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On Numerical Range of Multiplication Operator

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Abstract

Let H be an infinite dimensional complex Hilbert space and $A, B \in B(H)$ where $B(H)$ is the C^*-algebra of all bounded linear operators on H. Let $M_{AB} : B(H) \to B(H)$ be a multiplication operator induced by A and B defined by

$$M_{AB}(X) = AXB \tag{1}$$

In this paper we show that the numerical range of multiplication operator is given by

$V(M_{AB/B(B(H))}) = [\bigcup_{U\in U(H)} W(U^*AUB)^-]^-$ for all $A, B \in B(H)$

and U a unitary operator on the algebra $B(H)$ where V is the algebraic numerical range and W is the classical numerical range. The results obtained are an extension of the the work done by Barraa [4].

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1. Introduction

Multiplication operator

There are various settings for the definition of multiplication operator.
Let $\mathcal{A}$ be a unital Banach algebra.The multiplication operator $M_{a,b} : \mathcal{A} \to \mathcal{A}$ is defined by

$$M_{ab}(x) = axb \tag{2}$$

where $x \in \mathcal{A}$ and $a, b \in \mathcal{A}$ are fixed.

Numerical range

For operators in a Hilbert space H, the notion of numerical range (or field of values) is important in various applications in the study of operators. The numerical range of an operator $T \in B(H)$ is a subset of a complex plane $\mathbb{C}$ defined by
$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. This set is convex but not closed in general.For the multiplication operator we have

$W(M_{A,B}) = \{\langle AXBx, x\rangle : x \in H, \ \|x\| \le 1\}$ (3)

The algebraic numerical range of $a \in \mathcal{A}$ for a unital C^*-algebra $\mathcal{A}$ is defined by;

$V(a/\mathcal{A}) = \{f(a) : f \in \mathcal{A}^*, \|f\| = 1 = f(e)\}$ (4)

which is a closed convex set. Similarly,

$V(M_{AB/B(H)}) = \{f(M_{AB}) : f \in B(B(H)^*, \|f\| = 1 = f(AB)\}$ (5).

If $\mathcal{A} = B(X)$ where $B(X)$ is the algebra of bounded linear operators on a normed space X and $T \in B(X)$ then we have the spatial numerical range of T defined by;

$V_o(T) = \{f(Tx) : x \in X, f \in X^* \ with \ \|f\| = \|x\| = 1 = f(x)\}$. (6).

Theorem 1

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H.Then the numerical range $V(M_{AB})$ is a convex set.

Proof

We need to show that if $\alpha_1, \alpha_2 \in V(M_{AB})$ and $t \in (0, 1)$ then

$\alpha = t\alpha_1 + (1 - t)\alpha_2 \in V(M_{AB})$.

Let $\alpha_1, \alpha_2 \in V(M_{AB})$ the there exists states f_1 andf_2 such that

$\alpha_1 = f_1(M_{AB})$ and $\alpha_2 = f_2(M_{AB})$ where $M_{AB} \in B(B(H))$,

$f_1(AB) = 1 = \|f_1\|$ and $f_2(AB) = 1 = \|f_2\|$. We define f on $B(B(H))$ by

$f(M_{AB}) = tf_1(M_{AB}) + (1 - t)f_2(M_{AB})$. It suffices to show that f is a state by showing that it is linear, positive and its norm is one.

f is linear

Let $\mu_1, \mu_2 \in \mathbb{C}$ and $(M_{AB}) \in B(B(H))$ then

$f(\mu_1(M_{AB}) + \mu_2(M_{AB})) = tf_1(\mu_1(M_{AB}) + \mu_2(M_{AB})) + (1 - t)f_2(\mu_1(M_{A,B}) + \mu_2(M_{AB}))$

$= \{tf_1(\mu_1(M_{AB}))+(1-t)f_2(\mu_1(M_{AB}))\}+\{tf_1(\mu_2(M_{AB}))+(1-t)f_2(\mu_2(M_{A,B}))\}$

$= \{\mu_1(tf_1(M_{AB}))+\mu_1((1-t)f_2(M_{AB}))\}+\{\mu_2(tf_1(M_{AB})+\mu_2((1-t)f_2(M_{AB}))\}$

$= \mu_1\{tf_1(M_{AB}) + (1 - t)f_2(M_{AB})\} + \mu_2\{tf_1(M_{AB}) + (1 - t)f_2(M_{AB})\}$

$= \mu_1(f(M_{AB})) + \mu_2(f(M_{AB}))$

f is positive

$f((AXB)^*AXB) = tf_1((AXB)^*AXB) + (1-t)f_2((AXB)^*AXB) \geq 0$ Since
$f_1((AXB)^*AXB) = ((AXB)^*AXBx, x)$
$= (AXBx, AXBx) = \|AXB\|^2 = \|A\|^2\|B\|^2 \geq 0.$
Now,
$f(AB) = tf_1(AB) + (1-t)f_2(AB) = t + (1-t) = 1.$
$1 = |f(AB)| \leq \|f\||1| = \|f\|$ implying that $\|f\| \geq 1$. Also,
$|f(AB)| = |tf_1(AB) + (1-t)f_2(AB)|$
$\leq |tf_1(AB)| + |(1-t)f_2(AB)|$
$\leq |t|\|f_1\| + |1-t|\|f_2\| = 1$. Implying that $\|f\| \leq 1$. Thus f is a state in $B(B(H))^*$. Therefore, $f(M_{AB}) \in V(M_{AB})$ so $V(M_{AB})$ is convex.

Theorem 2
$V(M_{AB/B(B(H))}) = \overline{W(AXB)}$

Proof
Here,it will be shown that $V(M_{AB}) \subseteq \overline{W(AXB)}$ and $\overline{W(AXB)} \subseteq V(M_{AB})$.
Let $\alpha \in \overline{W(AXB)}$ then there exists a sequence $\{x_n\}_{n\geq 1}$ of unit vectors in H such that $lim_{n\to\infty}\langle AXBx_n, x_n\rangle = \alpha$ and $lim_{n\to\infty}\|AXBx_n\| = \|AXB\|$ for all $M_{AB} \in B(B(H))$.
We define a functional f on $B(B(H))$ by $f(AXB) = lim_{n\to\infty}\langle AXBx_n, x_n\rangle$ so that $f(AXB) = \alpha$
We will show that f is a state.
First, f is linear since if $(AXB) \in B(B(H))$ and $\lambda, \mu \in \mathbb{C}$ then,
$f(\lambda(AXB) + \mu(AXB)) = lim_{n\to}\infty\langle(\lambda(AXB) + \mu(AXB))x_n, x_n\rangle$
$= lim_{n\to\infty}\langle\lambda(AXB)x_n, x_n\rangle + lim_{n\to\infty}\langle\mu(AXB)x_n, x_n\rangle$
$= \lambda lim_{n\to\infty}\langle(AXB)x_n, x_n\rangle + \mu lim_{n\to\infty}\langle(AXB)x_n, x_n\rangle$
$= \lambda f(AXB) + \mu f(AXB).$
Also, f is positive since
$f(AXB)^*(AXB) = lim_{n\to\infty}\langle((AXB)^*AXB)x_n, x_n\rangle$
$= lim_{n\to\infty}\langle AXBx_n, AXBx_n\rangle$
$= \{lim_{n\to\infty}\|AXBx_n\|\}^2 = \|AXB\|^2 \geq 0.$
Finally, $|f(AXB)| = |lim_{n\to\infty}\langle AXBx_n, x_n\rangle|$

$$\leq lim_{n\to\infty}\|AXBx_n\|lim_{n\to\infty}\|x_n\| = \|AXB\|.$$

Thus $\|f\| \leq 1$.

Now, $1 = \|f(AB)\| \leq \|f\|\|AB\| = \|f\|$ so, that $\|f\| \geq 1$. Therefore $\|f\| = 1$ and so $\alpha \in V(M_{AB})$. Hence $\overline{W(AXB)} \subseteq V(M_{AB})$

Next we show that $V(AXB) \subseteq \overline{W(AXB)}$ See[1].

Let $\lambda \in V(AXB)$ and λ not in $W(AXB)$ and deduce a contradiction. Therefore, there exists a state $f \in B(B(H))^*$ such that $f(AXB) = \lambda$ and $f((AXB)^*AXB) \geq 0$. Since $W(AXB)$ is convex, then by rotating M_{AB}, we may assume that $Re\, W(AXB) \leq Re\,\lambda - \alpha$, $\alpha > 0$.

Let $G = \{x \in H : \|x\| = 1 \;\; and \;\; Re\,\langle AXBx, x\rangle \geq Re\,\lambda - \frac{\alpha}{2},\; \alpha > 0\}$ and $\vartheta = sup\{\|AXBx\| : x \in H\}$. Then $\vartheta < 0$.

The set G is nonempty because if it is not, then for all $x \in H$, $\|x\| = 1$ we shall have

$Re\langle AXBx, x\rangle < Re\,\lambda - \frac{\alpha}{2},\; \alpha > 0$.

But since f is a weak*-limit of convex combinations of vector states for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n > N$, $|f_n(AXB) - f(AXB)| < \varepsilon$.

Also we can find $M = M(\varepsilon)$ such that for all $n > M$,

$|f_n((AXB)^*AXB) - f((AXB)^*AXB)| < \varepsilon$.

Taking $\varepsilon < \frac{\alpha}{2}$ and $n > max(N, M)$ and since

$f_n(AXB) = \sum_{i=1}^{n} \alpha_i\omega_{x_i}(AXB) = \sum_{i=1}^{n} \alpha_i\langle AXBx_i, x_i\rangle$ for $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$ we have

$$Re\; f_n(x) = Re\; \sum_{i=1}^{n} \alpha_i\omega_{x_i}(AXB) = Re\; \sum_{i=1}^{n} \alpha_i\langle AXBx_i, x_i\rangle$$
$$= \sum_{i=1}^{n} \alpha_i Re\;\langle AXBx_i x_i\rangle \leq Re\,\lambda - \frac{\alpha}{2}.$$

But $f_n(x) > f(AXB) - \varepsilon$ and therefore $Re\; f_n(x) > Re\,\lambda - \varepsilon$ which implies that $\varepsilon > \frac{\alpha}{2}$. This is a contradiction.

Now, for all $x_i \in G$, we have that $\|AXBx_i\| \leq \vartheta$.

Since$f((AXB)^*AXB) < f_n((AXB)^*AXB) + \varepsilon$ and $0 \leq f((AXB)^*AXB$ we obtain

$$0 \leq f((AXB)^*AXB) < f_n((AXB)^*AXB)+\varepsilon = \sum_{i=1}^{n} \|AXBx_i\|^2+\varepsilon < \vartheta^2 < 0$$

which is also a contradiction. Thus λ not in $W(AXB)$ implies that λ is not in $V(AXB)$. Hence $\lambda \in V(AXB)$ implies that $\lambda \in W(AXB)$ and so $V(AXB) \subseteq W(AXB)$ and since $W(AXB)$ is convex, then $V(AXB) \subseteq \overline{W(AXB)}$.

Main result

Theorem 3

Let H be a complex Hilbert space and $B(H)$ a C^*-algebra of all bounded linear operators on H. Then,

$V(M_{AB/B(B(H))}) = [\bigcup_{U \in U(H)} W(U^*AUB)^-]^-$.

For all $A, B \in B(H)$ and U a unitary operator.

To prove this theorem we use the following Lemma 4

Lemma 4

Let A and B be elements in $B(H)$. Then, $W(AB) \subset V(M_{AB/B(B(H))})$ where $W(AB) = \{\langle ABx, x\rangle\}$.

Proof

Let $\alpha \in W(AB)$ then by definition of the classical numerical range, there exist $x \in H$ with $\|x\| = 1$ such that;

$\alpha = \langle ABx, x\rangle = tr(AB(x \otimes x))$ where tr(.) is a linear form trace.

We denote this linear form by $\Psi_{x\otimes x}$ and define it as

$\Psi_{x\otimes x}(X) = tr(X(x \otimes x)) = \langle Xx, x\rangle$ on $B(B(H))$. The linear form is bounded and its norm is equal to one that is;

$\|\Psi_{x\otimes x}\| = \|x \otimes x\| = 1$.

The form $\Psi_{x\otimes x}$ is also a state since

$\Psi_{x\otimes x}(I) = tr(x \otimes x) = \langle x, x\rangle = \|x\|^2 = 1$ and

$\Psi_{x\otimes x}X^*X = tr(X^*X(x \otimes x)) = \langle X^*Xx, x\rangle = \langle Xx, Xx\rangle = \|Xx\|^2 \geq 0$.

So $\Psi_{x\otimes x}(M_{AB}(I_H) \subset V(M_{AB/B(B(H))})$ and we have that

$\Psi_{x\otimes x}(M_{AB}(I_H) = \Psi_{x\otimes x}(AB) = tr(AB(x \otimes x) = \langle ABx, x\rangle = \alpha$.

Thus $W(AB) \in W(M_{AB}) \subset V(M_{AB/B(B(H))})$.

Let E be a Banach space. Then $T \in B(E)$ is said to be an isometry if $\|Tx\| = \|x\|$ for all $x \in E$. If T is an invertible isometry, then its inverse T^{-1} is also an isometry therefore,

$$V(TST^{-1}{}_{/B(E)}) = V(T^{-1}ST_{/B(E)}) = V(S_{/B(E)}) \quad (7)$$

for all $S \in B(E)$.
If $E = H$ then $T = U$ and $T^{-1} = U^*$. Thus from equation (7) we have that
$V(UAU^*{}_{/B(H)}) = V(U^*AU_{/B(H)}) = V(A_{/B(H)})$ (8)
for all $A \in B(H)$.
Given two isometries $U, V \in H$, then
$V(M_{U^*AU\ \ V^*BV/B(B(H))}) = V(M_{AB/B(B(H))})$ (9)
for all $A, B \in B(H)$.
Now, taking an invertible isometry R_{UV^*} with R_{U^*V} as its inverse, then
$V(M_{U^*AU\ V^*BV/B(B(H))}) = V(R_{UV^*}\ M_{AB}\ R_{U^*V?B(B(H))})$, and by lemma 4
$W(U^*AU\ V^*BV) \subset V(R_{UV^*}\ M_{AB}\ R_{U^*V/B(B(H))})$ and
$\bigcup_{U,V \in U(H)} W(U^*AU\ V^*BV) \subset V(M_{AB/B(B(H))})$.
Since the numerical range is closed and the product of two unitaries is also a unitary, then
$[\bigcup_{U \in U(H)} W(U^*AUB)^-]^- \subset V(M_{AB/B(B(H))})$ or
$[\bigcup_{V \in U(H)} W(V^*AVB)^-]^- \subset V(M_{AB/B(B(H))})$.
Next we proceed to show the inclusion
$V(M_{AB/B(B(H))}) \subset [\bigcup_{U \in U(H)} W(U^*AUB)^-]^-$.
Let $\mathcal{A}$ be a Banach algebra, then for any $a \in \mathcal{A}$;
$V(a/\mathcal{A}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|a - z\|\}$.(See [15]).
The norm of multiplication operator is defined by;

$$\|M_{AB}\| = Sup\{\|M_{AB}(X)\| :\ \|X\| = 1\}$$
$$= Sup\{\|AXB\| :\ X \in B(H), \|X\| \leq 1\}.$$

Theorem 5
Let $\mathcal{A}$ be C^*-algebra, then

$$\|M_{AB}\| = Sup\{\|M_{AB}(U)\| : U \in U(\mathcal{A})\}$$
$$= Sup\{\|AUB\| :\ U \in U(\mathcal{A})$$

where $U(\mathcal{A})$ denotes the set of unitaries in $\mathcal{A}$. (see [11]).
Now, if $\mathcal{A} = B(H)$ then $M_{AB}(U) = AUB$ for all $U \in U(H)$. Therefore,
$V(M_{AB/B(B(H))}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|M_{A,B} - z\|\}$. But

$$\|M_{AB} - z\| = Sup\{\|(M_{AB} - z)(U)\| : U \in U(H)\}$$
$$= Sup\{\|(AUB - z)U\| :\ U \in U(H)\}.$$

Since the unitary $U \in U(H)$ is an isometry, then
$\|M_{AB} - z\| = Sup\{\|U^*AUB - zI_H\| : \ U \in U(H)\}$.
So if $\mu \in V(M_{AB/B(B(H))})$ then for all $z \in \mathbb{C}$,
$\mu \in \{|\lambda - z| \leq \|M_{AB} - zI_{H/B(H)}\|\}$.
Taking a fixed $\varepsilon > 0$, there exists U_e such that
$\|M_{AB} - zI_{H/B(H)}\| < \|U_e^*AU_eB - zI_H\| + \varepsilon$ and by theorem 1.4 we have that,

$$W(U_e^*AU_e)^- = V(U_e^*AU_eB)$$
$$= \bigcap_{z\in\mathbb{C}}\{\lambda : |\lambda - z| \leq \|U_e^*AU_eB - zI_H\|\}$$

and so there exists $\lambda \in W(U_e^*AUB)$ such that $|\mu - \lambda| \leq \varepsilon$.
Since ε is arbitrary, $\mu \in [\bigcup_{U\in U(H)} W(U^*AUB)^-]^-$. □

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