ON SCHWARZ NORMS

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Abstract

Investigation of the properties of the numerical radius by Berger and Stampfli showed that indeed numerical radius norm is a Schwarz norm.Later on James P.Williams determined a family of distinct Schwarz norms by slightly modifying the Berger-Stampfli argument. In this thesis we have proved that by slight modification of the S_c class constructed by Williams ,we can obtain a class S_Q of Schwarz norms , for a positive hermitian operator Q where $Q = cI(c \geq 1)$.We have also determined the scope of the new class of Schwarz norms constructed in terms of the underlying space. Finally we have given the characterizations for the Hilbert space given a contraction;

$T \in \mathcal{B}(\mathcal{H})$, $||T|| \leq 1$

Chapter 1

Introduction

1.1 Background information

Suppose that f is an analytic function in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and is bounded ,i.e

$$\|f\|_\infty = \sup\{|f(z)| : z \in U\} < \infty.$$

If f has the following additional properties,

$$f(0) = 0, \ \|f\|_\infty < 1,$$

then the following lemma (Schwarz lemma) holds:

Lemma 1.1.1. *If f is analytic in the open unit disk as described above and,*

(i.) $|f(z)| \leq |z|$, $z \in U$.

(ii.) $|f^{'}(0)| \leq 1$,

and if the equality appears in (i) for one $z \in U - \{0\}$, *then* $f(z) = \alpha z$ *,where* α *is a complex constant with* $|\alpha| = 1$ *and also if the equality appears in (ii) ,*f *behaves similarly. In case of operators,we have that,if* $|T| \leq 1$, *then* $|f(T)| \leq \|f\|$ *for each* $f \in R(D)$ *such that* $f(0) = 0$*.Here* $R(D)$ *is the (sup-norm) algebra of the rational functions with no poles in the closed unit disk* D *and* $f(T)$ *defined by the usual Cauchy integral around a circle slightly larger than the unit circle.[5]*

We note here that a contraction (i.e an operator T such that $\|T\| < 1$) $T \in \mathcal{B}(H)$ has some relation with the closed unit disk of the complex plane, say for any contraction T and any complex-valued function $f(z)$ defined and analytic on the closed unit disk ,then by von Neumann [9],[11] the norm equality holds;

$$\|f(T)\| \leq \|f\|_\infty \equiv max_{|z|\leq 1}|f(z)|$$

where the operator $f(T)$ is defined by the usual functional calculus[10]. The above lemma has an interesting application in the theory of operators namely the following assertions hold :if f is analytic in the open unit disk and

$$f(0) = 0 \text{ with } \|f\|_\infty < 1,$$

then for any operator

$$T \in \mathcal{B}(\mathcal{H}), \|T\| < 1,$$

(Berger and Stampfli) [2] we have

$$\|f(T)\| < \|T\|.$$

Clearly if we have an equality for some T, then f is of the form

$$f(z) = \alpha z.$$

where α is a complex constant with $|\alpha| = 1$

A norm , say , $||.||^*$ on the algebra $\mathcal{B}(H)$ of all bounded operators T, is called a ***Schwarz*** norm if it is equivalent to the usual norm $||.||$ and the Schwarz lemma holds for it,i.e for any f analytic in the open unit disc U with $f(0) = 0$ and

$$||f||_\infty < 1,$$

and for any

$$T \in \mathcal{B}(H), ||T|| < 1 ,$$

we have

$$||T||^* < 1, ||f(T)||^* < 1.$$

1.2 Basic Concepts

We will in this section give the definitions that will be essential in our study. In the following $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Definition 1.2.1. For a set of points X ,the pair $(X, \mathbb{K})$ is called a linear space if for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$ then
$\alpha x + \beta y \in X$

In case $\mathbb{K} = \mathbb{R}$ then the pair is referred to as real linear space but if $\mathbb{K} = \mathbb{C}$ then it is a complex linear space.

Definition 1.2.2. Let $(X, \mathbb{K})$ be a linear space as defined above.A mapping $||.|| : X \mapsto \mathbb{R}$ is called a norm on X if it satisfies the following properties (norm axioms);

(i) $||x|| \geq 0$ for all $x \in X$ (non-negativity)

(ii) If $x \in X$ and $||x|| = 0$, then $x = \overline{0}$ (zero axiom)

(iii) $||\lambda x|| = |\lambda| ||x||$ for all $x \in X$ and $y \in \mathbb{K}$ (homogenity)

(iv) $||x + y|| \leq ||x|| + ||y||$ $\forall x.y \in X$ (triangular inequality)

The ordered pair $(X, ||.||)$ is called a normed linear space (n.l.s) over $\mathbb{K}$

Definition 1.2.3. Suppose property number (ii) (zero axiom) in the above definition fails , i.e if $x \in X$ and

$$||x|| = 0 \nRightarrow x = \overline{0}$$

then the function ,

$$\|.\| : X \mapsto \mathbb{R}$$

is referred to as seminorm on X.

Definition 1.2.4. Let $(X, \mathbb{K})$ be a linear space and $\|.\|_1, \|.\|_2$ be norms on X we say that

$$\|.\|_1 \text{ and } \|.\|_2$$

are equivalent if $\exists$ positive reals α,β such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \; \forall x \in X$$

The two norms generate the same open sets (same topology)

Definition 1.2.5. A sequence (x_n) is said to converge strongly in a normed linear space $(X, \|.\|)$ if $\exists x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

Definition 1.2.6. Let $(X, \|.\|)$ be a normed linear space and ρ be the metric induced by $\|.\|$.If (X, ρ) is a complete metric , then we call $(X, \|.\|)$ a Banach space or strongly complete normed linear space.
(A normed linear space $(X, \|.\|)$ is a Banach space if every strong Cauchy sequence of elements of X converges strongly in X)

Definition 1.2.7. Let $(X, \mathbb{K})$ be a linear space. If M is a subset of X such that $x, y \in M$ and

$$\alpha, \beta \in \mathbb{K} \Rightarrow \alpha x + \beta y \in M$$

then M is called a subspace of X

Definition 1.2.8. Let X be a linear space over $\mathbb{K}$ and $\langle , \rangle : X \mapsto \mathbb{K}$ be a function with,

(i) $\langle x, x\rangle \geq 0 \ \forall x \in X$

(ii) $\langle x, x\rangle = 0 \Rightarrow x = \overline{0}$

(iii) $\langle y, x\rangle = \overline{\langle x, y\rangle} or \langle x, y\rangle$ if $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ respectively for all $x, y \in X$. where $\overline{\langle x, y\rangle}$ denotes the conjugate of the complex number $\langle x, y\rangle$.

(iv) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in X$ and all $\lambda \in \mathbb{K}$.

(v) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$

The function $\langle . \rangle$ is called inner-product (i.p) function and the real or complex number

$$\langle x, y\rangle$$

is called the inner product of x and y (in this order). The ordered pair $(X, \langle . \rangle)$ is called an inner product space or pre-Hilbert space over $\mathbb{K}$. Let $(X, \langle . \rangle)$ be an inner-product space.The norm in X is given by

$$||x|| = \sqrt{\langle x, x\rangle}$$

for all $x \in X$ and is called the norm determined by (or induced by) the inner-product function of x. The metric ρ determined by this norm $||.||$ as defined above is

$$\rho(x, y) = ||x - y||$$

for all $x, y \in X$ is called the metric induced by the inner-product function $\langle . \rangle$. If with respect to this norm $\|x\|$,defined above,$(X, \|.\|)$ is strongly complete i.e $(X, \|.\|)$ is a Banach space,then we refer to $(X, \langle . \rangle)$ as a Hilbert space i.e a Hilbert space is a complete inner-product space.

Definition 1.2.9. Let $\mathcal{H}$ be a complex Hilbert space and T be a linear operator from $\mathcal{H}$ to $\mathcal{H}$. T is said to be positive if

$$\langle Tx, x \rangle \geq 0$$

for all $x \in \mathcal{H}$. This can be denoted by

$$T \geq 0 \text{ or } 0 \leq T.$$

T is said to be strictly positive or positive definite if

$$\langle Tx, x \rangle > 0$$

for all

$$x \in \mathcal{H} \setminus \{\overline{0}\}.$$

Definition 1.2.10. If $T \in \mathcal{B}(\mathcal{H})$, then the operator

$$T^* : \mathcal{H} \rightarrow \mathcal{H}$$

defined by

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

$\forall x, y \in \mathcal{H}$ is called the adjoint of T .

(T^* is also in $\mathcal{B}(H)$ and

$$||T^*|| = ||T||$$

Definition 1.2.11. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if

$$T^* = T$$

and if T is linear on a linear subspace M of a Hilbert space $\mathcal{H}$ into $\mathcal{M}$ then it is said to be Hermitian if in addition

$$\langle Tx, y\rangle = \langle x, Ty\rangle \ \forall x, y \in M.$$

Definition 1.2.12. Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$.Then there exists unique self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$T = A + iB$$

A and B are given by

$$A = \tfrac{1}{2}(T + T^*)\ ,\ B = \tfrac{1}{2i}(T - T^*)$$

so that A is called real part of T denoted by ReT and B the imaginary part of T denoted by ImT. Note that

$$\mathrm{Re}\langle Tx, x\rangle = \langle (ReT)x, x\rangle$$

for every $x \in \mathcal{H}$. Indeed

$$\langle Tx, x\rangle = \tfrac{1}{2}\langle (T + T^*)x, x\rangle + i\tfrac{1}{2}\langle (\tfrac{T-T^*}{2})x, x\rangle$$

and

$$\langle Tx, x\rangle$$

being a complex number we have

$$\langle Tx, x\rangle = a + ib,$$

where a, b are real numbers given by

$$a = \langle (ReT)x, x\rangle, b = \langle (ImT)x, x\rangle$$

Definition 1.2.13. Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$.The numerical range of T is the set

$$W(T) \subset \mathbb{C}$$

defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

Definition 1.2.14. The numerical radius $w(T)$ of an operator $T \in \mathcal{B}(H)$ is the number defined by the relation

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

Definition 1.2.15. Let X,Y be normed linear spaces over $\mathbb{K}$ and $T : X \to Y$ be a linear transformation,then T is said to be compact if for every bounded subset M of X,the image $\overline{T(M)}$ (strong closure of $T(M)$ in X) is compact or equivalently,if X,Y be normed linear spaces over $\mathbb{K}$ and $T : X \to Y$ be a linear transformation,then T is said to be compact if and only if for every bounded sequence (x_n) of elements of X ,the sequence $(T(x_n))$ has a subsequence which converges strongly in Y.The set $K(X, Y)$ of all compact linear operators $T : X \to Y$ is a linear subspace of $B(X, Y)$ which is a set of all bounded linear operators
$T : X \to Y$.

Definition 1.2.16. A Banach algebra $\mathcal{B}$ is a Banach space ($\mathcal{B}$,$||.||$) in which for every $x, y \in \mathcal{B}$ is defined a product $xy \in \mathcal{B}$ such that

(i) $(\lambda x)y = \lambda(xy) = x(\lambda y)\ forall\ \lambda \in \mathbb{K}$

(ii) $(x+y)z = xz + yz\ forall\ x, y, z \in \mathcal{B}$

(iii) $x(y+z) = xy + xz\ forall\ x, y, z \in \mathcal{B}$

(iv) $||xy|| \leq ||x||||y||\ forall\ x, y, z \in \mathcal{B}$

Definition 1.2.17. Suppose $\mathcal{A}$ is an arbitrary Banach algebra (commutative or not),a mapping $*$:$\mathcal{A} \rightarrow \mathcal{A}$ is called an involution of $\mathcal{A}$ or $\mathcal{A}$ is called an involutive Banach algebra if;

1. $(x+y)^* = x^* + y^*$
2. $(\lambda x)^* = \overline{\lambda} x^*\ \lambda \in \mathbb{C}$
3. $(xy)^* = y^* x^*$
4. $(x^*)^* = x\ forall\ x, y \in \mathcal{A}$

An involutive Banach algebra $\mathcal{A}$ is called a B^*-algebra if

$$||x^* x|| = ||x||^2\ forall$$

$x \in \mathcal{A}$

Definition 1.2.18. Let X be a linear space over $\mathbb{K}$ and M be a linear subspace of X.For each $x \in X$,we define

$$x + M = \{x + y : y \in M\},$$

and if $x, x' \in X$ then

$$x + M = x' + M$$

if and only if

$$x - x' \in M$$

(In this case we write $x \sim x'$ and the relation $\sim$ is an equivalence relation)

Let X/M or $X/\sim$ be the set of all equivalence classes; then if we define

(i) $(x + M) + (y + M) = x + y + M$

(ii) $\alpha(x + M) = \alpha x + M$

$x \in X$,$\alpha \in \mathbb{K}$.The sum + and scalar . are well defined and

$$(X/M, +, .)$$

is a linear space over $\mathbb{K}$,called Quotient space of X modulo M and is denoted by X/M

Definition 1.2.19. Let $(X, ||.||)$ be a normed linear space and M be a closed linear subspace of X. For each element$x + M$ in X/M,define a function:

$$|||x + M||| = \inf\{||x + y|| : y \in M\} = dis(x, M)$$

then $|||.|||$ is a norm in X/M, i.e

$$(X/M, |||.|||)$$

is a normed linear space.It is known that $(X/M, |||.|||)$ is a Banach space if $(X, ||.||)$ is a Banach space.

If M is not closed,then

$$|||x + M||| = 0 \nRightarrow x \in M$$

$$\therefore x + M \neq M,$$

the zero element of X/M.Therefore $|||.|||$ is a seminorm.

Definition 1.2.20. Suppose X in the above definition is $\mathcal{B}(H)$; i.e the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(H)$ the set of all compact operators on $\mathcal{H}$ which is norm closed in $\mathcal{B}(H)$.Then

$$\mathcal{B}(H)/\mathcal{K}(H) = \{T + \mathcal{K}(H) : T \in \mathcal{B}(H)\}$$

is called a Calkin algebra.

For each $T \in \mathcal{K}(H)$,there corresponds a unique in

$$\widehat{T}$$

in $\mathcal{B}(H)/\mathcal{K}(H)$ and this correspondence given by

$$T \mapsto \widehat{T}$$

and can also be given by

$$T \longmapsto (T + \mathcal{K}(H)) = \widehat{T}$$

Definition 1.2.21. For$T \in \mathcal{B}(X)$ where X is a Banach space .We define

$$e^T = I + T + \tfrac{T^2}{2!} + \tfrac{T^3}{3!} + ...$$

where the right hand side converges in the norm of $\mathcal{B}(X)$,for

$$||I|| + ||T|| + \tfrac{1}{2!}||T||^2 + ...$$

converges for real $||T||$ and

$$||I + T + \tfrac{1}{2}T^2 + ... + \tfrac{1}{n!}T^n \leq ||I|| + ||T|| + ||\tfrac{1}{2!}T^2|| + ... + ||\tfrac{1}{n!}T^n|| \leq$$
$$I + ||T|| + \tfrac{1}{2!}||T||^2 + ... + \tfrac{1}{n!}||T||^n$$

$\forall n \in \mathbb{N}$

If $T \in \mathcal{B}(X)$ then T is called Hermitian if

$$||e^{iT}|| = 1$$

Theorem 1.2.22. *If M is a linear subspace of a n.l.s X (real or complex) and f is a bounded linear functional on M , then f can be extended to a bounded linear functional F on X so that $\|F\| = \|f\|$*

We will state an important consequence of the above theorem.

Let X be a normed linear space over $\mathbb{K}$ and let M be a proper linear subspace of X and let x_o be a point in $X - M$ such that $d = dist(x_o, M) > 0$.Then there exists a bounded linear functional f on X such that

$$f(x) = 0 \text{ for all } x \in M$$

$$f(x_o) = d \text{ and } \|f\| = 1$$

1.3 Statement of the problem

In his work on Schwarz norms Williams [1] obtained a family

$$\{\|.\|_c : c \geq 1\}$$

$$\|T\|_c := inf\{\lambda : T \in \lambda S_c\}$$

of norms on $\mathcal{B}(\mathcal{H})$ and S_c is defined in Definition 2.0.5 , by slightly modifying the Berger-Stampfli argument [2].Now this family of Schwarz norms does not include all Schwarz norms on $\mathcal{B}(\mathcal{H})$,as remarked in [1].This suggests that the class of all Schwarz norms on $\mathcal{B}(\mathcal{H})$ is larger than S_c

1.4 Objectives of the study

The objectives of the study are:To

1. Construct new Schwarz norms
2. Characterise the new Schwarz norms
3. Determine the scope of the newly constructed norms

1.5 Significance of the study

This work on Schwarz norms is bound to expose other properties of contractions and spectral sets more so in the Harmonic Analysis of operators.

Chapter 2

Literature review

As defined in the background information above,a norm $||.||^*$ on $\mathcal{B}(\mathcal{H})$ which is equivalent to the operator norm $||.||$ is called a ***Schwarz*** norm if $||T|| \leq 1$ implies

$$||f(T)|| \leq ||f||_\infty \equiv max_{|z|\leq 1}|f(z)|.............(*)$$

for any analytic function f with

$$f(0) = 0 \text{ and } ||f||_\infty < 1$$

Von Neumann [11] first showed that if

$$T \in \mathcal{B}(\mathcal{H})$$

then the usual operator norm

$$||T|| = \sup\{\langle Tx, x\rangle : x \in \mathcal{H}, ||x|| = 1\}$$

is a Schwarz norm using the spectral representation of a unitary operator U i.e

$$f(U) = \int_0^{2\pi} f(e^{i\theta})dE(\theta)$$

generates a norm

$$\|f(U)x\|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 dE\|(\theta)\|^2$$

where $E(\theta)$ is a positive spectral measure of U

The inequality (*) above then follow from this norm.

Now the numerical radius of an operator

$$T \in \mathcal{B}(\mathcal{H})$$

is defined as

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

where $W(T)$ is the numerical range of T, i.e the set

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Berger and Stampfli [2] proved that the numerical radius $w(T)$ is a Schwarz norm using the theory of unitary dilations i.e

$$w(T) \leq 1$$

if and only if there is a unitary operator U on $\mathcal{K} \supset \mathcal{H}$ such that

$$T^n = 2PU^n/\mathcal{H} \text{ (n=1,2,...)}$$

Nagy and Foias [3] and later others papers improved on this to obtain the ρ-radius,$w_\rho(T)$ of an operator as

$$w_\rho(T) \equiv \inf\{\lambda > 0; \tfrac{1}{\lambda}T \in \mathcal{C}_\rho\}$$

where C_ρ is the class of operators with ρ-dilations. Thus for a complex valued function $f(z)$ defined and analytic on the closed unit disk with $f(0) = 0$,if T has a ρ-dilation U,then by series expansion,

$$f(T)^n = \rho P f(U)^n/\mathcal{H} \text{ (n=1,2,...)}$$

and it can then be proved that

$$w_\rho(f(T)) \leq \|f\|_\infty$$

so that the inequality (*) is achieved.

Using the two norms $\|T\|$ and $w(T)$ (as proved by Von Neumann and Berger-Stampfli to be Schwarz norms), Williams [1] constructed a class S_c of operators which he used to build a family of Schwarz norms.

Proposition 2.0.1. *If $T \in \mathcal{B}(\mathcal{H})$,then the following assertions hold:*

1. $\|T\| < 1$ *if and only if* $Re(I + zT)(I - zT)^{-1} \geq 0$ *for all z satisfying* $|z| < 1$,

2. $w(T) \leq 1$ *if and only if* $Re(I - zT)^{-1} \geq 0$ *for all z satisfying* $|z| < 1$

For the proof of this proposition 2.0.1,see [1]

From the form of the operators used for the characterization of the operators T for which $\|T\| \leq 1$ or $w(T) \leq 1$,we see that they are of the form

$$I + c\sum_{n=1}^{\infty} z^n T^n$$

and the conditions refer to such operators,indeed by Bonsall[6],[7] we have that if $\|T\| < 1$ and $|z| < 1$ then

$$(I - zT)^{-1} = I + \sum z^n T^n$$

i.e $c = 1$ whereas

$$(I + zT)(I - zT)^{-1}$$

$$= (I + zT)(I + \sum_{n=1}^{\infty} z^n T^n$$

$$= I + 2\sum z^n T^n.$$

where $c = 2$

(Convergence of the right hand side with respect to the norm of $B(\mathcal{H})$).

The following definition introduces the class of operators which plays a fundamental role in the construction of Schwarz norm.

Both

$$\|T\| \leq 1 \text{ and } w(T) \leq 1 \text{ imply that } \sigma(T) \subset U$$

while both

$$(I + zT)(I - zT)^{-1} \geq 0 \text{ and } (I - zT)^{-1} \geq 0 \text{ imply } Re(I + c\sum z^n T^n) \geq 0.$$

Definition 2.0.2. The S_c class of operators is the set of all operators $T \in \mathcal{B}(\mathcal{H})$ for which the following properties hold:

1. $\sigma(T) \subset U$
2. $Re(I + c\sum z^n T^n) \geq 0.$

where U is the open unit disk of the complex plane.

In this definition c is a positive number.From the definition and the proposition 1 we obtain the following results,

1. $\|T\| \leq 1$ if and only if $T \in \mathcal{S}_2$

2. $w(T) \leq 1$ if and only if $T \in \mathcal{S}_1$.

The following two propositions by Williams [1] and proved by Berger and Stampfli argument [8] ,[10] , gives information about the functional calculus (polynomial functional calculus) with operators in the S_c class.

Proposition 2.0.3. *If $T \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{S}_c$ then for any rational functional with no poles in the closed unit disk and with properties $f(0) = 0, \|f\|_\infty < 1$ we have $f(T) \in S_c$.*

To obtain Schwarz norms from these classes of operators we need more information about these classes.The most important is that S_c is a convex set for any $c > 1$

Proposition 2.0.4. *For the classes S_c , $c > 1$,of operators the following properties hold:*

(i.) $S_c = S_c^* = \{T^* : T \in S_c\}$

(ii.) $S_{c_1} \subset S_{c_2}$ *if* $c_2 < c_1$

(iii.) S_c *is a convex set if* $c \geq 1$

(iv.) For $c > 1$,$T \in S_c$ *if and only if* $(c-1)\|T\|^2 + |2-c|\|\langle Tx, x\rangle\| \leq \|x\|^2$ *for all* $x \in \mathcal{H}$.

$|2-c||\langle Tx, x\rangle| + (c-1)\|Tx\|^2$ over $|z| < 1$

By Williams [1] we next show that classes S_c are nonvoid and are strictly decreasing .For this consider the following example

Example 2.0.5. .For any $\lambda > 0$,we take the operator λA where A is the operator on a two dimensional space ℓ_2^2 with the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and we remark that λA is in S_c if and only if if

$$0 < Re(I + c\sum \lambda^n z^n A^n) = Re(I + c\lambda z A)$$

since $A^n = 0 \; n \geq 2$.

Hence the matrix of $Re(I + cz\lambda A)$ is

$$\begin{bmatrix} 1 & (\frac{c\lambda z}{2})^* \\ (\frac{c\lambda z}{2}) & 1 \end{bmatrix}$$

and consequently the spectrum of $(I + cz\lambda A)$ for all $|z| < 1$ is the set

$$\{1 + \tfrac{1}{2}|c\lambda z|, 1 - \tfrac{1}{2}|c\lambda z\}$$

and thus $\lambda A \in S_c$ if and only if$c\lambda \leq 2$.

Since the spectrum of $Re(I + cz\lambda A)$ is the set

$$\{1 + \tfrac{1}{2}|c\lambda z|, 1 - \tfrac{1}{2}|c\lambda z|\}$$

(where $|z| < 1$) it follows that

$$Re(1 + cz\lambda A) = I + Recz\lambda A$$

and by the spectral mapping theorem we have

$$\sigma(Rec\lambda z A) = \{-\tfrac{1}{2}|c\lambda z|, \tfrac{1}{2}|c\lambda z|\}$$

which is contained in U if and only if $c\lambda \leq 2$. From this we have that $\frac{2}{c}A \in S_c$. Hence if $c_1 > c_2$,we have

$$\tfrac{2}{c_2}A \in S_{c_2},$$

but $\frac{2}{c_2}A$ is not a member of S_c

(Note:$\frac{2}{c_2}c_1 > 2$).

Thus $S_{c_2} \not\supseteq S_{c_1}$.

The above example can be used to show that for $0 < c < 1$,S_c is not convex.For suppose that S_c is convex,then by property (i) we have that

$$\frac{1}{2}\{\frac{2}{c}A + \frac{2}{c}A^*\} \in S_c$$

and since this is equivalent to $\frac{2}{c}Re$ A which has the spectrum

$$\{-\frac{1}{c}, \frac{1}{c}\}$$

thus if $c < 1$,

$$\{-\frac{1}{c}, \frac{1}{c}\}$$

is not contained properly in U and the set S_c is not convex. The following lemma [1] summarizes the properties of the setS_c

Lemma 2.0.6. *The set S_c for $c \geq 1$ has the following properties*

(i.) S_c is bounded and closed.

(ii.) S_c is a circled convex set and is a neighborhood of zero.

The properties in this lemma permits us to define for each $c \leq 1$ a norm on $\mathcal{B}(\mathcal{H})$.

Definition 2.0.7. For any $c \geq 1$ the function on $\mathcal{B}(\mathcal{H})$ defined

$$\|T\|_c = inf\{\lambda : T \in \lambda S_c\}$$

is a norm equivalent to the usual norm $\|.\|$.

The fact that $||T||_c$ is a norm equivalent to $||.||$ follows from the properties of the S_c class indicated above.

We also note the following properties of the norm $||T||_c$ which follow directly from the above proposition.

(i.) $||T||_c = ||T^*||_c$

(ii.) If $c_1 < c_2$,then $||T||_{c_1} \leq ||T||_{c_2}$

(iii.) If $c \in [1, 2)$,$||T||_c = 1$.

Remark 2.0.8. In a paper [1],Williams express the opinion that the norm $||.||_c$ introduced above ,which are obvious Schwarz norms do not include all Schwarz norms on $\mathcal{B}(\mathcal{H})$.

Chapter 3

Results

3.1 New class of Schwarz norms

Proposition 3.1.1. *If $\|T\|_c$ is a norm and $\|\widehat{T}\|_c$ is a seminorm,then the sum is a Schwarz norm i.e taking the sum of two different Schwarz norm applied to T and to the image of T in the Calkin algebra.*

For any $c \geq 1$ we define on $B(\mathcal{H})$ the function

$$\|T\|_c^* = \|T\|_c + \|\widehat{T}\|_c$$

$\forall T \in \mathcal{B}(H)$ where $\widehat{T}$ denotes the image of T in the *Calkin* algebra and $\|\widehat{T}\|_c$ being a seminorm as indicated in definition 1.2.19.

Then

$$T \mapsto \|T\|_c^*$$

is a Schwarz norm on $B(\mathcal{H})$ and is not in the class constructed by Williams.

proof.

First we remark that we can construct a more general Schwarz norm on $B(\mathcal{H})$ by taking the sum of two different Schwarz norms applied to T and to the image of T in the *Calkin* algebra.Also since $\|T\|_c$ is a norm and $\|\widehat{T}\|_c$ is a seminorm,it follows that the sum is a Schwarz norm.

Suppose that Q is a positive hermitian operator with the property

$$0 < mI \leq Q \leq MI,$$

where

$$m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$$

$$M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$$

Then we can construct the operator $Q^{\frac{1}{2}}$ which is also positive and invertible. The following new class S_Q of operators is a generalization of the class S_c to which it reduces when $Q = cI$

Definition 3.1.2. If Q is a Hermitian operator $0 < mI < Q < MI$ then the class S_Q is the set of all operators $T \in \mathcal{B}(H)$ with the following properties

1. $\sigma(T)$ is in the unit disk.
2. $Re(I + \sum Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n) \geq 0$,For all $|z| < 1$

We can prove some results about this class as for the class S_c obtained by Williams.

Theorem 3.1.3. *If f is a rational function with no poles in the closed unit disk and $\|f\|_\infty < 1, f(0) = 0$ then for any $T \in S_Q$,*

$$f(T) \in S_Q$$

In this proof,we use the approach of Williams [1]:

Proof:

The function

$$z \mapsto \langle (1 + \sum_{n=1}^{\infty} Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n) x, x \rangle$$

is with real part positive.By the Herglotz theorem ,there exists a positive measure μ_x such that

$$\|x\|^2 + c \sum_{n=1}^{\infty} z^n \langle Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} x, x \rangle$$

$$= \int_0^{2\pi} \frac{1+ze^{it}}{1-ze^{it}} d\mu_x(t) \text{ for all } |z| < 1.$$

Now

$$\frac{1+ze^{it}}{1-ze^{it}} = (1 + ze^{it})(1 + \sum_{n=1}^{\infty} z^n e^{int})$$

$$= I + 2 \sum_{n=1}^{\infty} z^n e^{int}$$

since

$$|ze^{it}| < 1$$

by the above theorem,we have

$$c \langle Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} x, x \rangle = 2 \int_0^{2\pi} e^{int} d\mu_x(t) \text{ for } n = 1, 2, 3....$$

From these relations ,we obtain immediately that for any polynomial $p(z) = \sum a_i z^i$ and any $x \in \mathcal{H}$,

$$\langle p(Q^{\frac{1}{2}} T Q^{\frac{1}{2}}) x, x \rangle = 2 \int_0^{2\pi} p(e^{it}) d\mu_x(t)$$

and if we take $p^n(z)$,we obtain

$$\langle p^n(Q^{\frac{1}{2}} T Q^{\frac{1}{2}}) x, x \rangle$$

$$= 2 \int_0^{2\pi} p^n(e^{it}) d\mu_x(t).$$

This implies that if $\|p\|_\infty = 1$, $p^n(Q^{\frac{1}{2}}TQ^{\frac{1}{2}})$ is a bounded operator and for z ,$|z| < 1$, we obtain.

$$\langle 1 + c\sum_{n=1}^{\infty} z^n p^n(Q^{\frac{1}{2}}TQ^{\frac{1}{2}})x, x\rangle$$

$$= \|x\|^2 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} p^n(e^{it})d\mu_x(t)$$

$$= \int_0^{2\pi} \frac{1+zp(e^{it})}{1-zp(e^{it})} d\mu_x(t).$$

From this relation we obtain that $p(T) \in S_Q$ when p is a polynomial.Now if f is any functional which is rational and with no poles in the closed unit disk,then $f(T) \in S_Q$. Now this theorem shows that S_Q is a family of distinct Schwarz norms.

$$f(T) \in S_Q$$

Proposition 3.1.4. *The operator $T \in \mathcal{B}(H)$ is in S_Q if and only if :*

1. *$\sigma(T)$is in the unit disk*

2. $Re\langle(Q^{\frac{1}{2}}(I - zT)^{-1}Q^{\frac{1}{2}}x, x\rangle - \langle Qx, x\rangle + \|x\|^2 \geq 0$

Proof:

The condition,

$$Re\langle(I + \sum Q^{\frac{1}{2}}T^nQ^{\frac{1}{2}}z^{n \geq 0}) .$$

is equivalent to the following

$$Re\langle(I + \sum Q^{\frac{1}{2}}T^nQ^{\frac{1}{2}}z^n)x, x\rangle$$

$$= Re[\langle Q^{\frac{1}{2}}(I - zT)^{-1}Q^{\frac{1}{2}} - Q + I)x, x\rangle] \geq 0$$

which is our assertion.

From this characterization we obtain the following result.

Proposition 3.1.5. *If $Q \geq 1$, then $T \in S_Q$ if and only if*

1. *$\sigma(T)$ is in the unit disk*

2. *$Re\langle Q^{\frac{1}{2}}(I - zT)Q^{\frac{1}{2}}x, x\rangle \|Q^{\frac{1}{2}}x\|^2 - \|x\|^2 = \langle (Q - I)x, x\rangle$*

Proof:

This follows directly from the above proposition 3.1.4.

The following theorem gives information about the S_Q class which is similar to that given in proposition 2 for the S_c class.

Proposition 3.1.6. *If Q is a positive hermitian operator ,then the following assertions hold.*

1. *$S_Q = S_Q^* = \{T^* : T \in S_Q\}$*

2. *If $Q_1 < Q_2$ then $S_{Q_2} \subseteq S_{Q_1}$*

3. *For $Q \geq I$, S_Q is a convex bounded ,circled and weakly compact set in $(\mathcal{H})$ (it is also in the neighborhood of zero)*

Proof: Now we prove the assertion (1) above,Since $\sigma(T) \subset U$, it follows that $\sigma(T^*) \subset U$.

Indeed $\sigma(T^*) = (\sigma(T))^*$

(the star on the right side denotes the complex conjugation , i.e,

$$(\sigma(T))^* = \{z^* : z \in \sigma(T)\}.$$

Moreover ,since $|z| = |z^*| < 1$,for all $x \in \mathcal{H}$

$$\langle Q^{\frac{1}{2}}(I - zT)^{-1}Q^{\frac{1}{2}}x, x\rangle = \langle x, (Q^{\frac{1}{2}}(I - zT)^{-1}Q^{\frac{1}{2}})^{*}x\rangle$$

$$= \langle x, Q^{\frac{1}{2}}(I - z^{*}T^{*}Q^{\frac{1}{2}})^{-1}x\rangle$$

$$= \langle Q^{\frac{1}{2}}(I - z^{*}T^{*}Q^{\frac{1}{2}})^{-1}x, x\rangle$$

so

$$Re\langle Q^{\frac{1}{2}}(I - z^{*}T^{*})^{-1}Q^{\frac{1}{2}}x, x\rangle$$

$$= Re\langle Q^{\frac{1}{2}}(I - zT)^{-1}Q^{\frac{1}{2}}x, x\rangle \text{ for all } x \in \mathcal{H}$$

thus

$$T^{*} \in S_{Q},$$

i.e

$$S_{Q}^{*} \subset S_{Q}$$

,where $S_c^{*} = \{T^{*} : T \in S_c\}$.

Likewise $S_Q \subset S_Q^{*}$ and hence $S_Q = S_Q^{*}$.

To prove (2):let $Q_2 < Q_1$.Now $T \in S_{Q_1} \Rightarrow \sigma(T) \subset U$ and

$$(Q_1 - 1)\|Tx\|^2 + |2 - Q_1^{-1}||\langle Tx, x\rangle| \leq \|x\|^2$$

$$\Rightarrow (Q_2 - 1)\|Tx\|^2 + |2 - Q_2^{-1}||\langle Tx, x\rangle| \leq \|x\|^2.$$

Thus $T \in S_Q$. Hence $S_{Q_1} \subseteq S_{Q_2}$. To prove the convexity of S_c for $c \geq 1$,we use the property (iv).

If T_1 and T_2 are two operators and Q_2, Q_2 are their corresponding positive Hermitian operator as described just after proposition 3.1.1,then from

$$\|T_1 + T_2\|^2 \leq 2(\|T_1\|^2 + \|T_2\|^2).$$

Indeed $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$.

Also

$$(\|T_1\| - \|T_2\|)^2 \geq 0 \Rightarrow \|T_1\|^2 + \|T_2\|^2 \geq 2\|T_1\|\|T_2\| \text{ thus}$$

$$\|T_1x + T_2x\|^2 \leq \|T_1x\|^2 + \|T_2x\|^2 + 2\|T_1x\|\|T_2x\| \leq 2(\|T_1x\|^2 + \|T_2x\|^2).$$

Now if T_1 and T_2 are members of S_Q ,then using condition (2) in proposition 3.1.5, and a simple calculation, we have

$$\tfrac{1}{2}(T_1 + T_2) \in S_Q.$$

From the properties of S_Q in the proposition 3.1.6,we further obtain the following useful proposition.

Proposition 3.1.7. *For any bounded hermitian operator $Q > I$,the function,*

$$T \mapsto \|T\|_Q = \inf\{s : T \in sS_Q\}$$

is a Schwarz norm on $B(\mathcal{H})$.From this class of Schwarz norms,we can obtain ,using the Calkin algebra,another class of Schwarz norms.

Proposition 3.1.8. *Let Q_1 Q_2 be two bounded hermitian operators and $Q_i \geq I$ $i = 1, 2$. In this case the function on $B(\mathcal{H})$ defined by*

$$T \mapsto \|T\|_{Q_1} + \widehat{\|T\|}_{Q2}$$

where $\widehat{T}$ denotes the image of T in the Calkin algebra of H,is a Schwarz norm on $B(\mathcal{H})$

Remark 3.1.9. The above construction of Schwarz norms can be given in the case of B^*-algebras. For the construction of Schwarz norms we can use the representations of the B^*-algebra in the algebra $B(\mathcal{H})$ for some $\mathcal{H}$

3.2 Schwarz norms on Banach spaces

It is quite natural to investigate the problem about the existence of Schwarz norms on the algebra $B(X)$ of all bounded operators on a Banach space X. For this we recall that a function $[.]$ on $X \times X$ into $\mathbb{C}$ is called a semi-inner product if the following conditions are satisfied:

1. $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
2. $[ax, by] = ab^*[x, y]$
3. $|[x, y]| \leq \|x\|.\|y\|$
4. $[x, x] > 0$ for $x \neq \overline{0}$

for all $x_1, x_2, x, y \in X$ and a, b are complex numbers.

Theorem 3.2.1. *On every Banach space there exist a semi-inner product $[,]$ with the property*

$$[x, x] = ||x||^2$$

(i.e it is compatible with the norm)
Indeed for any $x \in X$ we define the functional $f_x \in X^*$. (where X^* denotes the space of all the bounded functionals on X)with the properties;

(i) $||f_x|| = ||x||$

(ii) $f_x(x) = ||x||^2$

The existence of the functional is guaranteed by Hahn-Banach theorem and we define

$$[x, y] = f_y(x) \text{ and } f_{\lambda x} = \lambda^* f_x$$

which satisfy the four conditions above,for each $\lambda \in \mathbb{C}, x \in X$

A operator $T \in B(\mathcal{X})$ is called hermitian if

$$\|e^{iT}\| = 1$$

for all real numbers t or equivalently,Bonsall[6] if

$$W(T) = \{[Tx, x] : \|x\| = 1\}$$

is a subset of real numbers.

An operator $T \in B(X)$ is called positive if T is hermitian and the spectrum of T is in the subset $\{x \in \mathbb{R} : x > 0\}$

Now the definition of the class S_Q can be as follows.

Definition 3.2.2. An operator $T \in S_Q$ if and only if

1. $\sigma(T) \subset U$

2. For any $x \in X$ and $|z| < 1$ $Re[(I + \sum Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n)x, x] \geq 0$

where Q is a hermitian operator such that $Q^{\frac{1}{2}}$ is also a hermitian operator.

The following results give indications about the possible existence of Schwarz norms.

Theorem 3.2.3. *There exists a Banach spaceX and an operator T such that*

$$Re[Tx, x] \geq 0$$

does not imply

$$Re[T^{-1}x, x] \geq 0.$$

As an example to illustrate this,we consider the Banach space ℓ_2^p of all pairs $x = (x_1, x_2)$ with the norm

$$x \mapsto \|x\|_p = \{|x_1|^p + |x_2|^p\}^{\frac{1}{p}},\ 1 < p < \infty.$$

In this case it can be seen that the semi-inner product compatible with the norm $[x, x] = \|x\|_p^2$ is given by

$$[x, y] = x_1|y_1|^{p-1} + x_2|y_2|^{p-2}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ We consider an operator on this space with the matrix

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}.$$

where the elements a, b, c are complex numbers .

We need to find conditions for the a, b, c such that $Re[Tx, x] \geq 0$.

A straight forward but complicated computation shows that these are :

1. $Rea \geq 0,\ Reb \geq 0$
2. $|c| \leq (pRea)^{\frac{1}{p}}(qReb)^{\frac{1}{q}}\ (\frac{1}{p} + \frac{1}{q} = 1)$

. and the condition for

$$Re[T^{-1}x, x] \geq 0$$

is

$$|\tfrac{c}{ab}| \geq (pRea^{-1})^{\frac{1}{p}}(qReb^{-1})^{\frac{1}{q}}$$

and thus if

$$Re[Tx,x] > 0 \text{ then } Re[T^{-1}x,x] > 0$$

if and only if

$$|c| \leq |a|^{1-\frac{2}{p}}|b|^{1-\frac{2}{q}}(Repa)^{\frac{1}{p}}(Reqb)^{\frac{1}{q}}$$

and this gives that$Re[Tx,x] \geq 0$ does not imply that $Re[T^{-1}x,x] \geq 0$.

Remark 3.2.4. In the case of Hilbert space (and invertible) operators,the condition ,$ReT \geq 0$ implies the condition $ReT^{-1} \geq 0$

We now give an example of a Banach space with the property that the induced norm on $B(X)$ is not a Schwarz norm.

Example 3.2.5. If $X = \ell_2^1$ then the induced norm on $B(X)$ is not a Schwarz norm.We consider the operator T with the matrix(triangular)

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

and a simple computation shows that

$$\|T\| = max\{|a| + |c|, |b|\}.$$

We now take $0 < a < 1$ and in this case the operator with the matrix

$$\begin{bmatrix} a & 0 \\ 1-a & 1 \end{bmatrix}$$

is a contraction operator.An elementary computation shows that for $|\alpha| < 1$,the conformal map/function

$$\varphi_\alpha(z) = (z-\alpha)(1-\overline{\alpha}z)^{-1}$$

for all $z \in \mathbb{C}$,take contractions to contractions; now consider the function $f_\alpha(z) = \frac{z-\alpha}{1-\hat{\alpha}z}$.

So

$$f_\alpha(T) = (1 - \overline{\alpha}T)^{-1}(T - \alpha I).$$

The computation of the norm of the operator $f_\alpha(T)$ shows that this is given by

$$\|f_\alpha(T)\| = a|\alpha + a| + (1 - a)|\frac{1+\alpha+a(1+\overline{\alpha})}{(1+\overline{\alpha}a)(1+\alpha)}|$$

and thus for $\|f_\alpha(T)| \leq 1$,where α is a real number,we obtain

$$a|\alpha + a| + (1 - a)(1 + a) \leq |1 + \alpha a|$$

which is not true for $\alpha = -\frac{1}{2}(a + 1)$.

In view of the results of this section,the following result is of interest.

Proposition 3.2.6. *If X is a complex Banach space and for any contraction T,$f(T)$ is also a contraction for all $|f| \leq 1$,then X is a Hilbert space.*

proof:

Let $x_o \in X$ be arbitrary $x_o \in X$ such that

$$\|x_o\|\|x_o^*\| \leq 1$$

and define the operator on X by the relation

$$Tx = x_o^*(x)x_o.$$

It is clear that T is a contraction.

From the hypothesis it follows that for any f_α

$$f_\alpha(T)$$

is also a contraction.

This gives the relation

$$\|(T+\alpha)(I+\alpha^*T)^{-1}x\| < \|x\| \text{ ¡}$$

which is equivalent to the relation

$$\|(T+\alpha)x\| \leq \|(I+\alpha^*T)x\|.$$

From the form of the operator T it follows that

$$\|x_o^*(x)x_o + x\| \leq \|x + \alpha^* x^*(x)x_o\|.$$

Now if $x, y \in X$ and $\|x\| \geq \|y\| > 0$,we obtain from the H-Banach theorem that there exists $x_o^* \in X^*$ such that

$$\|x_o^*\| = \|x\|^{-1},\ x_o^*(x) = 1.$$

We take $x_o = y$ and remark that the operator T constructed with these element gives us

$$\|y + \alpha x\| \leq \|x + \alpha^* y\| \ |\alpha| < 1$$

and from the continuity argument,it follows that this relation holds for $|\alpha| = 1$. Now if $\|x\| = \|y\|$,changing the role of x with y and α with α^*,we obtain

$$\|x + \alpha^* y\| \geq \|y + \alpha x\|.$$

Thus we have the equality $\|x + \alpha^* y\| = \|y + \alpha x\|$. Now if $|\alpha| > 1$ then for $\beta = \frac{1}{\alpha}$ we have by the above result

$$\|x + \alpha^* y\| = |\alpha||\beta x + y| = |\alpha|\|x + \beta^* y\| = \|\alpha x + y\|$$

and thus the relation is true for any α . Now for $\alpha = \frac{p}{q}$,p and q being real numbers,we obtain that

$$\|px + qy\| = |q|\|\tfrac{p}{q}t + x\| = |q|\|y + \tfrac{p}{q}x\| = \|qy + px\|$$

and thus for any x and y,$\|x\| = \|y\|$ and any p,q real numbers we obtain that

$$\|px + qy\| = \|qx + py\|$$

and by a famous result of F.A.Ficken,this relation is characteristic for a norm to be inner product norm,i.e,there exists an inner product $\langle , \rangle$ on X such that for all $x \in X$

$$\|x\|^2 = \langle x, x \rangle$$

Chapter 4

Summary and Conclusion

We therefore have as a conclusion that,a Schwarz norm can be constructed from the sum of a norm and a seminorm and that Schwarz norms are are easily realizable in the Hilbert space context.

4.1 Recommendation

We will finally note that there could be other classes of Schwarz norms which are not related to the class S_Q.For some directions with regard to this conjecture,the reference [10] could be exploited.

Bibliography

[1] J.P Williams,(1968),Schwarz norms for operators, *Pacific Journal of Mathematics.* **24**, No.1

[2] C.A Berger and Stampfli,(1967), Mapping theorems for the numerical range, *to appear in American J. Math.* **26**, 247-250.

[3] B.S.Z Nagy and C.Foias,(1983), On certain classes of power-bounded operators, *Acta.Sci. Math. Ser.* **III** 18(38) 317-320.

[4] C.Foias,(1957), Sur certains theoremes de von Neumann concernant les ensemplesspectraux , *Acta.Math.Sci.*(Szeged) **85** 15-20.

[5] J.G Stampfli,(1966),Normality and the numerical range of an operator,*Bull Amer.Math.Soc.*. **72** 1021-23.

[6] F.F. Bonsall, J. Duncan, *Numerical Ranges of operators on Normed spaces and elements of Normed algebras*, London Math. Soc. Lecture Notes series 2, Cambridge University Press, London-New York, 1971.

[7] F.F. Bonsall, J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture notes Series 10, Cambridge University Press, London-New York, 1973.

[8] E.Kreyszig, *Introduction to functional analysis with applications*, University of Windsor,1978.

[9] J.A.R Holbrook,Inequalities of von Neumann type for small matrices, *Function Spaces*(ed.K.Jarosz),Marcel Dekker,1992,273-280

[10] T.Ando Construction of Schwarz norms,*Operator Theory.Advances and Application.*,**127**(2001) 29-39

[11] J.Von Neumann,Eine Spektraltheorie fur allgemeine Operatoren eines Unitaren Raumes,Math.Nach., 4 (1951),258-281

[12] C.A Berger,A strange dilation theorem (Abstract),Amer.Math.Soc.Notice, 12 (1965)590