplus \sum_{\substack{\lambda\in P\sigma(T)\\ \lambda\neq 0}} \oplus\eta_{T-\lambda I} = \sum_{\lambda\in\mathbb{C}} \oplus\eta_{T-\lambda I} = \sum_{\lambda\in\mathbb{C}} \oplus\eta_{T^*-\bar{\lambda} I} .$$

ii. $$\overline{\Re_T} = \overline{\Re_{T^*}} = \sum_{\substack{\lambda\in\mathbb{C}\\ \lambda\neq 0}} \oplus\eta_{T-\lambda I} = \sum_{\substack{\lambda\in\mathbb{C}\\ \lambda\neq 0}} \oplus\eta_{T^*-\bar{\lambda} I} .$$

iii. $$T = \sum_{\substack{\lambda\in\mathbb{C}\\ \lambda\neq 0}} \lambda P_\lambda = \sum_{\substack{\lambda\in P\sigma(T)\\ \lambda\neq 0}} \lambda P_\lambda .$$

Where P_λ is the orthogonal projector on H onto $\eta_{\lambda I-T}$ and $\sum_{\lambda\in\mathbb{C}} P_\lambda = I$. (See [4]).

Remark:

T is compact $\Leftrightarrow T^*$ is compact, T is normal $\Leftrightarrow T^*$ is normal

T is compact and normal $\Leftrightarrow T^*$ is compact and normal.

In the rest of this section use will be made of the weak operator topology on the algebra $B(H)$ of all operators on H. A net $\{T_\alpha\}$ converges to an operator T in this topology if and only if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in H$. A typical neighborhood U of T is determined by vectors $x_1,\cdots,x_n,\ y_1,\cdots,y_n \in H$ as follows:

$$U = \left\{S : \left|\left\langle (T-S)x_i, y_i\right\rangle\right| \le 1 \ \ for\ 1 \le i \le n\right\}$$

Proposition 2.7.16

The unit ball of $B(H)$ is compact in the weak operator topology.

Proof:

Since H is reflexive, the ball $B_r = \left\{x : x \in H\ and\ \|x\| \le r\right\}, r \ge 0$ is a weakly compact subset of H by Alaoglu's Theorem ([7] and [12]). Therefore $p = \Pi\left\{B_{\|x\|} : x \in H\right\}$ is compact in the cartesian product topology. This product consist of the functions $f : H \to H$ such that $\|f(x)\| \le \|x\|$ for all $x \in H$, and consequently contains the unit ball of $B(H)$. It is almost obvious that the unit ball of $B(H)$ is a closed subset of p on which the product topology and weak operator topology coincide, and the proposition follows.

□

In outline, the method for obtaining invariant subspaces is to produce projections E_n whose ranges are almost invariant, in a suitable sense, and then to consider a cluster point of $\{E_n\}$ in the weak operator topology (which must exist by proposition 2.7.16). Unfortunately such a cluster point need not be a projection. In fact:

Proposition 2.7.17

If H is infinite-dimensional, the closure in the weak operator topology of the set ε of all projections is the set ξ^+ of all positive contractions (This result is due to Halmos [26] and Nagy [32]).

By arguing in the same way with unitary dilations, it may be shown that the closure in the weak operator topology of the unitary operators is the entire unit ball of $B(H)$. The construction of invariant subspaces begins with the following lemma.

Lemma 2.7.18

Let $e \in H, T \in B(H), R_n$ the projection on $[e, Te, \ldots T^n e]$, and $d_n = \|T^n e - R_{n-1} T^n e\|$ the distance from $T^n e$ to $[e, \cdots, T^{n-1} e]$. Then

$$\|TR_n - R_n TR_n\| = d_{n+1} / d_n .$$

Proof:

Fix $x \in H$ and let $R_n x = a_0 e + \cdots + a_n T^n e$. Then

$$TR_n x = a_0 Te + \cdots + a_n T^{n+1} e,$$

$$R_n TR_n x = a_0 Te + \cdots + a_{n-1} T^n e + a_n R_n T^{n+1} e .$$

$$TR_n x - R_n TR_n x = a_n \left(T^{n+1} e - R_n T^{n+1} e\right)$$

$$\|TR_n x - R_n TR_n x\| = |a_n| d_{n+1} = \left(d_{n+1} / d_n\right)\left(d_n |a_n|\right).$$

But

$$|a_n| d_n = \|a_n T^n e - R_{n-1}\left(a_n T^n e\right)\|$$

$$= \|R_n x - R_{n-1} R_n x\|$$

$$= \|R_n x - R_{n-1} x\|,$$

and the lemma follows.

□

Proposition 2.7.19

If e is a cyclic vector for T such that $\lim_{\inf} \|T^n e\|^{\frac{1}{n}} = 0$, then there are finite-dimensional projections $P_1 \leq P_2 \leq \cdots$ such that $P_n \to I$ strongly and

$$\|TP_n - P_n TP\| \to 0 .$$

If Q_n is a projection such that $Q_n \leq P_n$ and $Q_n \in H$ is invariant for $T_n = P_n TP_n$, then $\|Q_n TQ_n - TQ_n\| \to 0$. (The proof follows from lemma 2.7.18 (see [17], [26] or [49])).

Proposition 2.7.20

The space $M=\{x:Qx=x\}$ is closed, invariant, and distinct from H.

Proof:

Obviously M is a closed subspace, and $M\neq H$ since $Q\neq I$. Let $\{Q_\alpha\}$ be a subnet of $\{Q_n\}$ which converges weakly to Q. If $x\in M$, then $\|Q_\alpha x-x\|^2=\langle x,x\rangle-\langle Q_\alpha x,x\rangle\to 0$, and hence $\|TQ_\alpha x-Tx\|^2\to 0$. But

$$\left|\langle Q_\alpha TQ_\alpha x-QTx,y\rangle\right|\leq\left|\langle Q_\alpha TQ_\alpha x-Q_\alpha Tx,y\rangle\right|+\left|\langle Q_\alpha Tx-QTx,y\rangle\right|$$

$$\leq\|TQ_\alpha x-Tx\|\|y\|+\left|\langle (Q_\alpha-Q)Tx,y\rangle\right|,$$

so that $Q_\alpha TQ_\alpha x\to QTx$ weakly. On the other hand, $Q_\alpha TQ_\alpha x\to TQx$ weakly by proposition 2.7.19, so that $QTx=TQx=Tx$ and $Tx\in M$.

□

The only point remaining is whether $M\neq\{0\}$, and this is where a compactness hypothesis is used.

Proposition 2.7.21

Let T be an operator such that

(i) there is a non-zero vector e with $\liminf\|T^n e\|^{1/n}=0$, and

(ii) the norm-closed algebra generated by T and I contains a non-zero compact operator C. Then T has a nontrivial invariant subspace.

Corollary 2.7.22

If T is quasinilpotent and if the norm-closed algebra generated by T and I contains a non-zero compact operator, then T has nontrivial invariant subspaces.

Proposition 2.7.21 and Corollary 2.7.22 are due to Arveson and Feldman [3]; the proof is based on earlier work of Bernstein and Robinson [6] and Halmos [22]. From the fact that: if $p(T)$ is compact for some polynomial $p\neq 0$, then T has a nontrivial invariant subspace [7] we have

Proposition 2.7.23

Any compact operator has a nontrivial invariant subspace (the proposition is due to Neumann, Aronszajn, and Smith [2] on any Banach Space).

CHAPTER THREE

3.1 LOMONOSOV'S THEOREM AND K-PARANORMAL OPERATORS

Let X be a complex Banach space and $B(X)$ be the set of all bounded linear operators on X. The invariant subspace problem has the following form; "does every bounded operator T have a non-trivial invariant subspace?" A similar assertion is concerned with the case of hyperinvariant subspaces i.e does every bounded operator T have a non-trivial hyperinvariant subspace?"

Definition 3.1.1

If A is a set in $B(X)$ and M is a closed subspace of X, we say that M is ***invariant for*** A if M is invariant for all operators in A. A closed sub algebra A of $B(X)$ is called ***transitive*** if the only subspaces invariant for A are X and $\{\bar{0}\}$. A is called ***intransitive*** if it has non-trivial invariant subspaces. We quote the famous;

Proposition 3.1.2 (Schauder's fixed point theorem)

If K is a convex and bounded subset of a Banach space X and $f: K \to K$ is a continuous mapping (generally non linear), then there exists a point $x_0 \in K$ such that $f(x_0) = x_0$ (for a proof see [7] or [12]). The following is a consequence of proposition 3.1.2

Lemma 3.1.3

Let K be a closed convex subset of a Banach Space X and $f: K \to K$ be continous. If $f(K)$ is contained in a compact subset of K, then there exists a point $x_0 \in K$ such that $f(x_0) = x_0$.

Proof:

Since $f(K)$ is contained in a compact subset of K, $f(K)$ is compact. Thus the set $\overline{convf(K)}$ is compact and contained in K. Since $f(K) \subset C$, where C is the compact set which exists, we have $f(convC) \subset convC$. And proposition 3.1.2 applied to the convex set C gives the assertion.

□

Proposition 3.1.4

Let X be a Banach space and A a transitive sub-algebra of $B(X)$ and T be an arbitrary non-zero compact operator in $B(X)$. Then there exists an element of A, say S, such that ST has an eigenvalue 1, and similarly, there exists $S_1 \in A$, such that TS_1 has an eigenvalue 1.

Proof:

Without loss of generality, we may assume that $\|T\| = 1$. Let x_0 be any element of X such that $\|Tx_0\| > 1$. Clearly this implies that $\|x_0\| > 1$. Let

$$B = \{x \in X : \|x - x_0\| < 1\}$$

and let $B_T = \overline{T(B)}$. It is obvious that B_T is compact and convex and

i. $\bar{0} \notin B$

ii. $\bar{0} \notin B_T$.

Let $R \in A$ and the set U_R be defined by $U_R = \{y \in X : \|Ry - x_0\| < 1\}$. Clearly U_R is an open set and we see that $\underset{R \in A}{\cup} U_R = X - \{\bar{0}\}$. Indeed, let y be fixed and consider the subspace $\{Ry : R \in A\}$, which from the definition is clearly invariant for A. Since A is a transitive algebra, it follows that this subspace is X. Thus we can find $R \in A$ such that $\|Ry - x_0\| < 1$. This shows that B_T is contained in $X - \{\bar{0}\} = \underset{R \in A}{\cup} U_R$. Now the set B_T is compact, and thus we can find a finite number of operators, say $R_1, R_2, \cdots, R_n$ in A such that

$$B_T \subset U_{R_1} \cup U_{R_2} \cup U_{R_3} \cup \cdots \cup U_{R_n}.$$

We now define the following functions on the set B_T. For $i, 1 \le i \le n$

$$\varphi_i(y) = \max\{0, 1 - \|R_i y - x_0\|\}$$

for all $y \in B_T$. From the definition it follows that these are continuous functions and

iii. $0 < \varphi_i < 1$

iv. $\sum \varphi_i(y) > 0$

In this case we may define the functions ψ_j by the relations

$$\psi_j(y) = \frac{1}{\left(\sum \varphi_i(y)\right)} \varphi_j(y)$$

Clearly, these are also continuous functions and $\sum_j \psi = 1$. We now define a function on B_T with values in X as follows;

$$f : B_T \to X, \ f(y) = \sum \psi_j(y) R_j(y)$$

since $x_0 = \sum \psi_j(y) x_0$, we have that the values of f are in B. Indeed,

$$\| f(y) - x_0 \| = \left\| \sum \left(R_j(y) - x_0 \right) \psi_j(y) \right\| \leq \sum \psi_j(y) = 1$$

and the assertion is proved. We now define a function f_1, on B as

$$f_1(y) = f(Ty).$$

This function maps B into itself. Since f is continuous, f_1 is continuous and since T is a compact operator, the range of f_1 is contained in a compact set. In this case by Lemma 3.1.3, there exists a non-zero element in B, say x_1 such that $f_1(x_1) = x_1$. If we consider the linear operator S where $S = \sum \psi_j(\mathrm{T}x_1) R_j$ then, obviously $STx_1 = x_1$ which proves the first part of the proposition. For the second part we define the function f_2 on B_T by

$$f_2(y) = Tf(y)$$

which is also continuous and maps B_T into itself. From proposition 3.1.2 we can find an element $z_1 \in B_T$ such that $Tf(z_1) = z_1$. Now if we set

$$S_1 = \sum \psi_j(z_1) R_j$$

we find that $TS_1(z_1) = z_1$, i.e. S_1, satisfies the last part of the proposition.

□

Using proposition 3.1.4, we propose to prove

Proposition 3.1.5 (Lomonosov's theorem)

Let $T \in B(X)$ be any operator which is non-scalar and commutes with a non-zero compact operator K. Then T has a non-trivial hyperinvariant subspace.

Proof:

For any algebra A in $B(X)$,

$$A' = \{S : S \in B(X), ST = TS \text{ for all } T \in A\}$$

A' is called the ***commutant*** of A (and $(A')' = A''$ is called the ***bicommutant*** of A). We now consider the algebra A of all polynomials in T and the closure in $B(X)$ of this algebra which is a commutative Banach algebra. The assertion of the proposition is in fact that; A' is a nontransitive algebra. Suppose that this is not so. Then A' is transitive and by proposition 3.1.4, we can find an operator S in A' such that $\lambda = 1$ is an eigenvalue of the operator SK. Now since T commutes with SK and SK has a finite dimensional eigenspace (for SK is compact) which is obviously invariant for T, it follows that T has an eigenvalue. Since T is not a scalar, the associated eigenspace is not X; since this eigenspace is invariant for A', we have a contradiction. This proves the proposition.

□

We now obtain a refinement of proposition 3.1.5 through

Proposition 3.1.6

Let A, B and T be in $B(X)$ and $C = TA - BT$ be of rank 1(by the rank of an operator we mean the dimension of its range). If the largest invariant subspace of A contained in the kernel of C is $\{\bar{0}\}$ and the smallest invariant subspace of B containing the range of C is X, then T is one-to-one or T is with dense range.

Proof:

Suppose that the assertion of this proposition is not true. Then T has a non-trivial kernel and a non-dense range. Let $x \in kerT$ and $x \neq \bar{0}$. In this case $Cx = (TA - BT)x = TAx$ and clearly this yields

i. $Cx \neq \bar{0}$ (since the largest invariant subspace of A contained in the kernel of C is $\{\bar{0}\}$ or

ii. $Ax \in \ker T$.

Now if (ii) is true and since C is of rank 1, we obtain that $TA^2x = CAx$ and thus $CAx \neq \bar{0}$ or $A^2x \in \ker T$. We can continue in this manner and since $\{A^n x : n \in \mathbb{N}\}$ cannot be an invariant subspace for A contained in the kernel of C, we can find $y \in \ker T$ such that $Cy \neq \bar{0}$. Since $Cy = TAy$ and the kernel of A^*T^* annihilates the range of C, which is one-dimensional we obtain that $\ker A^*T^* \subset \ker C^*$. Since $A^*T^* - T^*B^* = C^*$ we obtain $\ker A^*T^* \subset \ker T^*B^*$. Now if $x^* \in \ker A^*T^*$ then

$$\left(A^*T^*\right)B^*x^* = A^*\left(T^*B^*\right)x^* = 0$$

And thus $\ker A^*T^*$ is an invariant subspace for B^* contained in the null-space of C^*. This implies that the closure of the range of TA is an invariant subspace for B containing the range of C and this gives the assertion of the proposition.

□

We are now in a position to prove

Proposition 3.1.7

Let X be a Banach space and $T \in B(X)$ and there be a non-zero compact operator K such that $C = TK - KT$ is of rank 1. Then T has a non-trivial invariant subspace.

Proof:

Suppose that T has no non-trivial invariant subspaces then by proposition 3.1.4 there exists an operator B commuting with T such that $BKy = y$ for some $y \in X$, $y \neq \bar{0}$. Since we have

$$BC = B(TK - KT) = BTK - BKT = TBK - BKT$$
$$= T\left(BK - I\right) - \left(BK - I\right)T$$

and BC has rank 1 (for C has rank 1), $BK - I$ has a non-trivial kernel and thus has non-dense range. This contradiction proves the proposition.

□

Definition 3.1.8

An element x in a normed linear space X is called a ***maximal vector*** for the operator $T \in B(X)$ if $\|Tx\| = \|T\|\|x\|$.

Proposition 3.1.9

If $T \in B(H)$, H is a Hilbert space and T is k-paranormal then the set of all maximal vectors for T is an invariant subspace.

Proof:

We can assume, without loss of generality, that $\|T\| = 1$. Since $\|Tx\| = \|x\|$, we obtain (assuming x to be a maximal vector)

$$\begin{aligned}\|T^*Tx - x\|^2 &= \|T^*Tx\|^2 - 2\operatorname{Re}\langle T^*Tx, x\rangle + \|x\|^2 \\ &= \|T^*Tx\|^2 - 2\|x\|^2 + \|x\|^2 = \|T^*Tx\|^2 - \|x\|^2 \le 0.\end{aligned}$$

For $\|Tx\| = \|x\|$ implies $\operatorname{Re}\langle T^*Tx, x\rangle = \operatorname{Re}\langle Tx, Tx\rangle = \operatorname{Re}\|Tx\|^2 = \|Tx\|^2 = \|x\|^2$ and $\|T^*Tx\|^2 \le \|x\|^2$ i.e $\|T^*Tx\|^2 - \|x\|^2 \le 0$, but $\|.\| \ge 0$ and hence $\|T^*Tx\|^2 - \|x\|^2 = 0 \Rightarrow T^*Tx = x$. This shows that the set of maximal vectors is a closed subspace. If x is a maximal vector and since T is k-paranormal we may assume that $\|x\| = 1$. Now from $\|T^k x\| \ge \|Tx\|^k = (\|T\|\|x\|)^k = \|T\|^k$ we obtain that

$$\begin{aligned}\|T(Tx)\| &\le \|Tx\|\|T\| \le \|T\|\|x\|\|T\| = \|T\| = \frac{1}{\|T\|^{k-2}}\|T\|^k \le \frac{1}{\|T\|^{k-2}}\|T^k\|^2 \\ &= \frac{1}{\|T\|^{k-2}}\|T^{k-2}(T^2 x)\| \le \frac{1}{\|T\|^{k-2}}\|T^{k-2}\|\|T^2 x\| = \|T(Tx)\|\end{aligned}$$

and hence $\|T(Tx)\| = \|Tx\|\|T\| = \|Tx\|$. This shows that Tx is a maximal vector and proves the proposition

□

Corollary 3.1.10

If T is an operator on a Hilbert space H and T is k-paranormal and has maximal vectors, then T has invariant subspaces.

Proof:

From the result of proposition 3.1.9 we can consider only the case when the set of all maximal vectors in H. In this case T is a multiple of an isometry and clearly there exist invariant subspaces.

□

3.2 REDUCING SUBSPACES AND POLYNOMIALLY COMPACT OPERATORS

Proposition: 3.2.1

If $(e_i)_{i=1}^{\infty}$ is an orthonormal basis for a Hilbert space H and $A \in B(H)$ is not a multiple of I and n, m fixed integers, then for any real $\varepsilon > 0$ there exists a unitary operator U such that

$$\|I-U\| < \varepsilon \text{ and } \langle U^* AU\, e_n, e_m \rangle \neq 0.$$

Proof:

If A is not a multiple of I and $\langle Ae_n, e_m \rangle \neq 0$ for fixed integers n, m we may take $U_1 = I$. Suppose that $\langle Ae_n, e_m \rangle = 0$. If Ae_n is not a multiple of e_n, we define $U_1 = I$. If $Ae_n = ze_n$, we choose k such that $Ae_k \neq ze_k$ and for real d close to 0 we note that, $(1-d^2)^{\frac{1}{2}} e_n + de_k$ is not an eigenvector of A. Let d be such that $\left(1-(1-d^2)^{\frac{1}{2}}\right)^2 + d^2 < \varepsilon^2$ and set $d' = (1-d^2)^{\frac{1}{2}}$ .We now define

$$U_1 e_n = d'e_n + de_k,\ U_1 e_k = -de_n + d'e_k,\ U_1 e_i = e_i \ \ if\ \ i \neq n, k.$$

Clearly : $U_1 : H \to H$ is onto. We compute the inner products $\langle U_1 e_i, U_1 e_j \rangle$.

If $i = j$ and $i \neq n, k$, then $\langle U_1 e_i, U_1 e_j \rangle = \langle e_i, e_i \rangle = 1$

If $i = j = n$, then $\langle U_1 e_i, U_1 e_j \rangle = \langle d'e_n + de_k, d'e_n + de_k \rangle = d'^2 + d^2 = 1$

If $i = j = k$, then $\langle U_1 e_i, U_1 e_j \rangle = d'^2 + d^2 = 1$

If $i = n, k$ and $j \neq n, k$ then $\langle U_1 e_i, U_1 e_j \rangle = 0$

If $i = k$ and $j = n$, then $\langle U_1 e_i, U_1 e_j \rangle = 0$

If $i = n$ and $j = k$, then $\langle U_1 e_i, U_1 e_j \rangle = 0$

If $i \neq j$ and i, j are both not n, k then $\langle U_1 e_i, U_1 e_j \rangle = \langle e_i, e_j \rangle = 0$

If $i \neq n, k$ and $j = n, k$ then

$$\langle U_1 e_i, U_1 e_j \rangle = \langle e_i, d'e_n + de_k \rangle \ or \ \langle e_i, -de_n + d'e_k \rangle$$

$$= 0 \text{ in both cases.}$$

If $x, y \in H$ with representations $x = \sum_{i=1}^{\infty} x_i e_i$, $y = \sum_{j=1}^{\infty} y_j e_j$, then

$$\begin{aligned}\langle U_1 x, U_1 y\rangle &= \sum_i \sum_j x_i \overline{y_j} \langle U_1 e_i, U_1 e_j\rangle \\ &= \sum_i x_i \overline{y_i}\end{aligned}$$

On using the inner products computed above since $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$, we obtain $\langle U_1 x, U_1 y\rangle = \langle x, y\rangle$ for all $x, y \in H$ and since range $U_1 = H$, it follows that U_1 is unitary.

Now, with $x = \sum_{i=1}^{\infty} x_i e_i \in H$, we have

$$\begin{aligned}(I - U_1)x &= \sum_{i=1}^{\infty} x_i e_i - U_1\left(\sum_{i=1}^{\infty} x_i e_i\right) \\ &= \sum_{i=1}^{\infty} x_i e_i - \left[\sum_{i \neq n,k} x_i e_i + x_n(d'e_n + de_k) + x_k(-de_n + d'e_k)\right] \\ &= \{x_n(1-d') + x_k d\} e_n + \{x_k(1-d') - x_n d\} e_k\end{aligned}$$

Since $e_n \perp e_k$, we have

$$\begin{aligned}\|(I - U_1)x\|^2 &= \{|x_n(1-d') + x_k d|\}^2 + \{|x_k(1-d') - x_n d|\}^2 \\ &= \{d^2 + (1-d')^2\}^{*}(|x_n|^2 + |x_k|^2) \\ &< \varepsilon^2 \|x\|^2 \quad \forall x \in H\end{aligned}$$

Thus $\|(I - U_1)\| < \varepsilon$. Now, if $\langle AU_1 e_n, U_1 e_m\rangle \neq 0$, U_1 satisfies the proposition. Suppose that this is not so, then for some m, n we have

$$\langle AU_1 e_n, U_1 e_m\rangle = 0 .$$

Since A is not a multiple of I, for some $i \neq n$, $AU_1 e_i$ is not a multiple of $U_1 e_i$. This clearly implies that $i \neq m$. We now have two cases to discuss:

CASE 1:

$n \neq m$. In this case we define the unitary operator U_2 as follows:

$$\begin{aligned}&U_2 U_1 e_m = d'U_1 e_m + dU_1 e_i \\ &U_2 U_1 e_i = -dU_1 e_m + d'U_1 e_i \\ &U_2 U_1 e_p = U_1 e_p \quad \textit{for } p \neq i, m\end{aligned}$$

The operator $U = U_2U_1$ is clearly unitary and since

$$\begin{aligned}\langle U^*AUe_n, e_m\rangle &= \langle AU_2U_1e_n, U_2U_1e_m\rangle \\ &= \langle AU_1e_n, d'U_1e_m + dU_1e_i\rangle \\ &= 0 + d\langle AU_1e_n, U_1e_i\rangle \neq 0\end{aligned}$$

CASE 2:

$n = m$. If we consider the fourth roots of the unity, we define the elements f_j by the relations

$$f_j = d'U_1e_n + w_j dU_1e_i, \quad w_j^4 = 1, \quad j = 1,2,3,4$$

and we can find a j such that $\langle Af_j, f_j\rangle \neq 0$ since

$$4dd'\langle AU_1e_p, U_1e_i\rangle = \sum w_i \langle Af_j, f_j\rangle.$$

For this j we define the unitary operator U_2 as follows:

$$\begin{aligned}&U_2U_1e_n = f_j \\ &U_2U_1e_i = -\overline{w_j}de_n + d'U_1e_i \\ &U_2U_1e_p = U_1e_p \quad for\ p \neq i, n\end{aligned}$$

It is then seen that $U_2U_1 = U$ is unitary and satisfies the relation $\|I - U\| < \varepsilon$ and also satisfies the proposition.

□

Proposition 3.2.2

If (A_i) is a countable set of elements in $B(H)$ none of which is a multiple of I, then there exists an orthonormal basis H such that $\langle A_ie_n, e_m\rangle \neq 0$.

Proof:

For a fixed orthonormal basis of H, say (f_j), we consider the following unitary operators

$$U_{i,n,m} = \{U : U\ unitary\ and\ \langle U^*A_iUf_n, f_m\rangle = 0\}$$

and it is obvious that these are closed sets in $B(H)$. We show now that these sets are nowhere dense in the set of unitary operators considered as a metric subspace of $B(H)$.

Indeed, let $U \in U_{i,n,m}$ and by proposition 3.2.1 applied to the orthonomal basis (Uf_i), for each $\varepsilon > 0$ we can find a unitary operator V such that $\|I - V\| < \varepsilon$ *and* $\langle V^* \Lambda_i V U f_n, U f_m \rangle \neq 0$.

Since

$$0 \neq \langle U^* V^* \Lambda_i V U f_n, f_m \rangle = \langle (VU)^* \Lambda_i V U f_n, f_m \rangle$$

the assertion is proved since $\|U - VU\| < \varepsilon$. We can apply the Baire Category theorem[4] and we obtain that there exists a unitary operator $\overline{U}$ which is not in the union of the sets $U_{i,n,m}$. We now define $e_i = \overline{U} f_i$, and this satisfies the proposition.

□

Proposition 3.2.3

If M is a finite or countable set of nonscalar operators on a Hilbert space H, then there exists a hermitian operator $K \in B(H)$ such that no member of M leaves invariant a nontrivial subspace of K.

Proof:

By proposition 3.2.2 there exists a basis (e_i) of H such that every entry in the matrix of $A \in M$ with respect to $\{e_i\}$ is non zero. If K is the hermitian operator defined on H whose matrix with respect to $\{e_i\}$ is $diag\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ then clearly K is a compact operator and since the invariant subspaces of K are of the form $\underset{j \in B}{U}\left[\{e_j\}\right]$ *or* $\underset{j \in B}{\oplus}\left[\{e_j\}\right]$ where B is a subset of natural numbers, for any $A \in M$, A and K have no common invariant subspaces since no entry in the matrix of A is zero.

□

Proposition 3.2.4

Every operator $T \in B(H)$ is the sum of two irreducible operators.

Proof:

If $T \in B(H)$ is scalar, then we take an arbitrary irreducible operator C, and $T = (T - C) + C$ gives the desired decomposition. Now if T is not scalar, $T = \operatorname{Re} T + i \operatorname{Im} T$ and we have two cases;

CASE I:

The operators $\operatorname{Re}T$ and $\operatorname{Im}T$ are nonscalar operators. In this case we can apply proposition 3.2.3, for $M = \{\operatorname{Re}T, \operatorname{Im}T\}$ and we find the compact operator K (Hermitian) and

$$T_1 = (\operatorname{Re}T - K) - iK,\ T_2 = K + i(\operatorname{Im}T + K)$$

satisfies the proposition.

CASE II:

If either $\operatorname{Re}T$ or $\operatorname{Im}T$ is a scalar, we may suppose that $\operatorname{Re}T = cI$ and we can apply proposition 3.2.3, for $M = \{\operatorname{Im}T\}$, and we find the compact operator K. The operators.

$$T_1 = K + cI + \frac{i}{2}\operatorname{Im}T,\ T_2 = -K + \frac{i}{2}\operatorname{Im}T$$

satisfy the proposition.

□

Proposition 3.2.5

The set of all irreducible operators in $B(H)$ is dense in $B(H)$

Proof:

Let $T \in B(H)$ be arbitrary. By the spectral theorem (proposition 2.7.15) we can find a hermitian operator K whose matrix is diagonal with respect to an orthonormal basis $\{e_i\}$ and K satisfying the inequality

$$\left\| K - \frac{T^* + T}{2} \right\| < \frac{\varepsilon}{4}\ .$$

Now we can find another hermitian operator K_1, diagonal with respect to $\{e_i\}$, all the eigenvalues of K_1 are distinct, and $\|K - K_1\| < \frac{\varepsilon}{4}$. If K_2 is any hermitian operator such that $\|K_2 - \operatorname{Im}T\| < \frac{\varepsilon}{2}$ with all entries non zero (with respect to $\{e_i\}$), the operator $K_1 + \varepsilon K_2$ is irreducible and

$$\|T - (K_1 + \varepsilon K_2)\| < \varepsilon\ .$$

□

Definition 3.2.6

An operator $T \in B(H)$ is called ***completely reducible*** if the restriction of T to any reducing subspace has a reducing subspace. As an example of a completely reducible operator, we can take every normal operator.

Proposition 3.2.7

If T is a completely reducible operator on a finite dimensional space, then T is normal

Proof:

Since T is compact, we can find invariant subspaces

$$M_1 \subset M_2 \subset \cdots \subset M_n = H \ (\dim H = n)$$

In this case $\dim M_i = i$ and M_1 is a reducing subspace. Also, in $M_2 = M_1 \oplus M_1^{\perp}$, where the orthogonal complement is taken with respect to M_2. Clearly $M_1^{\perp}$ is a reducing subspace. We can continue in this way to obtain that T is normal. However, there exist completely reducible operators which are not normal. Indeed, let $H = L^2[0,1]$ and let $T : H \to H$ be the operator defined by $Tf(x) = xf(x)$ which is clearly self-adjoint. On the space $H \times H$ with the inner product

$$\langle\langle (x,y),(x',y') \rangle\rangle = \langle x,x' \rangle + \langle y,y' \rangle .$$

Consider the operator $\tilde{T}$ with the matrix

$$\tilde{T} = \begin{bmatrix} 0 & T \\ 0 & I \end{bmatrix}.$$

Then,

$$\tilde{T}^* = \begin{bmatrix} 0 & T \\ 0 & I \end{bmatrix}, \ \tilde{T}^*\tilde{T} = \begin{bmatrix} 0 & 0 \\ 0 & T^2 + I \end{bmatrix}.$$

Clearly, $\tilde{T}$ is not normal. We shall show that every reducing subspace of $\tilde{T}$ is of the form $N \oplus N$, where N is an invariant subspace of T. Let M be any invariant subspace which is reducing for $\tilde{T}$ and let N be the invariant subspace of T generated by elements f of L^2 for which there exist g$\in L^2$ such that (f, g)$\in$M or $(g, f) \in M$. From the definition of N it is clear that $M \subset N \oplus N$ We now show the inclusion in the reverse direction. Let $(h, k) \in M$. Since for any positive integer n,

$$\left(\tilde{T}^*\tilde{T}\right)^n(h,k)=\begin{bmatrix}0 & 0\\ 0 & T^2+I\end{bmatrix}^n\begin{bmatrix}h\\ k\end{bmatrix}=\begin{bmatrix}0\\ \left(T^2+I\right)^n k\end{bmatrix}$$

it follows that $\{\bar{0}\}\oplus\vee\{(T^2+I)k\}\in M$. We consider the algebra of polynomials without constant term having as variable $z=x^2+1$ which satisfies the conditions of the Stone-Weierstrass approximation theorem [4] and thus is dense in $C[0,1]$. From this we obtain that $(0,k)\in M$ and thus $(h,0)$ is in M since $(h,k)\in M$.. In this case

$$\{\bar{0}\}\oplus\vee\{T^{2^n}k\},\ \{\bar{0}\}\oplus\vee\{T^{2^n+1}k\},\ \vee\{T^{2^n}h\}\oplus\{\bar{0}\}\text{ and }\vee\{T^{2^n+1}h\}\oplus\{\bar{0}\},$$

are in M by applying $\tilde{T}$ and $\tilde{T}^*$ to the elements $(h,0)$ and $(0,k)$, since $(h,k)\in M$, it follows that $(h,0)$ and $(0,k)$ are in M and the assertion is proved.

□

Proposition 3.2.8

If $T\in B(H)$ is a hyponormal operator and M is an invariant subspace of T such that $S=TP$ (Where P is the orthogonal projector of H and M) is a normal operator, then M reduces T.

Proof:

Since T is hyponormal, we have

$$\|T^*x\|\le\|Tx\|\ for\ all\ x\in H \qquad (3.1).$$

Since $S=TP$ is normal, we have

$$\|Sx\|=\|S^*x\|\ for\ all\ x\in H,$$

i.e.

$$\|TPx\|=\|P^*T^*x\|=\|PT^*x\|\ for\ all\ x\in H \qquad (3.2).$$

Let $x\in M$. Since M is invariant under T, $Tx\in M$. Also $Px=x$ (proposition 2.2.2). So (3.2) yields

$$\|Tx\|=\|PT^*x\|\ for\ all\ x\in M \qquad (3.3).$$

By (3.1), we obtain, using (3.3)

$$\|T^*x\|\le\|PT^*x\|\ for\ all\ x\in M \qquad (3.4).$$

Since P is an orthogonal projection $\|Py\| \le \|y\| \ \forall y \in H$ (proposition 2.2.2). Hence (3.4) must imply

$$\|PT^*x\| \le \|T^*x\| \ for\ all\ x \in M \qquad (3.5).$$

(3.4) and (3.5) yield

$$\|PT^*x\| = \|T^*x\| \ for\ all\ x \in M$$

This implies that $T^*x \in M$, and hence M reduces T.

□

Proposition 3.2.9

Suppose $T \in B(H)$ is normal and has one of the following properties.

1. T is compact.
2. $\operatorname{Re} T$ or $\operatorname{Im} T$ is a compact operator,

then every invariant subspace of T is reducing.

Proof:

Let M be an invariant subspace for T and P be the orthogonal projection on H onto M. Consider the operator $S = TP$. Now S is hyponormal. Indeed, if $x \in M$, then

$$\|S^*x\| = \|(TP)^* x\| = \|P^*T^*x\| = \|PT^*x\| \le \|T^*x\| = \|Tx\|$$

(for P is an orthogonal projection implies $P^* = P$ and $Px = x$ (proposition 2.2.2)) whereas $\|Sx\| = \|TPx\| = \|Tx\|$. Thus $\|S^*x\| \le \|Sx\|$, when $x \in M$.. If $x \in M^{\perp}$, then since $M^{\perp}$ is invariant under T^* (proposition 2.2.9), we have $T^*x \in M^{\perp}$, so $PT^*x = \bar{0}$, i.e. $S^*x = \bar{0}$ if $x \in M^{\perp}$. Since $Sx = \bar{0}$ for $x \in M^{\perp}$ we get $\|S^*x\| = \|Sx\| = 0$ if $x \in M^{\perp}$. Hence

$$\|S^*x\| \le \|Sx\| \text{ for all } x \in H.$$

If T satisfies condition 1, then S is compact. Now S is hyponormal and compact $\Rightarrow S$ is normal (see problem 163 [21]) and hence it follows from proposition 3.2.8 that M reduces S. If either $\operatorname{Re} T$ or $\operatorname{Im} T$ is compact, then since $\operatorname{Im}(iT) = \operatorname{Re} T$, we see that T is compact and the assertion of the proposition follows as in the case of condition 1.

□

Definition 3.2.10

Let X be a Banach space and $T \in B(X)$. T is called ***polynomially compact*** if there exists a nonzero polynomial $p(z)$ such that $p(T)$ is a compact operator. Of course, we can suppose that $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ and in this case the polynomial is unique as is easy to see. We call this polynomial the ***minimal polynomial*** for T.

Averson and Feldman [3] proved that,

Proposition 3.2.11

If T is polynomially compact, then T has a non-trivial invariant subspace.

Also see [22] and [6]. A good account is also found in [17].

Proposition 3.2.12

If T is a compact operator on a Hilbert space and (M_α) is a family of invariant subspace which is totally ordered and $\|P_{M_\alpha} T\| = \|T\|$. Then we have $\|P_{M_0} T\| = \|T\|$ where $M_0 = \underset{\alpha}{\vee} M_\alpha$.

Proof:

Let $\varepsilon > 0$. For each α there exists a vector x_α of norm 1 such that

$$\left\|\left(P_{M_0} - P_{M_\alpha}\right) T x_\alpha\right\| > \left\|(P_{M_0} - P_{M_\alpha}) T\right\| - \frac{\varepsilon}{2}.$$

Since $\{x_\alpha\}$ is a bounded set and T is a compact operator we can find a sequence in $\{Tx_\alpha\}$ which is convergent (see proposition 2.7.3), say to x. Since P_{M_0} is the strong limit of $\{P_{M_\alpha}\}$, there exists μ such that

$$\left\|\left(P_{M_{\alpha_\mu}} - P_{M_0}\right) x\right\| < \frac{\varepsilon}{4}, \quad \left\|Tx_{\alpha_\mu} - x\right\| < \frac{\varepsilon}{4}$$

and hence

$$\begin{aligned}
\left\|\left(P_{M_0} - P_{M_\alpha}\right) T\right\| &< \left\|\left(P_{M_0} - P_{M_\alpha}\right) T x_{\alpha_\mu}\right\| + \frac{\varepsilon}{4} \\
&< \left\|\left(P_{M_0} - P_{M_\alpha}\right)\left(Tx_{\alpha_\mu} - x\right)\right\| + \left\|\left(P_{M_0} - P_{M_{\alpha_\mu}}\right) x\right\| + \frac{\varepsilon}{4} \\
&< 2\left\|Tx_{\alpha_\mu} - x\right\| + \left\|\left(P_{M_0} - P_{M_{\alpha_\mu}}\right) x\right\| + \frac{\varepsilon}{4} \\
&< \varepsilon
\end{aligned}$$

Since $\varepsilon < 0$ is arbitrary, we have $\left\|P_{M_0}T\right\| = \|T\| = \lim\left\|P_{M_{\alpha_\mu}}T\right\|$ and the proposition is proved.

□

Proposition 3.2.13

Suppose $T \in B(H)$ and

1. T is polynomially compact.
2. T has only reducing subspaces,

then T is a normal operator.

Proof:

It is clear that for every point in the point spectrum of T say λ_0 the subspace $\eta_{\lambda_0}(T) = \{x : Tx = \lambda_0 x\}$ is reducing. Also it is easily seen that the family of subspaces $\{\eta_\lambda(T) : \lambda \in \sigma_p(T)\}$ is an orthogonal family of subspaces, and if we consider the space $H_0 = \sum \oplus \{\eta_\lambda(T) : \lambda \in \sigma_p(T)\}$, then this is a reducing subspace and the restriction of T to this subspace is a normal operator. Denote this restriction by T_0. On the orthogonal complement of H_0, which is also a reducing subspace, the operator TQ, where Q is the orthogonal projection of H onto the orthogonal complement of H_0, has the form $p(TQ) = p(T)Q$ and so it is a polynomially compact operator of course, $P(T)Q$ is not zero (in the contrary case there is nothing to prove). The operator TQ being polynomially compact has nontrivial invariant subspace which is also reducing for T (proposition 3.2.9 and proposition 3.2.11). Consider the family $\Im$ of all invariant subspaces of T for which

$$\|P_M p(T)\| = \|p(T_1)\| \text{ for all } M \in \Im,$$

where T_1 is the restriction of T to M. Using Zorn's lemma [13], we can see that the family $\Im$ contains a minimal element, say M_0. From the condition about norms it is clear that $M_0 \neq \{\bar{0}\}$. Now if the dim $M_0 \geq 2$, the restriction of T has again an invariant subspace which is also reducing. Let N be this invariant subspace. Then we have

$$\|p(T_1)\| = \max\{\|P_N p(T_1)\|, \|(P_{M_0} - P_N)p(T_1)\|\}$$

and this shows that either N or $M_0 \setminus N^{\perp}$ is in $\Im$, contradicting the minimality of M_0. In this case $\dim M_0 = 1$, the operator T has an eigenvalue, and this contradicts the definition of H_0. Thus on H_0 the operator T is zero and the proposition is proved.

□

Using invariant subspaces we obtain the structure of polynomially compact operators on Banach spaces. Let X be a Banach space and $T \in B(X)$ be polynomially compact; so there exists a nonzero polynomial $p(z)$ such that $p(T)$ is a compact operator. We can suppose that $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ is the minimal polynomial for T.

Proposition 3.2.14

Let T be a polynomially compact operator and $p(z) = (z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_k)^{n_k}$ its minimal polynomial. In this case the Banach space X has decomposition in the direct sum of invariant subspaces for T

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_k$$

and T has the form

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_k$$

such that for each integer $i \in [1, K]$, T_i is the restriction of T to X_i and all operators $(T_i - z_i I)^{n_i}$ are compact. The spectrum of T consists of a countable set of points having as possible limit points $z_1, z_2, \cdots, z_k$ and every z_i is either a limit point or $(T - z_i)^{n_i}$ is a quasi-nilpotent operator.

Proof:

First we show that every z_i is in $\sigma(T)$. Suppose that it is not so. Thus $(T - z_i)^{-1}$ exists and $p(T)(T - z_i)^{-n_i}$ is a compact operator and a polynomial in T. Clearly this contradicts the minimality of $p(T)$. Since by the spectral mapping theorem (proposition 2.4.2)

$$\sigma(p(T)) = p(\sigma(T)).$$

It follows that $\sigma(T)$ is a finite or a countable set. Now since the only limits points or $\sigma(p(T))$ is possibly 0, it follows that the possible limit points of $\sigma(T)$ are $z_1, z_2, \cdots, z_k$. Let z be a complex number which is an isolated point in $\sigma(T)$ and $p(z) \neq 0$. We show that z is an eigenvalue and $\eta_z(T) = \{x : Tx = zx\}$ is a finite dimensional subspace. Let D be a domain in the complex plane such that $z \in D$ and $\overline{D} \cap \sigma(T) = \{z\}$. We define the projection

$$P_D = \frac{1}{2\pi i} \int_{\partial D} (T - \lambda I)^{-1} d\lambda$$

and set $X_D = P_D X$, which is an invariant subspace for T, and $\sigma\left(T|_{X_D}\right) = \{z\}$. Since $p(z) \neq 0$, it follows that $p\left(T|_{X_D}\right)$ is an invertible operator and also a compact operator. This is possible if and only if X_D is a finite-dimensional subspace (by the Riesz Lemma [4]). But $\{z\} = \sigma\left(T|_{X_D}\right)$ and it follows that $X_D = \{x : (T - zI)^n x = 0\}$ for a sufficiently large integer. We now consider the case of a point $z \in \sigma(T)$ and $p(z) = 0$, and also an isolated point in $\sigma(T)$. Let D_1 and D_2 be two domains in the complex plane such that $\overline{D_1} \cap \overline{D_2} \neq \phi$ and also

1. $D_1 \cup D_2 \supset \sigma(T)$.
2. $z \in D_1$
3. $\overline{D_1} \cap \sigma(T) = \{z\}$.

We define

$$X_{D_1} = P_{D_1} X, \quad X_{D_2} = P_{D_2} X$$

which have the property that $X = X_1 \oplus X_2$, $T = T_{D_1} \oplus T_{D_2}$ where T_{D_i} are the corresponding restrictions of T. Since z is exactly the spectrum of T_{D_1} and $z \notin \sigma\left(T_{D_1}\right)$, we obtain that $\left(T_{D_2} - z\right)^{-1} p\left(T_{D_2}\right)$ is a compact operator, and since the above polynomial is minimal, we obtain that X_{D_1} is an infinite dimensional subspace and $\left(T_{D_1} - z\right)$ is a quasi-nilpotent operator. Now for the roots of p, let D_i be domains in the complex plane such that

1. $z_i \in D_i \ (i = 1,...k)$
2. ∂D_i is a rectifiable Jordan curve
3. $D_i \cap D_j = \phi$ for $i \neq j$.
4. If z_i is an isolated point in $\sigma(T)$, then $D_i \cap \sigma(T) = \{z_i\}$
5. $\sigma(T) \subset \bigcup_{i=1}^{k} D_i$.

For each j we define the projections P_j by

$$P_j = \frac{1}{2\pi i} \int_{\partial D_j} (T - \lambda I)^{-1} \, d\lambda$$

and $X_i = P_i X$ which are invariant subspaces for T. If T_i denotes the restriction of T to X_i, from properties 1 to 5 of D_i we have

1. $X = \sum_{i=1}^{k} \oplus X_i$
2. $T = \sum_{i=1}^{k} \oplus T_i$
3. $\sigma(T_i) = \sigma(T) \cap D_i$

and also the operators $(T_i - z_i)^{ni}$ are compact.

□

From this proposition we obtain in the case of normal operators on Hilbert spaces the following:

Proposition 3.2.15

If $T \in B(H)$ is normal and also polynomially compact and $p(z) = (z - z_1)^{n_1} (z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$ is its minimal polynomial then the following assertions hold:

1. $\sigma(T)$ is a countable set.
2. Some z_i are eigenvalues of finite multiplicity.

3. If z_i is not as in 2, then they are either isolated eigenvalues of infinite multiplicity or limits of eigenvalues.
4. $H = \sum H_i$, direct orthogonal sum, $T = \sum T_i$ and each H_i is reducing and $T_i - z_i I$ are compact operators then $T - z_i I$ is compact.

Proof:

All the assertions follow from proposition 3.2.14 except the last assertion. Since $(T_i - z_i I)^{n_i}$ are compact operators and T is normal, we have for any x, $\|x\| = 1$.

$$\left\|(T_i - z_i I)^{n_i} x\right\| \geq \left\|(T_i - z_i I)x\right\|^{n_i} \text{ i.e. } \left\|(T - z_i I)^{n_i} x\right\| \geq \left\|(T - z_i I)x\right\|^{n_i}$$

and this implies the assertion. (Note; T is normal $\Rightarrow T - z_i I$ is normal $\Rightarrow (T - z_i I)^{n_i}$ is normal [4]. For a normal $S \in B(H)$ we have the result $\|Sx\|^k \leq \|S^k x\|$ for all $x \in H$ and $\|x\| = 1$ [4]. Take a sequence (x_n) of unit vectors in H converging weakly to 0. Then by compactness of $(T_i - z_i I)^{n_i}$, we have $(T_i - z_i I)^{n_i} x_n \xrightarrow{s} \bar{0}$. Hence

$$\left\|(T_i - z_i I)^{n_i} x_n\right\| \to 0 \text{ or } \left\|(T_i - z_i I)x_n\right\|^{n_i} \to 0$$

i.e.

$$\left\|(T - z_i I)^{n_i} x_n\right\| \to 0 \text{ or } \left\|(T - z_i I)x_n\right\|^{n_i} \to 0$$

i.e

$$(T - z_i I)^{n_i} x_n \xrightarrow{s} \bar{0} \text{ or } (T - z_i I)x_n \xrightarrow{s} \bar{0}$$

that is $T - z_i I$ is compact.

□

CHAPTER FOUR

4.1 SUBNORMAL OPERATOR OF H WITH A CYCLIC VECTOR

Definition 4.1.1

An operator T on H (i.e. $T \in B(H)$) is called subnormal if there exists a normal operator N on a Hilbert space K such that $K \supseteq H$ (i.e. H is a subspace of K), the subspace H is invariant under N and the restriction of N to H coincides with T. N is called a ***normal extension*** of T and is called a ***minimal normal extension*** of T if there is no reducing subspace of N between H and K, i.e., K is the smallest subspace which contains H and reduces N. Halmos [26] proved;

Proposition 4.1.2

A necessary and sufficient condition for an operator T on H to be subnormal is that, for every finite set $x_0, x_1, \cdots, x_r$ in H

$$\sum_{m=0}^{r}\sum_{n=0}^{r}\left\langle T^n x_m, T^m x_n \right\rangle \geq 0$$

(See [26] . Bram [8] showed;

Proposition 4.1.3

A necessary and sufficient condition for $T \in B(H)$ to be subnormal is that, for any finite set $x_0, x_1, \cdots, x_r$ in H

$$\sum_{m=0}^{r}\sum_{n=0}^{r}\left\langle T^{n+1} y_m, T^{m+1} y_n \right\rangle \leq \|T\|^2 \sum_{m=0}^{r}\sum_{n=0}^{r}\left\langle T^n y_m, T^m y_n \right\rangle$$

See [8].

We consider a subnormal operator with a cyclic vector. In this case the extension space K is simplified and we can use the spectral decomposition of the normal extension of T (proposition 2.7.15).

Definition 4.1.4

For an operator T on H, $x \in \mathrm{H}$ is said to be a ***cyclic vector*** of T if H is the smallest subspace which is invariant under T and contains x. This means $H = \vee\left\{T^n x : n = 0,1,2,\ldots\right\}$.

Lemma 4.1.5

If T is a subnormal operator on H with a cyclic vector x and if N is its minimal normal extension on K, then $K = \vee\left\{N^{*m}N^n x \quad ; \quad m,n = 0,1,2,\ldots\right\}$.

Proof:

Since x is a cyclic vector of T, $H = \vee\left\{T^n x : n = 0,1,2,\ldots\right\}$ and let

$$M = \vee\left\{N^{*m}N^n x \quad ; \quad m,n = 0,1,2,\ldots\right\}$$

Then $M \subseteq K$ and $N^n x = T^n x$ for all $n = 0,1,2,\ldots$ and hence $H \subseteq M$. Clearly M reduces N and, by the minimality of N, $K \subseteq M$. Therefore $M = K$.

□

Proposition 4.1.6

Under the same conditions as in Lemma 4.1.5 above, K and H are identifiable with $L^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ and $H^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ respectively, where $\{E_z\}$ is the resolution of the identity for N, $L^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ = the L^2- closure of the set $L^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right\}$ of all bounded Borel functions on $\sigma(N)$ and $H^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ = the L^2-closure of the set $\wp$ of all complex polynomials in z.

Proof:

Let

$$D = \left\{f(N)x : f \in L^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right)\right\}.$$

Then, by proposition 4.1.2, $\overline{[D]} = K$. For any $f(N)x \in D$, let $Wf(N)x = f$. Then

$$\|Wf(N)x\|^2 = \|f\|^2 = \int_{\sigma(N)} |f(z)|^2 d\|E_z x\|^2 = \|f(N)x\|^2$$

i.e. W is a well defined isometric transformation from D onto $L^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ and K is identifiable with $L^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$. Next, since $H = \vee\left\{T^n x : n = 0,1,2,\ldots\right\}$, for any $y \in H$, there exist $p_n \in \wp$ such that $\|y - p_n(T)x\| \to 0$ as $n \to \infty$. Since $p_n(T)x = p_n(N)x$ for all $n = 0,1,2\cdots$,

$$\int_{\sigma(N)} |p_n(z)-p_m(z)|^2 d\|E_z x\|^2 = \|p_n(N)x-p_m(N)x\|^2$$

$$= \|p_n(T)x-p_m(T)x\|^2 \to 0 \text{ as } m,n\to\infty$$

and there exists a $f_y \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ such that

$$\int_{\sigma(N)} |f_y(z)-p_n(z)|^2 d\|E_z x\|^2 \to 0 \text{ as } n\to\infty$$

Since the existence of f_y is independent of the choice of $p_n \in \wp$, we may define V by $Vy = f_y$ and we obtain

$$\|Vy\|^2 = \|f_y\|^2 = \int_{\sigma(N)} |f_y(z)|^2 d\|E_z x\|^2 = \lim_{n\to\infty}\int_{\sigma(N)} |p_n(z)|^2 d\|E_z x\|^2 = \lim_{n\to\infty}\|p_n(N)x\|^2$$

$$= \lim_{n\to\infty}\|p_n(T)x\|^2 = \|y\|^2.$$

Then V is well defined and is an isometric transformation from H into $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. Conversely, for any $f \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ there exist $p_n \in \wp$ such that

$$\int_{\sigma(N)} |f(z)-p_n(z)|^2 d\|E_z x\|^2 \to 0 \text{ as } n\to\infty$$

by the definition of $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ and $\left(p_n(T)x\right)$ is a Cauchy sequence in H because $p_n(T)x = p_n(N)x$. Hence there exists a $y \in H$ such that

$$\|p_n(T)x - y\| \to 0 \text{ as } n\to\infty.$$

This means that V is an isometric transformation from H onto $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. Therefore H is identifiable with $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$.

□

Proposition 4.1.7

Let $T \in B(H)$ be a subnormal operator with a cyclic vector x and let $S \in B(H)$ satisfy $ST = TS$. Then S is subnormal and there exists a $f_s \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ such that

$Sy = f_s(N)y$ for all $y \in H$, where N is the minimal normal extension on K of T, $\{E_z\}$ is a resolution of the identity for N and

$$H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) = H^2\left(d\|E_z x\|^2;\sigma(N)\right) \cap L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$

Proof:

Since $H = \vee\{T^n x : n = 0,1,2\ldots\}$, there exist $p_n \in \wp$ such that $\|Sx - p_n(T)x\| \to 0$ as $n \to \infty$ and since $p_n(T)x = p_n(N)x$ for $n = 0,1,2,\ldots$

$$\int_{\sigma(N)} |p_n(z) - p_m(z)|^2 d\|E_z x\|^2 = \|p_n(N)x - p_m(N)x\|^2$$

$$= \|p_n(T)x - p_m(T)x\|^2 \to 0 \text{ as } m,n \to \infty$$

and there exists a Borel measurable function $f_s \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ such that

$$\int_{\sigma(N)} |f_s(z)|^2 d\|E_z x\|^2 < \infty$$

and

$$\int_{\sigma(N)} |f_s(z) - p_n(z)|^2 d\|E_z x\|^2 \to 0 \text{ as } n \to \infty,$$

and hence

$$x \in D(f_s(N)) = \left\{ y \in K : \int_{\sigma(N)} |f_s(z)|^2 d\|E_z x\|^2 < \infty \right\}$$

and

$$Sx = f_s(N)x \qquad (4.1).$$

Since, by proposition 4.1.6, K is identifiable with $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$, $D(f_s(N))$ is also identifiable with

$$D(f_s) = \left\{ g \in L^2\left(d\|E_z x\|^2;\sigma(N)\right); f_s(z)g(z) \in L^2\left(d\|E_z x\|^2;\sigma(N)\right) \right\}.$$

If, for any $g_n \in D(f_s)$, there exist g and h in $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ such that

$$\int_{\sigma(N)} |g_n(z) - g(z)|^2 d\|E_z x\|^2 \to 0$$

and

$$\int_{\sigma(N)} |f_s(z)g_n(z)-h(z)|^2 d\|E_z x\|^2 \to 0 \text{ as } n\to\infty$$

then

$$\int_{\sigma(N)} |f_s(z)g_n(z)-h(z)| d\|E_z x\|^2 \le \|x\| \left\{ \int_{\sigma(N)} |f_s(z)g_n(z)-h(z)|^2 d\|E_z x\|^2 \right\}^{1/2} \to 0$$

as $n\to\infty$ and

$$\int_{\sigma(N)} |f_s(z)g(z)-f_s(z)g_n(z)|\, d\, \|E_z x\|^2$$

$$\le \left\{ \int_{\sigma(N)} |f_s(z)|^2\, d\|E_z x\|^2 \right\}^{1/2} \left\{ \int_{\sigma(N)} |g(z)-g_n(z)|^2\, d\|E_z x\|^2 \right\}^{1/2} \to 0 \; as\; n\to\infty$$

and hence

$$\int_{\sigma(N)} |f_s(z)g(z)-h(z)|\, d\|E_z x\|^2 \le \int_{\sigma(N)} |f_s(z)g(z)-f_s(z)g_n(z)|\, d\|E_z x\|^2$$

$$+ \int_{\sigma(N)} |f_s(z)g_n(z)-h(z)|\, d\, \|E_z x\|^2 \to 0 \; as\; n\to\infty$$

Therefore $f_s(z)\, g(z)=h(z)$ almost everywhere $\left(d\|E_z x\|^2\right)$. Since

$$h(z)\in L^2\left(d\|E_z x\|^2;\sigma(N)\right),\; f_s(z)\,g(z)\in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$$

and $g(z)\in D(f_s)$. Therefore the graph of the multiplication operator f_s on $D(f_s)$ is closed and the graph of $f_s(N)$ is also closed (4.2).

Next, since $ST=TS$, for any $p\in\wp$

$$Sp(T)x = p(T)Sx = p(N)f_s(N)x \text{ (by (4.1))}$$

$$= f_s(N)\, p(N)\, x = f_s(N)\, p(T)x \qquad (4.3)$$

and since, for any $y\in H$ there exist $q_n\in\wp$ such that $\|y-q_n(T)x\|\to 0 \; as\; n\to\infty$, we have, by (4.3)

$$\|f_s(N)q_n(T)x-Sy\| = \|Sq_n(T)x-Sy\| < \|S\|\,\|q_n(T)-y\| \to 0 \; as\; n\to\infty$$

and, by (4.2) $y\in D\left(f_s(N)\right)\; and\; Sy=f_s(N)y$. Therefore

$$H \subseteq D(f_s(N) \; and \; f_s(N)\, y = Sy \; for \; all \; y \in H \qquad (4.4).$$

Since $SH \subseteq H$, $f_s(N)H = SH \subseteq H \subseteq D(f_s(N))$ and $H \subseteq D(f_s(N)^2)$ and $f_s(N)^2 y = S^2 y$ for all $y \in H$. By repeating this argument, generally we have, for each $n = 0,1,2,\cdots$

$$H \subseteq D(f_s(N)^n) \; and \; f_s(N)^n y = S^n y \; for \; all \; y \in H \qquad (4.5).$$

Clearly

$$D(f_s(N)^n) = \left\{ y \in K; \int_{\sigma(N)} |f_s(z)^n|^2 \, d\|E_z y\|^2 < \infty \right\} = D(f_s(N)^{*n}),$$

and then, for $y_0, y_1, \cdots, y_r \in H$

$$\sum_{m=0}^{r}\sum_{n=o}^{r} \langle S^n y_m, S^m y_n \rangle = \sum_{m=0}^{r}\sum_{n=0}^{r} \langle f_s(N)^n y_m, f_s(N)^m y_n \rangle$$

$$= \left\| \sum_{m=0}^{r} f_s(N)^{*m} \, y_m \right\|^2 \geq 0$$

and, by proposition 4.1.2, S is subnormal. Lastly, let $M = \vee \{ f_s(N)^{*n} y; y \in H, n = 0,1,2,\cdots \}$. Then, clearly $H \subseteq M \subseteq K$, and since S is subnormal, we get (see [8]), for any finite set $y_o y_1, \ldots\ldots y_r$ in H

$$\sum_{m=0}^{r}\sum_{n=0}^{r} \langle S^{n+1} y_m, S^{m+1} y_n \rangle \leq \|S\|^2 \sum_{m=0}^{r}\sum_{n=0}^{r} \langle S^n y_m, S^m y_n \rangle .$$

Then, by (4.5),

$$\sum_{m=0}^{r}\sum_{n=0}^{r} \langle f_s(N)^{n+1} y_m, f_s(N)^{m+1} y_n \rangle < \|S\|^2 \sum_{m=0}^{r}\sum_{n=0}^{r} \langle f_s(N)^n{}_s y_m, f_s(N)^m y_n \rangle$$

and

$$\left\| f_s(N) \sum_{m=0}^{r} f_s(N)^{*m} y_m \right\|^2 \leq \|S\|^2 \left\| \sum_{m=0}^{r} f_s(N)^{*m} y_m \right\|^2$$

and hence $f_s(N)$ *is* bounded on a dense linear subset of M. Since by (4.2), the graph of $f_s(N)$ is closed,

$$M \subseteq D(f_s(N)) \; and \; \|f_s(N) y\| \leq \|S\| \, \|y\| \; \forall \; y \in M \qquad (4.6)$$

and then M reduces $f_s(N)$. It is clear that M is invariant under N and since, for any $y \in M$,

$$\|f_s(N)N^{*n}y\| = \|N^{*n}f_s(N)y\| = \|N^{n}f_s(N)y\| = \|f_s(N)N^{n}y\|$$

$$\leq \|S\|\|N^{n}y\| \text{ (by (4.6) because } N^{n}y \in M \text{)}$$

$$= \|S\|\|N^{*n}y\|$$

we have for each $n = 1,2,\cdots,$ $N^{*n}M \subset D(f_s(N))$ and $\|f_s(N)y\| \leq \|S\|\|y\|$ for all $y \in N^{*n}M$ and hence by (4.2) $\overline{[N^{*n}M]} \subset D$ and

$$\|f_s(N)y\| \leq \|S\|\|y\| \text{ for all } y \in \overline{[N^{*n}M]} \qquad (4.7).$$

Since M reduces $f_s(N)$, each $\overline{[N^{*n}M]}$ also reduces $f_s(N)$. Let

$$\aleph = \vee\{N^{*n}y : y \in M, n = 0,1,2,\ldots\ldots\}.$$

Then, clearly $M \subseteq \aleph \subseteq K$. Since for each $k = 0,1,2,\cdots$

$$\overline{[N^{*k}M]} \subseteq \sum_{j=0}^{r}\overline{[N^{*j}M]}$$

we can represent for any $y_k \in \overline{[N^{*k}M]}$.

$$\sum_{k=0}^{r} y_k = x_0 \oplus x_0^{\perp} \text{ for some } x_0 \in M \text{ and } x_0^{\perp} \in \sum_{k=0}^{r}\overline{[N^{*k}M]}\Theta M$$

and, for each $k = 0,1,2,\cdots,r$ and for any $n = 0,1,2,\cdots$

$$x_{k-1+(r+1)n}^{\perp} = x_{k+(r+1)n} \oplus x_{k+(r+1)n}^{\perp}$$

where

$$x_{k+(r+1)n} \in \overline{[N^{*k}M]}$$

and

$$x_{k+(r+1)n}^{\perp} \in \sum_{j=0}^{r}\overline{[N^{*j}M]}\Theta\overline{[N^{*k}M]}$$

and hence

$$\left\|\sum_{k=0}^{r} y_k\right\|^2 = \|x_0\|^2 + \|x_0^{\perp}\|^2 = \|x_0\|^2 + \|x_1\|^2 + \|x_1^{\perp}\|^2 = \cdots = \sum_{k=0}^{n}\|x_k\|^2 + \|x_n^{\perp}\|^2.$$

If there exists a positive integer n_0 such that $\|x_n^\perp\| = \|x_{n_0}^\perp\| \neq 0$ for all $n = n_0, n_0+1, \cdots$, then $x_n = 0$ for all $n = n_0+1, n_0+2, \cdots$ and $x_n^\perp = x_{n_0}^\perp$ for all $n = n_0, n_0+1, \cdots$ and hence

$$x_{n_0}^\perp \in \sum_{j=0}^{r} \overline{\left[N^{*j}M\right]} \Theta \overline{\left[N^{*k}M\right]} \text{ for all } k = 0,1,2,\cdots,r.$$

Then $x_{n_0}^\perp = 0$ which is a contradiction. Therefore $\sum_{k=0}^{r} y_k = \sum_{k=0}^{\infty} x_k$ and $\left\|\sum_{k=0}^{r} y_k\right\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2$.

Since, for each $k = 0,1,2,\cdots,r$, $\overline{\left[N^{*k}M\right]}$ reduces $f_s(N)$, each $\sum_{j=0}^{r} \overline{\left[N^{*j}M\right]} \Theta \overline{\left[N^{*k}M\right]}$ reduces $f_s(N)$ and

$$\begin{aligned}
\left\|f_s(N)\sum_{k=0}^{r} y_k\right\|^2 &= \left\|f_s(N)\{x_0 \oplus x_0^\perp\}\right\|^2 = \left\|f_s(N)x_0 \oplus f_s(N)x_0^\perp\right\|^2 \\
&= \|f_s(N)x_0\|^2 + \|f_s(N)x_0^\perp\|^2 = \|f_s(N)x_0\|^2 + \left\|f_s(N)\{x_1 \oplus x_1^\perp\}\right\|^2 \\
&= \|f_s(N)x_0\|^2 + \|f_s(N)x_1 \oplus f_s(N)x_1^\perp\|^2 \\
&= \|f_s(N)x_0\|^2 + \|f_s(N)x_1\|^2 + \|f_s(N)x_1^\perp\|^2 \\
&= \cdots \\
&= \sum_{k=0}^{\infty} \|f_s(N)x_k\|^2 \le \sum_{k=0}^{\infty} \|S\|^2 \|x_k\|^2 \text{ (by (4.6) and (4.7))} \\
&= \|S\|^2 \sum_{k=0}^{\infty} \|x_k\|^2 = \|S\|^2 \left\|\sum_{k=0}^{r} y_k\right\|^2
\end{aligned}$$

and hence $f_s(N)$ is bounded on a dense linear subset of $\aleph$. Since by (4.2), the graph of $f_s(N)$ is closed.

$$\aleph \subseteq D(f_s(N)) \text{ and } \|f_s(N)y\| \le \|S\|\|y\| \ \forall y \in \aleph.$$

Clearly $\aleph$ reduces N and, by minimality of N, $\aleph = K$ and then $D(f_s(N)) = K$. Therefore $f_s(N)$ is bounded on K and $f_s \in L^\infty\left(d\|E_z x\|^2; \sigma(N)\right)$ since $f_s \in H^2\left(d\|E_z x\|^2; \sigma(N)\right)$ described as before, $f_s \in H^\infty\left(d\|E_z x\|^2; \sigma(N)\right)$.

□

Corollary 4.1.8

If T is a subnormal operator on H with a cyclic vector x and if $S \in \{T, T^*\}'$, then S is normal

Proof:

Since $ST = TS$, by proposition 4.1.7, S is subnormal. Let M on K be its minimal normal extension and P be the orthogonal projection from K on H. Then, for any $y \in H$, $Sy = My$ and

$$\|Sy\| = \|My\| = \|M^* y\| \geq \|PM^* y\| = \|S^* y\|.$$

On the other hand, since $S^*T = TS^*$, we have, by the same argument as above

$$\|S^* y\| \geq \|Sy\| \text{ for all } y \in H.$$

Thus $\|Sy\| = \|S^* y\|$ for all $y \in H$, i.e., S is normal.

□

Proposition 4.1.9

Under the same conditions as in proposition 4.1.7 let

$$L_{\{T\}'} = \left\{ f_s \in H^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right); S \in \{T\}' \right\}.$$

Then

$$L_{\{T\}'} = H^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right)$$

Proof:

By the definition of $H^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right)$, for any $f \in H^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right)$, there exist $p_n \in \wp$ such that

$$\int_{\sigma(N)} |f(z) - p_n(z)|^2 d\left(\|E_z x\|^2\right) \to 0 \text{ as } n \to \infty$$

Since $f \in L^\infty\left(d\|E_z x\|^2 ; \sigma(N)\right)$ we can define the bounded linear operator L_f on $L^2\left(d\|E_z x\|^2 ; \sigma(N)\right)$ as the multiplication operator by $f(z)$ on the functions in

$L^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$ and since by proposition 4.1.6, K is identifiable with $L^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$, we can define the bounded linear operator $f(N)$ on K such that

$$\left\|p_n(N)x - f(N)x\right\| \to 0 \text{ as } n \to \infty .$$

Since

$$p_n(N)x = p_n(T)x \in \mathrm{H} \ , f(N)x \in H \qquad (4.8).$$

By assumption $H = \vee\left\{\mathrm{T}^n x; n = 0,1,2,\cdots\right\}$ and since, for any $q \in \wp$

$$f(N)q(T)x = f(N)q(N)x = q(N)f(N)x = q(T)f(N)x \in H$$

(by (4.1)), we have, by the boundedness of $f(N)$,

$$f(N)y \in H \text{ for all } y \in H \text{ i.e. } f(N)H \subseteq H .$$

Clearly $f(N)|_H \in \{T\}'$ and $f \in L_{\{T\}'}$. Therefore $L_{\{T\}'} = H^\infty\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$.

□

Corollary 4.1.10

If T is a normal operator on H with a cyclic vector x, then

$$L_{\{T\}'} = H^\infty\left(d\left\|E_z x\right\|^2;\sigma(T)\right) = \overline{H^\infty\left(d\left\|E_z x\right\|^2;\sigma(T)\right)}$$

where $\{E_z\}$ is the resolution of the identity for T.

Proof:

The minimal normal extension of T is T itself and if $S \in \{T\}'$, then, by Prop 4.1.7 there exists an $f_s \in H^\infty\left(d\left\|E_z x\right\|^2;\sigma(T)\right)$ such that

$$Sy = f_s(T)y \text{ for all } y \in H .$$

Since by Fuglede's Theorem (proposition 2.3.12) $S^* \in \{T\}'$ there also exists an $f_{s^*} \in H^\infty\left(d\left\|E_z x\right\|^2;\sigma(T)\right)$ such that $S^* y = f_{s^*}(T)y$ for all $y \in H$, and then, for any $u, v \in H$

$$\left\langle f_{s^*}(T)u, v\right\rangle = \left\langle S^* u, v\right\rangle = \left\langle u, Sv\right\rangle = \left\langle u, f_s(T)v\right\rangle = \left\langle f_s(T)^* u, v\right\rangle .$$

Therefore $f_{s^*}(T) = f_s(T)^*$ and $f_{s^*}(z) = \overline{f_s(z)}$ by proposition 4.1.7, since by proposition 4.1.9,

$$L_{\{T\}'} = H^\infty\left(d\|E_z x\|^2;\sigma(T)\right).$$

□

We have the conclusion: let $T \in B(H)$ and $\dim H > 1$. We are interested whether there is a non-trivial (different from $\{\overline{0}\}$ and H itself) invariant subspace of H. The answer is true and trivial when H is finite dimensional for T has an eigenvector which spans a one-dimensional invariant subspace. The answer is also true when H is non-separable. In fact, for any non-zero element x of H, the closure of the subspace spanned by $\{x, Tx, T^2x, \cdots\}$ is an invariant subspace of T larger than $\{\overline{0}\}$ and smaller than H because it is separable. Thus the problem concerns operators in separable, infinite dimensional Hilbert spaces.

4.2 SPACE $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ AND $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$

In order to show the existence of non-trivial closed invariant subspaces of an operator $T \in B(H)$, we may assume that T has a cyclic vector $x \in H$ because otherwise $\vee\{T^n x : n = 0,1,2,\cdots\}$ is clearly desired invariant subspace. We shall in the sequel assume that T is a subnormal operator of H with a cyclic vector x, N is the minimal normal extension on K of T and $\{E_z\}$ is the resolution of the identity for N.

Proposition 4.2.1

For $\phi_n \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ the following statements are equivalent

i. $\phi_n \to \phi$ as $n \to \infty$ in the weak operator topology on $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$, i.e.

$$\langle(\phi_n - \phi) f, g\rangle = \int_{\sigma(N)} \{\phi_n(z) - \phi(z)\} f(z)\overline{g(z)} d\|E_z x\|^2 \to 0 \; as \; n \to \infty$$

for all $f, g, \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$

ii. $\phi_n \to \phi$ in the weak $*$ - topology, that is

$$\int\{\phi_n(z)-\phi(z)\}h(z)\,d\|E_z x\|^2 \to 0 \text{ as } n\to\infty$$

$$\text{for all } h\in L^1\left(d\|E_z x\|^2;\sigma(N)\right).$$

Proof:

$(i)\Rightarrow(ii)$:

For any $h\in L^1\left(d\|E_z x\|^2;\sigma(N)\right)$, let $f=\frac{h}{|h|}|h|^{1/2}$, where $\frac{h}{|h|}=0$ if h vanishes, and $g=|h|^{1/2}$. Then $f,g\in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ because $\frac{h}{|h|}\in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and

$$\int_{\sigma(N)}\{\phi_n(z)-\phi(z)\}h(z)\,d\|E_z x\|^2 = \int_{\sigma(N)}\{\phi_n(z)-\phi(z)\}f(z)\overline{g(z)}\,d\|E_z x\|^2$$

$$=\langle(\phi_n-\phi)f,g\rangle\to 0 \;\; as \;\; n\to\infty \;\; by \;\; (i)$$

$(ii)\Rightarrow(i)$:

Since, for any $f,g\in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$, $f\overline{g}\in L^1\left(d\|E_z x\|^2;\sigma(N)\right)$ and

$$\langle(\phi_n-\phi)f,g\rangle=\int_{\sigma(N)}\{\phi_n(z)-\phi(z)\}f(z)\overline{g(z)}\,d\|E_z(x)\|^2\to 0\; as\; n\to\infty\; by\; (ii)$$

□

Remark:

$L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is weak $*$-closed because it is the dual space of $L^1\left(d\|E_z x\|^2;\sigma(N)\right)$ and, by proposition 4.2.1, it is also closed in the weak operator topology on $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$

Proposition 4.2.2

$H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is a weak $*$-closed sub algebra of $L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$

Proof:

Since, for any $\phi,\psi\in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$, $\phi\psi\in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and

$$\phi\psi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)\cdot H^2\left(d\|E_z x\|^2;\sigma(N)\right) \subseteq H^2\left(d\|E_z x\|^2;\sigma(N)\right).$$

We have

$$\phi\psi \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \cap H^2\left(d\|E_z x\|^2;\sigma(N)\right) = H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$

and $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is thus an algebra. Next let

$$\phi_n \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right), \phi_n \to \phi \text{ as } n \to \infty$$

in the weak $*$ – topology. Then, for all $f \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ and all

$$g \in L^2\left(d\|E_z x\|^2;\sigma(N)\right) \Theta H^2\left(d\|E_z x\|^2;\sigma(N)\right),$$

we have $\langle(\phi_n - \phi) f, g\rangle \to 0 \text{ as } n \to \infty$. By proposition 4.2.1 and since $\phi_n f \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$,

$$\phi f \in H^2\left(d\|E_z x\|^2;\sigma(N)\right) \ \forall\ f \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$$

and $\phi = \phi \cdot 1 \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ because $1 \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. On the other hand, by the remark made after proposition 4.2.1, $\phi \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and

$$\phi \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \cap H^2\left(d\|E_z x\|^2;\sigma(N)\right) = H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$

and hence $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is weak-*-closed.

□

Proposition 4.2.3

Let $\phi \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and let $f \in L^1\left(d\|E_z x\|^2;\sigma(N)\right)$. Then, for each α such that $0 < \alpha < 2$

$$\frac{|f|}{|\lambda - \phi|^{2-\alpha}} \in L^1\left(d\|E_z x\|^2;\sigma(N)\right) \text{ for m-a.e. } \lambda \in \mathbb{C}$$

where m denotes the planar Lebesgue measure. In the case where $\alpha \geq 2$, the above assertion is clear since $\dfrac{1}{|\lambda - \phi|^{2-\alpha}} \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$.

Proof:

Let $B_R(0) \supseteq \sigma(N) \cup \{\textit{the essential range of } \phi\}$ where

$B_r(\lambda) = \{z \in \mathbb{C} : |z - \lambda| < r\}, r > 0$.

Then, for any $\lambda \in B_{3R}(0)$

$$|\phi(z) - \lambda| < 4R \quad \textit{for } d\|E_z x\|^2 - a.e.\ z$$

and hence

$$\int_{\sigma(N)} |f(z)| \left\{ \int_{B_{3R}(0)} \frac{1}{|\lambda - \phi(z)|^{2-\alpha}} dm(\lambda) \right\} d\|E_z x\|^2$$

$$\leq \int_{\sigma(N)} |f(z)| \left\{ \int_{B_{4R}(\phi(z))} \frac{1}{|\lambda - \phi(z)|^{2-\alpha}} dm(\lambda) \right\} d\|E_z x\|^2$$

$$\leq \int_{\sigma(N)} |f(z)| \left\{ \int_0^{2\pi} d\theta \int_0^{4R} \frac{rdr}{r^{2-\alpha}} d\|E_z x\|^2 \right\} \text{(we set here } \lambda - \phi(z) = re^{i\theta}\text{)}$$

$$= 2\pi \int_{\sigma(N)} |f(z)| \left\{ \int_0^{4R} r^{\alpha-1} dr \right\} d\|E_z x\|^2 = 2\pi \frac{(4R)^\alpha}{\alpha} \int_{\sigma(N)} |f(z)| d\|E_z x\|^2 < \infty,$$

since $f \in L^1\left(d\|E_z x\|^2; \sigma(N)\right)$. Hence, by Fubini's theorem [12]

$$\int_{B_{3R}(0)} \left\{ \int_{\sigma(N)} \frac{|f(z)|}{|\lambda - \phi(z)|^{2-\alpha}} d\|E_z x\|^2 \right\} dm(\lambda)$$

$$= \int_{\sigma(N)} |f(z)| \left\{ \int_{B_{3R}(0)} \frac{1}{|\lambda - \phi(z)|^{2-\alpha}} dm(\lambda) \right\} d\|E_z x\|^2 < \infty \qquad (4.9).$$

Therefore

$$\int_{\sigma(N)} \frac{|f(z)|}{|\lambda - \phi(z)|^{2-\alpha}} d\|E_z x\|^2 < \infty \text{ for } m - a.e.\ \lambda \in B_{3R}(0)$$

and

$$\frac{|f(z)|}{|\lambda - \phi(z)|^{2-\alpha}} \in L^1\left(d\|E_z x\|^2; \sigma(N)\right) \text{ for } m - a.e.\ \lambda \in B_{3R}(0).$$

If $\lambda \notin B_R(0)$, then $\lambda - \phi(z)$ is invertible and

$$\frac{1}{|\lambda - \phi(z)|^{2-\alpha}} \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$

and hence

$$\frac{|f(z)|}{|\lambda - \phi(z)|^{2-\alpha}} \in L^1\left(d\|E_z x\|^2;\sigma(N)\right).$$

Therefore

$$\frac{|f(z)|}{|\lambda - \phi(z)|^{2-\alpha}} \in L^1\left(d\|E_z x\|^2;\sigma(N)\right) \text{ for } m-a.e. \quad \lambda \in \mathbb{C}$$

□

Proposition 4.2.4

For $\phi \in L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and $f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ let

$$\int_{\sigma(N)} \frac{f(z)}{\lambda - \phi(z)} d\|E_z x\|^2 = 0 \text{ for } m-a.e. \ \lambda \in \mathbb{C}.$$

Then

$$\int_{\sigma(N)} f(z)\left\{4R^2 - |\phi(z) - \omega|^2\right\} d\|E_z x\|^2 = 0 \text{ for all } \omega \in B_R(0),$$

where $B_R(0) \supseteq \sigma(N) \cup \{\text{the essential range of } \phi\}$, and hence $f \perp \phi$ and $f \perp \overline{\phi}$

Proof:

For any fixed $\omega \in B_R(0)$, we have

$$\int_{B_{2R}(\omega)} \left\{ \int_{\sigma(N)} |\lambda - \omega| \frac{|f(z)|}{|\lambda - \phi(z)|} d\|E_z x\|^2 \right\} dm(\lambda)$$

$$\leq 2R \int_{B_{2R}(\omega)} \left\{ \int_{\sigma(N)} \frac{|f(z)|}{|\lambda - \phi(z)|} d\|E_z x\|^2 \right\} dm(\lambda) \quad (\text{since } |\lambda - \omega| \leq 2R)$$

$$\leq 2R \int_{B_{3R}(0)} \left\{ \int_{\sigma(N)} \frac{|f(z)|}{|\lambda - \phi(z)|} d\|E_z x\|^2 \right\} dm(\lambda) < \infty$$

by (4.9) in the proof of proposition 4.2.3 (since $|\lambda-\omega|\le 2R$ *and* $|\omega|<R$ *imply* $|\lambda|\le 3R$, because we have only to set $\alpha=1$). Hence, by Tonelli's theorem [12] and by the assumption,

$$\int_{\sigma(N)} f(z)\left\{\int_{B_{2R}(\omega)} \frac{\lambda-\omega}{\lambda-\phi(z)}dm(\lambda)\right\}d\|E_z x\|^2$$

$$=\int_{B_{2R}(\omega)}(\lambda-\omega)\left\{\int_{\sigma(N)}\frac{f(z)}{\lambda-\phi(z)}d\|E_z x\|^2\right\}dm(\lambda)=0$$

Since

$$\int_{B_{2R}(\omega)}\frac{\lambda-\omega}{\lambda-\phi(z)}dm(\lambda)=\int_{B_{2R}(\omega)}\left\{1+\frac{\phi(z)-\omega}{\lambda-\phi(z)}\right\}dm(\lambda)$$

$$=\int_{B_{2R}(\omega)}dm(\lambda)+(\phi(z)-\omega)\int_{B_{2R}(\omega)}\frac{1}{\lambda-\phi(z)}dm(\lambda)\ (\text{here we set } \lambda-\omega=re^{i\theta})$$

$$=4\pi R^2+(\phi(z)-\omega)\int_0^{2R}\int_0^{2\pi}\frac{rdrd\theta}{re^{i\theta}+\omega-\phi(z)}$$

$$=4\pi R^2+(\phi(z)-\omega)\int_0^{2R}\frac{1}{i}\left\{\int_0^{2\pi}\frac{ie^{i\theta}d\theta}{e^{i\theta}\left(e^{i\theta}+\frac{\omega-\phi(z)}{r}\right)}\right\}dr$$

$$=4\pi R^2+(\phi(z)-\omega)\int_0^{2R}\frac{1}{i}\left\{\oint_{|\xi|=1}\frac{d\xi}{\xi\left(\xi+\frac{\omega-\phi(z)}{r}\right)}\right\}dr\ (\text{here we set } \xi=e^{i\theta})$$

$$=4\pi R^2+\int_0^{2R}ir\left\{\oint_{|\xi|=1}\left(\frac{1}{\xi}-\frac{1}{\xi+\frac{\omega-\phi(z)}{r}}\right)d\xi\right\}dr$$

$$=4\pi R^2+\int_0^{2R}ir\left\{2\pi i-\oint_{|\xi|=1}\left(\frac{d\xi}{\xi+\frac{\omega-\phi(z)}{r}}\right)\right\}dr=4\pi R^2-2\pi\int_0^{|\omega-\phi(z)|}rdr.$$

because, by Cauchy's integral formula [4]

$$\oint_{|\xi|=1} \frac{d\xi}{\xi + \frac{w-\phi(z)}{r}} = \begin{cases} 2\pi i & if \left|\frac{w-\phi(z)}{r}\right|<1 \\ 0 & if \left|\frac{w-\phi(z)}{r}\right| \geq 1 \end{cases})$$

$$= 4\pi R^2 - \pi|\omega - \phi(z)|^2 = \pi\left\{4R^2 - |\omega - \phi(z)|^2\right\},$$

we have

$$\int_{\sigma(N)} f(z)\left\{4R^2 - |\omega - \phi(z)|^2\right\} d\|E_z x\|^2 = 0 \ for\ all\ \omega \in B_R(0) \qquad (4.10).$$

If we set $\omega = 0$ in (4.10), then

$$\int_{\sigma(N)} f(z)\, d\|E_z x\|^2 = \frac{1}{4R^2} \int_{\sigma(N)} f(z)|\phi(z)|^2\, d\|E_z x\|^2$$

and then

$$\left|\int_{\sigma(N)} f(z)\, d\|E_z x\|^2\right| \leq \frac{\|\phi\|_\infty^2}{4R^2} \int_{\sigma(N)} f(z)\, d\|E_z x\|^2 \to 0 \quad as \quad R \to \infty$$

and hence

$$\int_{\sigma(N)} |f(z)|\, d\|E_z x\|^2 = 0 \qquad (4.11).$$

By (4.10) and (4.11), we have

$$-\int_{\sigma(N)} |f(z)|\phi(z)|^2\, d\|E_z x\|^2 + 2\int f(z)\,\mathrm{Re}\left(\omega\overline{\phi(z)}\right) d\|E_z x\|^2 = 0 \qquad (4.12)$$

for all $\omega \in \mathrm{B}_R(0)$. If we set $\omega = 0$ in (4.12), then

$$\int_{\sigma(N)} |f(z)|\phi(z)|^2\, d\|E_z x\|^2 = 0 \qquad (4.13).$$

By (4.12) and (4.13), we have

$$\int_{\sigma(N)} f(z)\,\mathrm{Re}\left(\omega\overline{\phi(z)}\right) d\|E_z x\|^2 = 0\ \forall\ \omega \in B_R(0) \qquad (4.14).$$

By taking suitable nonzero $\omega \in B_R(0)$ in (4.14) and taking notice of the argument of $\phi(z)$, we have

$$\int_{\sigma(N)} f(z)\operatorname{Re}(\phi(z))d\|E_z x\|^2 = 0$$

and

$$\int_{\sigma(N)} f(z)\operatorname{Im}(\phi(z))d\|E_z x\|^2 = 0 \qquad (4.15).$$

Therefore

$$\langle f,\phi\rangle = \int_{\sigma(N)} f(z)\overline{\phi(z)}d\|E_z x\|^2$$

$$= \int_{\sigma(N)} f(z)\operatorname{Re}(\phi(z))d\|E_z x\|^2 - i\int_{\sigma(N)} f(z)\operatorname{Im}(\phi(z))d\|E_z x\|^2 = 0$$

(by (4.15)) and also

$$\langle f,\overline{\phi}\rangle = \int_{\sigma(N)} f(z)\phi(z)d\|E_z x\|^2$$

$$= \int_{\sigma(N)} f(z)\operatorname{Re}(\phi(z))d\|E_z x\|^2 + i\int_{\sigma(N)} f(z)\operatorname{Im}(\phi(z))d\|E_z x\|^2 = 0$$

(by (4.15)) and hence $f \perp \phi$ and $f \perp \overline{\phi}$.

□

Proposition 4.2.5

For every $\phi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and for each

$$f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)\Theta H^2\left(d\|E_z x\|^2;\sigma(N)\right),$$

let

$$\int_{\sigma(N)} \frac{\overline{f(z)}}{\lambda-\phi(z)} d\left(\|E_z x\|^2;\sigma(N)\right) = 0 \text{ m-a.e } \lambda \in \mathbb{C}.$$

Then $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ acting on $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ is an abelian von Neumann algebra and hence there is a non-trivial $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$- invariant subspace in $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$.

Proof:

For each $f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)\Theta\, H^2 d\|E_z x\|^2;\sigma(N)$, we have $\overline{f} \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$

and by proposition 4.2.4, $f \perp \phi$ and $f \perp \overline{\phi}$ for all $\phi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and hence

$$H^2\left(d\|E_z x\|^2;\sigma(N)\right) \supset H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \cup \overline{H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)}.$$

Then

$$H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) = H^2\left(d\|E_z x\|^2;\sigma(N)\right) \cap L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$
$$\supseteq H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \cup \overline{H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)}.$$

Therefore

$$H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) = \overline{H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)} \qquad (4.16).$$

Since, by propositions 4.2.1 and 4.2.2, the algebra $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is closed in the weak operator topology on $L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ and since $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is a *-algebra by (4.16), it is a von Neumann algebra because it clearly contains the identity. Clearly $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ is abelian. For a non-constant function $\phi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$, the multiplication operator M_ϕ by $\phi(z)$ on $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$is a non-trivial normal operator and hence its non-trivial spectral subspaces are the desired $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$-invariant subspaces in $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$.

□

Proposition 4.2.6

For some $\phi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and for some

$$f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right) \Theta H^2\left(d\|E_z x\|^2;\sigma(N)\right),$$

let

$$\int_{\sigma(N)} \frac{\overline{f(z)}}{\lambda - \phi(z)} d\|E_z x\|^2 \neq 0 \text{ for m-a.e } \lambda \in \mathbb{C}$$

Then there exists $g \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$ with $g \geq 1$ such that there is a non-trivial $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$-invariant subspace M in $H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$ with $M \cap L^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \neq \{\overline{0}\}$, and there is a non-trivial $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$-invariant subspace in $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$.

Proof:

Since $f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right)$,

$$\int_{\sigma(N)} |f(z)|^{3/2} d\|E_z x\|^2 \leq \left\{ \int_{\sigma(N)} |f(z)| d\|E_z x\|^2 \right\}^{\frac{1}{2}} \left\{ \int_{\sigma(N)} |f(z)|^2 d\|E_z x\|^2 \right\} < \infty$$

and so $|f|^{3/2} \in L^1\left(d\|E_z x\|^2;\sigma(N)\right)$ and hence

$$\left|\frac{f}{\lambda-\phi}\right|^{3/2} \in L^1\left(d\|E_z x\|^2;\sigma(N)\right) \text{ for m-a.e. } \lambda \in \mathbb{C}$$

by taking $\alpha = \frac{1}{2}$ in proposition 4.2.3, i.e.

$$\int_{\sigma(N)} \left|\frac{f}{\lambda-\phi(z)}\right|^{3/2} d\|E_z x\|^2 < \infty \text{ for m-a.e. } \lambda \in \mathbb{C} \qquad (4.17).$$

Since $1 \in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$ and since $f \in L^2\left(d\|E_z x\|^2;\sigma(N)\right) \Theta H^2\left(d\|E_z x\|^2;\sigma(N)\right)$, $f \perp 1$ and $\phi(z) \neq$ constant, because in the case where $\phi(z) =$ constant, we have

$$0 \neq \int_{\sigma(N)} \overline{f(z)} d\|E_z x\|^2 = \langle 1, f \rangle \text{ by (1).}$$

By taking $\alpha = 1$ in proposition 4.2.3, we have $\left|\frac{f}{\lambda-\phi}\right| \in L^1\left(d\|E_z x\|^2;\sigma(N)\right)$ for m-a.e. $\lambda \in \mathbb{C}$. Letting $g(z) = \left(1 + \left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{1/2}$, we have $g(z) \geq 1$ and

$$g \in L^2\left(d\|E_z x\|^2;\sigma(N)\right) \text{ for m-a.e. } \lambda \in \mathbb{C} \qquad (4.18).$$

Choose $\lambda \in \mathbb{C}$ satisfying (4.16), (4.17) and (4.18) at the same time and define the linear functional F on $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ by

$$F(\psi)=\int_{\sigma(N)}\frac{\overline{f(z)}}{\lambda-\phi(z)}\psi(z)\,d\|E_z x\|^2 \text{ for } \psi\in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$$

then

$$|F(\psi)|\le\int_{\sigma(N)}\left|\frac{f(z)}{\lambda-\phi(z)}\right||\psi(z)|\,d\|E_z x\|^2$$

$$\le\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)|\psi(z)|\,d\|E_z x\|^2$$

$$\le\left\{\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{\frac{1}{2}}\cdot\left\{\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{1/2}|\psi(z)|^2\,d\|E_z x\|^2\right\}$$

(by Schwarz's inequality [4])

$$=\left\{\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{1/2}\left\{\langle\psi,\psi\rangle_g\right\}^{1/2}$$

$$=\left\{\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{\frac{1}{2}}\|\psi\|_g.$$

Since, by Minkowskii's inequality [4] and by (4.17),

$$\left\{\int_{\sigma(N)}\left(1+\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{2/3}$$

$$\le\left\{\int_{\sigma(N)}d\|E_z x\|^2\right\}^{2/3}+\left\{\int_{\sigma(N)}\left(\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{2/3}$$

$$=\|x\|^{\frac{4}{3}}+\left\{\int_{\sigma(N)}\left(\left|\frac{f(z)}{\lambda-\phi(z)}\right|\right)^{3/2}d\|E_z x\|^2\right\}^{2/3}<\infty,$$

we have

$$\left\{ \int_{\sigma(N)} \left(1 + \left| \frac{f(z)}{\lambda - \phi(z)} \right| \right)^{\frac{3}{2}} d\|E_z x\|^2 \right\} < \infty$$

and F is bounded with respect to the norm $\|\cdot\|_g$ and is a linear functional on $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ and hence it can be extended by continuity to a bounded linear functional on $H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$. Then, by the Riesz Representation theorem (proposition 2.1.8), there exists a unique $h_\lambda \in H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$ such that

$$F(\psi) = \langle \psi, h_\lambda \rangle_g .$$

Since

$$\langle 1, h_\lambda \rangle_g = F(1) = \int_{\sigma(N)} \frac{\overline{f(z)}}{\lambda - \phi(z)} d\|E_z x\|^2 \neq 0,$$

(by (4.17)) we have $h_\lambda \neq 0$. Let

$$M = L^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right) - \text{closure of } \left\{(\lambda-\phi)\psi : \psi \in H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)\right\}.$$

Then

$$\{\bar{0}\} \subset M \subseteq H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$$

because $\phi(z) \neq$ constant as mentioned above and M is $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$- invariant.

Since

$$\begin{aligned}
\langle (\lambda-\phi)\psi, h_\lambda \rangle_g &= F((\lambda-\phi)\psi) \\
&= \int_{\sigma(N)} \frac{\overline{f(z)}}{\lambda - \phi(z)} (\lambda - \phi(z))\psi(z) d\|E_z x\|^2, \\
&= \int_{\sigma(N)} \overline{f(z)}\psi(z) d\|E_z x\|^2 = \langle \psi, f \rangle = 0,
\end{aligned}$$

we have $\{\bar{0}\} \subset M \subseteq H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$, and $\bar{0} \neq \lambda - \phi \in M \cap L^\infty\left(d\|E_z x\|^2;\sigma(N)\right)$ implies $M \cap L^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \neq \{\bar{0}\}$. Next, if $1 \in M$, then $H^\infty\left(d\|E_z x\|^2;\sigma(N)\right) \subseteq M$

because M is $H^{\infty}\left(d\|E_z x\|^2;\sigma(N)\right)$- invariant and which contradicts $M\subset H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$, and hence $1\notin M$. Since $1\in H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$ because $H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$ is the $L^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)$ - closure of $H^{\infty}\left(d\|E_z x\|^2;\sigma(N)\right)$, let

$$y = \text{the projection of 1 from } H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right) \text{ onto}$$

$$H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)\Theta M$$

Then $y\neq\bar{0}$. Since $g\geq 1$,

$$H^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right)\subseteq H^2\left(d\|E_z x\|^2;\sigma(N)\right),$$

because

$$\int_{\sigma(N)}|u(z)|^2 g(z)d\|E_z x\|^2\geq\int_{\sigma(N)}|u(z)|^2 d\|E_z x\|^2$$

and $y\in H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. If $(\lambda-\phi)y=\bar{0}$, let

$$\aleph=\left\{u\in H^2\left(d\|E_z x\|^2;\sigma(N)\right);(\lambda-\phi)u=\bar{0}\right\}.$$

Then $\aleph\neq\{\bar{0}\}$ and, clearly, $\aleph$ is $H^{\infty}\left(d\|E_z x\|^2;\sigma(N)\right)$ - invariant subspace of $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. If $\aleph=H^2\left(d\|E_z x\|^2;\sigma(N)\right)$, then $\lambda-\phi=(\lambda-\phi)1=\bar{0}$, which contradicts $\phi\neq$ constant and hence $\aleph\subset H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. Therefore, $\aleph$ is the desired subspace. If $(\lambda-\phi)y\neq\bar{0}$, let

$$M=\text{the } L^2\left(g(z)d\|E_z x\|^2;\sigma(N)\right) \text{ - closure of}$$

$$\left\{\psi(\lambda-\phi)y:\psi\in H^{\infty}\left(d\|E_z x\|^2;\sigma(N)\right)\right\}$$

then, $M\neq\{\bar{0}\}$ and, clearly, M is $H^{\infty}\left(d\|E_z x\|^2;\sigma(N)\right)$ - invariant subspace of $H^2\left(d\|E_z x\|^2;\sigma(N)\right)$. Let $\frac{y}{y}=0$ if y vanishes, then

$$\frac{y}{y} \in L^{\infty}\left(d\left\|E_z x\right\|^2;\sigma(N)\right) \text{ and } \left(\frac{y}{y}\right)g \in L^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right).$$

Let $v =$ the projection of $\left(\frac{y}{y}\right)g$ from $L^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$ onto $H^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$.

Then

$$\langle y, v\rangle = \left\langle y, \left(\frac{y}{y}\right)g\right\rangle = \int_{\sigma(N)} y(z)\frac{\overline{y(z)}}{y(z)} g(z)\,d\left\|E_z x\right\|^2 = \langle y, y\rangle_g = \|y\|_g^2 \neq 0$$

and $v \neq 0$, and since

$$\begin{aligned}\langle \psi(\lambda-\phi)y, v\rangle &= \left\langle \psi(\lambda-\phi)y, \left(\frac{y}{y}\right)g\right\rangle \\ &= \int_{\sigma(N)} \psi(z)(\lambda-\phi(z))y(z)\frac{\overline{y(z)}}{y(z)} g(z)\,d\left\|E_z x\right\|^2 \\ &= \langle \psi(\lambda-\phi), y\rangle_g = 0\end{aligned}$$

for all $\psi \in H^{\infty}\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$ because $\psi(\lambda-\phi) \in M$ and because $y \in H^2\left(g(z)d\left\|E_z x\right\|^2;\sigma(N)\right)\Theta M$, we have $M \subseteq H^2\left(d\left\|E_z x\right\|^2;\sigma(N)\right)$ and M is the desired subspace.

□

Thus, if we assume that a subnormal operator T on a Hilbert space H has a cyclic vector x, then propositions 4.1.7 and 4.1.9 hold. Therefore the cases of propositions 4.2.5 and 4.2.6 are satisfied and we see that the subnormal operator has non-trivial invariant subspace.

CHAPTER FIVE

5.1 CONCLUDING REMARKS AND RECOMMENDATIONS

The hope that non-trivial invariant subspaces always exist(i.e. that, except on a one-dimensional space, every operator is intransitive) is perhaps still alive in the hearts of some. That existence theorem would be the first step towards a detailed structure theory for operators (possibly a generalization of the theory of the Jordan form in the finite-dimensional case). Repeated failure has convinced most of those who tried that the truth lies in the other direction, but the proof of that is, so far just as elusive.

There are many special classes of operators that have been proved intransitive. The problem: if an intransitive operator has an inverse, is its inverse also intransitive? (due to Douglas [21]) seems to be the simplest problem of its kind (derive the intransitivity of something from that of something else). For the strong kind of invariance (that is, for reduction) the problem is easy: if a reducible operator has an inverse then the inverse is also reducible. Indeed more is true ; if a subspace M reduces an invertible operator A, then the same M reduces A^{-1}, (if a projection commutes with A then it commutes with A^{-1}). For plain invariance then more is false; the invariance of a subspace under an invertible operator A does not imply its invariance under A^{-1} (consider the bilateral shift).

A related question concerns square roots instead of inverses. Squaring preserves invariance (and reduction); what about the formation of square roots? If A^2 is reducible, is A reducible? The answer is no (consider weighted shits). If A^2 is intransitive, is A intransitive? The answer is not known. Equally unknown is the answer to a mixed question (proposed by Pearcy [19]): if A^2 is reducible, is A at least intransitive? The only thing along these lines that can be said has an undesirably special hypothesis that is; If A^2 is normal, then A is intransitive [19].

If even the relatively elementary algebraic questions are unanswered, there is no immediate hope for the much deeper questions of perturbation (in imitation of the theory of spectral perturbation) but they deserve to be put on record. What is known is that if A and B are Hermitian and B has rank 1, then $A+iB$ is intransitive. The presently hoped for generalization is obtained by replacing "rank 1" by "compact". Although large classes of compact operators are admissible here [24], not all are known to be. The mildest unproved perturbation statement appears to be this: if A is normal and B has rank 1, then $A+B$ is intransitive. The generalization of this perturbation problem to arbitrary intransitive operators in place of normal ones in not going to be easy to settle. Indeed: if A is an arbitrary operator, and if P is a projection of rank 1, then $A = AP + A(I-P)$: the first summand has rank 1, and the second has a non-trivial kernel. In chapter three we obtained a refinement of Lomonosov's theorem (proposition 3.1.5) and then used this refinement to show that; if X is a Banach space and $T \in B(X)$ and there be a non-zero compact operator K such that $C = TK - KT$ is of rank 1, then T has a non-trivial invariant subspace.

If $T \in B(H)$, H is a Hilbert space and T is k-paranormal then the set of all maximal vectors for T is an invariant subspace (proposition 3.1.9). Using this result, we obtained the result; if T is an operator on a Hilbert space H and T is k-paranormal and has maximal vectors, and then T has invariant subspaces (corollary 3.1.10). For the strong kind of invariance (that is, for reduction) and as concerns irreducible operators we have that; every operator $T \in B(H)$ is the sum of two irreducible operators (proposition 3.2.4) and the set of all irreducible operators in $B(H)$ is dense in $B(H)$ (proposition 3.2.5). Since if T is a completely reducible operator on a finite dimensional space, then T is normal (proposition 3.2.7), it follows that: if $T \in B(H)$ is a hyponormal operator and M is an invariant subspace of T such that $S = TP$ (Where P is the orthogonal projector of H and M) is a normal operator, then M reduces T (proposition 3.2.8). Using these results we showed that; if $T \in B(H)$ is normal and has one of the following properties;

1. T is compact
2. $\operatorname{Re} T$ or $\operatorname{Im} T$ is a compact operator,

then every invariant subspace of T is reducing (proposition 3.2.9). Finally in chapter three, since if T is polynomially compact, then T has a non-trivial invariant subspace, using invariant subspaces we obtained the structure of polynomially compact operators on Banach spaces.

Replacement of the algebraic condition of normality by the analytic condition of compactness yields a famous and useful intransitivity theorem: every compact operator (on a space of dimension greater than 1, of course) has a non-trivial invariant subspace [2]. Do two commutative compact operators always have a non-trivial invariant subspace in common? The answer is not known.

For normal operators the spectral theorem yields many invariant subspaces. The step from normal to subnormal is, however, large: it is not known whether every subnormal operator is intransitive. In chapter four we considered subnormal operators. In order to show the existence of non-trivial closed invariant subspaces of a subnormal operator $T \in B(H)$ we assumed that T has a cyclic vector $x \in H$ because otherwise $\vee\{T^n x : n = 0,1,2,\cdots\}$ is clearly desired invariant subspace. In this case the extension space K is simplified and we can use the spectral decomposition of the normal extension of T. Thus, if we assume that a subnormal operator T on a Hilbert space H has a cyclic vector x, then propositions 4.1.7 and 4.1.9 hold. Therefore the cases of propositions 4.2.5 and 4.2.6 are satisfied and we see that the subnormal operator has non-trivial invariant subspaces.

Two more invariant subspace results deserve mention; they are elementary, but they are sufficiently different from the preceding ones and from each other that they might suggest a new idea to someone. Firstly the strictly algebraic version of the invariant subspace problem has a positive solution that is; *every linear transformation (on a vector space of dimension greater than 1) has a non-trivial invariant linear manifold* [23]. Secondly we have; *for every operator A there exists a hyperplane M (subspace of co-dimension 1) such that the compression of A to M has an eigenvector. (The compression of A to M is the operator B on M defined as follows: if P is the projection with range M, then $Bf = PAf$ for each f in M). In fact more is true: for each non-zero vector f, there exists a hyperplane M containing f such that the compression of A to M has f as an eigenvector*. The proof is due to Wallen [1]. The result was suggested by the following Apolstol's theorem; *if A is an operator such that 0 is in the spectrum of A but $\mathrm{Ker}A = \mathrm{Ker}A^* = 0$, then there exists an infinite dimension subspace M, such that the compression of A to M is compact* [1].

All the results reported so far concern intransitivity; they assert that under certain conditions subspaces are invariant. The last two results to be mentioned in this context go in the direction of transitivity; certain subspaces are not invariant. Firstly; *for each countable set of non-trivial subspaces, there exists an operator that leaves none of them invariant* [34]. Secondly; *there exists a linear transformation on a Hilbert space H that leaves no non-trivial subspace of H invariant* (warning: this does not pretend to be a solution of the invariant subspace problem. "Subspace" here means closed linear manifold, as always, but "linear transformation" does not mean operator i.e., the linear transformation whose existence is asserted is not necessarily bounded. There is however no fudging about domains: the assertion is about linear transformations that act on the entire space H). The result is due to Shields [25].

In this thesis we have thus solved the invariant subspace problem in the affirmative for the case of a commutator operator, a k-paranormal operator and a subnormal operator. For the strong kind of invariance i.e. reduction we have proved that if a normal operator has one of the following properties; it is compact or its real part or imaginary part is a compact operator, then every invariant subspace of the operator is reducing. Finally if an operator is polynomially compact, then it has a non-trivial invariant subspaces, using invariant subspaces we have obtained the structure of polynomially compact operators on Banach spaces. Our idea in this thesis was to capture the attention of those who may not be well versed in functional analysis and operator theory but are interested in knowing the present ramifications of the invariant subspace problem without taxing much on their indulgence. At the same time we have provided an overview of the subject which even a specialist would relish. The results we have proved in this thesis give more insight into the invariant subspace problem and contribute immensely towards finding an affirmative answer to the problem. That existence theorem would be the first step towards a detailed structure theory for operators.

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