APPLICATION OF DISCRETE AND CONTINOUS TIME MODELS IN VALUATION OF CREDIT INSURANCE FOR ASSET-BASED LENDING COMPANIES

BY

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DECLARATION

This research is my own work and has not been presented for a degree award in any other institution.

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DEDICATION

I dedicate this project to my parents, Mr. and Mrs Etyang and my best friends Okama A. Edwin, Anastacia Lwoba and Etyang Frankline for bearing the costs incurred during the entire research process, also to my friends who offered their prayers and support towards successful completion of this project.

ABSTRACT

Asset-based lending companies and other loan providers are exposed to risk of loan defaults by borrowers. To reduce this risk, these companies acquire credit insurance. Thus when the borrower defaults in payment, the insurance company covers a percentage of the outstanding balance which generates a way to lessen and spread credit risk that the lender incurs. Therefore there are a number of methods put in place such as frequency-serverity and hazard rate models used to value credit insurance. Valuing of credit insurance for asset-based lending companies is a challenging task especially in Kenyan market, where in the case of a borrower's default, the process for recovering of the collateral will last a longer period of more than a year and where data on the borrower's behaviour of payment is of poor quality or generally unavailable. The existing methods do not consider the time to repossession of the collateral in case of loan default. Our proposed model takes into account time to repossession of the collateral and can be used in emerging market economies where other available methods may be either unsuitable or are too complex to implement due to lack of enough data. Therefore, this project aims to incorporate the discrete and continuous time models to forecast loss reserves in credit insurance for asset-based lending companies. First, we established a discrete-time model to describe delinquency of credits in loan insurance product. Martingale properties, Replicating of asset portfolio strategy and Ito's calculus are used to obtain results on expected values of future losses of credit insurance products. Secondly, we used the Black-Scholes model to develop a continuous-time model to forecast future losses in credit insurances. This is constructed by linking it from the discrete-time model using the methods of stochastic calculus. We estimated the loss reserves by first applying the Geometric Brownian Motion simulation to predict the probability of default of the borrower. The probability of default was then multiplied by the simulated outstanding balances, a factor that considers the time to repossession of the collateral and the assumed percentage coverage of the insurance company to obtain estimates of loss reserves in credit insurance.

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INTRODUCTION

1.1 Background of the study

Credit Insurance is used to pay out a loan balance or to postpone debt payments on the customer's behalf in the event of disability or job loss. Credit insurance can be purchased to insure all kinds of consumer loans including car loans, loans from finance companies, and home mortgage borrowing. Credit insurance is as a result of credit risk.

Whilst credit risk has not been primary area of consideration by Actuarial profession in history, Actuaries have nevertheless made important contributions in the development of modern credit risk modeling methods. The most common credit risk models used are frequency-severity or hazard rate models. As credit risk became an increasing concern in recent years, various advanced methods have been put in place to widely measure the spread of credit risk. Therefore there is a need for actuaries to familiarize with these popular methods and their advantages and disadvantages, in order to compete effectively in this rapidly growing area.

Currently, reduced and structural form models constitute to two classes of credit risk modeling methods. The structural approach provides an explicit connection between default risk and capital structure, while the reduced form approach models credit defaults as exogenous events driven by a stochastic process (such as a Poisson jump process). In this case, most actuarial models used for credit risk measurement lie within the reduced form class. Structural models, pioneered by [7], employ modern option pricing theory in corporate debt valuation. Merton model [7] was the first structural model and has served as the cornerstone for all other structural models.

Structural approach, led by Merton model [7], has a very nice feature of connecting credit risk to underlying structural variables. It provides both an intuitive economic interpretation and an endogenous explanation of credit defaults, and allows for applications of option pricing methods. As a result, structural models not only facilitate security valua-

tion, but also address the choice of financial structure.

For this study we first focus on the binomial option pricing model then link it to the Black-Scholes model. The binomial option pricing model uses an iterative procedure allowing for the specification of nodes, or points in time, during the time span between the option valuation date and the option's expiration date.

1.1.1 Definition of key terms

(i) **Loss Reserves**

Loss reserve is an estimate of an insurer's liability from future claims. Loss reserves are typically comprised of liquid assets, and they allow the insurer to cover claims made against policies that it underwrites. Forecasting loss reserves can be a complex undertaking. Insurers must consider the duration of the insurance contract, the type of insurance offered and the odds of a claim being resolved quickly. Insurers have to adjust their loss reserve calculations under different circumstances.

When an insurer underwrites a new policy, it records a premium receivable (which is an asset) and a claim obligation (which is a liability). The liability is considered part of the unpaid losses account, which represents the loss reserve.

(ii) **Asset -based lending**

With asset-based lending, in an instance where an individual borrows money to buy, for example, a home or even a car, the house or the vehicle serves as collateral for the loan. If the loan is not repaid in the agreed time period, it falls into default, and the lender may then recover the car or the house in order to pay off the amount of the loan.

1.2 Statement of the Problem

Asset-based lending companies and other loan providers are exposed to risk of loan defaults by borrowers. To lessen this risk,these companies acquire credit insurance. Therefore when the borrower defaults in his or her credit payment, the insurance company covers a percentage of the outstanding balance and the rest of the balance is taken care of by the insured lender through repossession of the collateral. Although data might be

available, the pricing of credit insurance for asset based lending companies is a challenging task and is even more challenging in the case of emerging markets like Kenya, where, in the case of a borrower's default, the process of repossession of loan collateral may last a longer period of time of more than a year and where data on payment behavior is generally unavailable or of poor quality. The current valuation methods do not consider the time to repossession of an asset in case of loan default. Credit insurance for the case of asset-based lending differs from the other types of insurance contracts in several ways. As a result of this, it is difficult to use the conventional techniques for valuing credit insurance contracts.

First, in life insurance, risk increases with time while in credit insurance, the risk involved decreases over the time because of spreading payments over multiple period. Secondly, in credit insurance, prepayments and default rates depend on macroeconomic factors that include interest rates and individual income among others hence a significant amount of systematic risk is involved in the credit insurance. In contrast, insurers can lessen the risks of conventional insurance policies through geographic diversification

Thirdly, casualty insurance contracts use historical performance to cover subsequent periods since they only consider a single period. However, credit insurance for asset-based lending companies cover multiple periods. Therefore,it is impossible to use this information on historical experience to determine the premiums for credit insurance. Premiums for life credit insurances are determined at the date of inception of the policy unlike the other types of insurance policies.

Finally, credit insurance only covers the risk for the lender and not the borrower's risk. Because of these factors, the valuation of premiums for credit insurances requires extensive research with new approaches. A more appropriate model for pricing should give more emphasis on minimizing default risk. My proposed model applies methods of stochastic calculus to come up with a forecast of loss reserves in credit insurances for asset-based lending companies.

1.3 Objectives of the study

1. To develop a decrete time model to model the delinquency of credit.

2. To construct a continuous time model to predict future losses in credit Insurances and how time to repossession affects the future loss reserves estimates.

1.4 Significance of the study

- Instead of replacing currently used models, this will also present an alternative method for insurers looking for a reasonable check for their reserves and premiums levels. Our proposed projection techniques can be applied in any market, especially in emerging market economies where other existing methods may be either unsuitable or are too difficult to use due to inadequate significant data.

- Using our discrete-continuous time model, all insurers in Kenya even those with absence of resident Actuaries can use historical data from existing insurers and their office experience data to accurately value credit insurance contracts. This will in turn grow the number of credit insurance companies hence bringing foward healthy competition and improved services.

- Financial institutions will be protected from the risk of loan defaults.

1.5 Basic Concepts

1. **Brownian Motion.**

Brownian motion $(B_t\,, t \geq 0)$ is a continuous time stochastic process with a continuous state space and has the following properties;

- $B_0 = 0$
- B_t has independent increments i.e $B_t - B_s$ is independent of $(B_r, r \leq s)$ whenever $s < t$. Thus the changes in the value of the process over any two non-overlapping periods are statistically independent.
- B_t has stationary increment, i.e the distribution of $B_t - B_s$ depends only on t-s.
- B_t has Gaussian increment, i.e the distribution of $B_t - B_s$ is N(0,t-s).

- B_t has continuous sample paths $t \longrightarrow B_t$,i.e the graph of B_t as a function of t, does not have any breaks in it [12]

2. **Martingales.**

   A stochastic process X is called an $\mathcal{F}_t$ martingale, if the following conditions hold:

   - X is adapted to filtration $\mathcal{F}_t$ $t \geq 0$
   - For all t, $E(|X_t|) < \infty$
   - For all s and t with $s \leq t$, the following relation holds:

   $$E(|X_t|\mathcal{F}_s) = X_s$$

   - A process satisfying, for all s and t with $s \leq t$, the inequality $E(X_t|\mathcal{F}_s) \leq X_s$ is called a supermatingale and a process satisfying $E(X_t|\mathcal{F}_s) \geq X_s$ is called a submatingale [11]

3. **Equivalent probability measure $\mathbb{P}$ and $\mathbb{Q}$.**

   Two probability measure are equivalent if they are defined on the same sample space and have the same null sets (i.e sets that have the probability zero.)

   Mathematically $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if $\mathbb{P}(A) > 0 \Leftrightarrow \mathbb{Q}(A) > 0$, where $\mathbb{P}(A)$ denotes probability under measure $\mathbb{P}$ and $\mathbb{Q}(A)$ denotes probability under measure $\mathbb{Q}$.

4. **The Black Scholes model(Continuous model).**

   Is a pricing model used to determine the fair price or theoretical value for a call or a put option

   Assumptions

   i) Price of the underlying share follows a geometric Brownian motion, i.e the share price changes continuously through time according to the stochastic differential equation.

   $$dS_t = S_t(\mu dt + \sigma dZ_t)$$

   where; S_t is the share price, μ is the drift, σ is the volatility and Z_t is a standard Brownian motion.

ii) The risk-free rate of interest is constant, the same for all maturities and same for borrowing or lending.

iii) There are no risk-free arbitrage opportunities.

iv) The underlying asset can be traded continuously and infinitesimally.

v) No taxes or transaction costs.

vi) Unlimited short selling is allowed.

5. **The Binomial Model (discrete model)**

Is an option pricing model that uses an iterative procedure, allowing for the specification of nodes or points in time during the time span between the time of valuation and the options expiration date.

The model reduces possibilities of price changes and removes the possibility of arbitrage.

Assumptions

i) There are no trading costs or taxes.

ii) There are no minimum or maximum units of trading.

iii) Stock and bonds can only be bought and sold at discrete times 1,2,3..

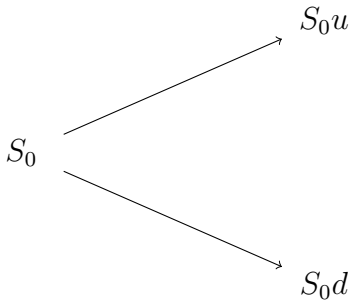
iv) The principle of no arbitrage applies

Consider one period binomial model, we start at time 0, when the stock price = S_0.

The stock price will do one of two things;

- Jump upwards to S_0u
- Jump downwards to S_0d

This is represented as;



Where u is the proportion of upward price movement and d is the proportion of downward price movement.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The application of derivatives for risk mitigation and uncertainty inherent in financial instruments has become a very relevant component in the global market. Based on the similar financial contracts with other type of contracts, there have been more research on their applications to other types of contracts. In recent times, the most commonly used derivatives are the options because they are not obligatory in nature. Options give the right but not the obligation to sell or buy the underlying asset or security at a specified price on a specified future date. Baxter M. and Rennie A. [1] presented the idea of hedging and pricing by arbitrage in the discrete-time setting by binary trees. The key probabilistic concepts of conditional expectation, martingales, change of probability measure and representation are all introduced. They also presented the concepts of expectation pricing versus arbitrage. Expectation was used as a tool for risk-free construction. The concepts of backward induction (extending the construction portfolio back one tick at a time from claim to required starting place) are also looked at.

Baxter M. and Rennie A. [1] also brought to table the idea of hedging and pricing by arbitrage in the continuous time setting. Brownian motion is brought out as well as the Ito calculus needed to manipulate it culminating in a derivation of the Black Scholes formula. Pricing of an individual asset subject to credit risk has been extensively studied in the literature we refer to Duffie D.and K.Singleton(1996) [3] for the survey of such pricing models. Among them, Jarrow, R. and S. M Turbull [4] assumed that the payoffs upon default are expressed as an exogenous fraction of the claim and they showed, under some regularity conditions, the price is given by the expected discounted payoffs under the risk neutral probability measure. Duffie D. and K. Singleton(1996) [3] proposed another model in which the payoffs are discounted by an interest rate that is adjusted so as to reflect the effect of default risk. The Health-Jarrow-Morton type model of defaultable

term structures with multiple ratings was proposed by T.R Bielecki and M.Rutkowski [5] and P.Schonbuncher [6]. This approach:

- Formulates sufficient consistency conditions that tie together credit spreads and recovery rates in order to construct a risk neutral probability Q* and the corresponding risk neutral intensities of credit events.
- Shows how the statistical probability P and risk neutral probability Q* are connected via the market price of interest rate risk and the market price of credit risk.
- Combines market and credit risk.

In 1973, Merton developed the contingent claims model that provides the motivation for the behavior of the borrower using the options theory Black.F & Scholes [7]. Most studies initially used this approach in the valuation of mortgages by focusing default and prepayment as individual risks. For example, Cunningham,D.F [2] used the Black-Scholes option-pricing model to value the risk of default by considering the default risk as a put option sold by the FHA and purchased by the buyer of a home for the protection of risk of default to the lender.

Ashok Bardhan and collegues in their Journal of Real Estate Finance and economics(2006) [8], developed a new option-based method for valuation of mortgage insurance contacts in closed economy where agents are risk neutral. As an application, they priced a typical Serbian government backed mortgage insurance contract. Zeyep Corpur [9] used the concept of Geometric Brownian motion to describe the random behaviour of the asset price S_t over time.In my case, I examine the random behaviour of the delinquency index Y_t.

CHAPTER THREE

RESEARCH METHODOLOGY

We acknowledge Oscar Perez et al [8] in his study on Stochastic Calculus applied to estimation of loss reserves in mortgage insurance, for the concept is utilized in this methodology. In his study he focused on mortgage backed securities in Mexico by applying the discrete and continous time model to forecast loss reserves in mortgage insurance.

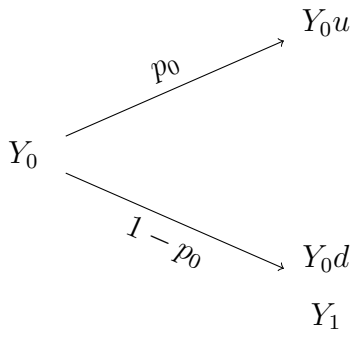
3.1 Credit Risk Modeling

Here, we will model credit risk and present Insurance function in order to obtain the expected present value of the payments of the Insurance. This will be done by applying the binomial model(discrete model). Consider a borrower with a loan term of N months given by a financial institution. Let a stochastic process Y represent the number of defaultable periods (delinquency index) as of time *t*. Then,

$$Y = (Y_t)_{t \in T}, \text{ for } T = (0, 1, 2, ..., N-1)$$

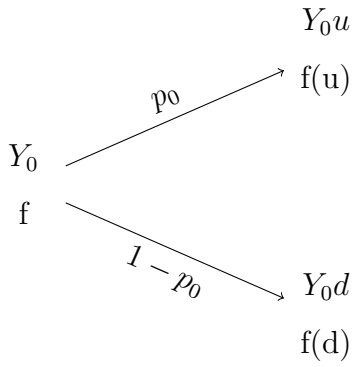
Under certain circumstances, the value of Y_t can be negative. In such scenarios we treat Y as absolute value. Let p_i represent the probability that the borrower fails to make his credit payment for time *i*. Denote a function f(Y_t) which is a function that depends on the delinquency index. Consider cash bond B_t to be another stochastic process to represent the time-value of money. Assume a constant risk-free rate r. Thus $B_t = B_0 e^{rt}$ with condition that B_0=1. We thus analyze the simple case of the process Y_t by applying binomial trees. That is, when N=2.

Consider the following;



Where;

u represents the value of the delinquency index Y_1 when the borrower defaults his payment and d represents the value the delinquency index Y_1 when the borrower pays his credit payment. p_0 denotes the probability that borrower defaults payments for time 0. Consider $f(Y_t)$ defined above to represent compensation by the insurer due to the value of the delinquency index at time 1. Thus we have the following figure:



From the above figure,we can obtain the expected present value of the compensation by the insurance using the following formula:

$$E = E[f(Y_t)] = e^{-r}[p_0 * f(u) + (1 - p_0) * f(d)] \tag{3.1}$$

Where;

$f(u)$ repesents compensation paid by an Insurance company in which the borrower defaults payment and

$f(d)$ repesents compensation paid by an Insurance company in which the borrower pay the corresponding payment of his credit.

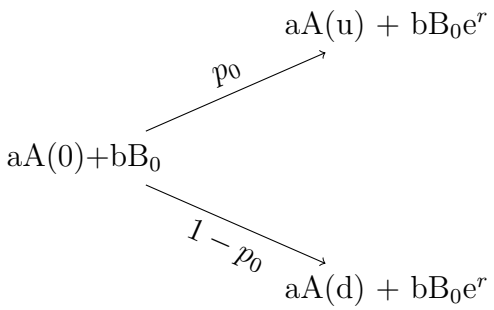
Thus by Kolmogorov's strong law of large numbers, if the insurance company has a large number of portfolio, then that company can expect a loss given by formula (3.1).

3.2 Change of probability measure and free arbitrage valuation

We can also arrive at equation (3.1) through construction of replicating assets portfolio. The technique of construction of replicating asset portforlios is widely used in the valuing of financial derivatives. An extension of this is explained further in the subsequent sections. Based on the examples of the binomial tree shown previously, we assume that there exists a financial asset A(Y_t) that depends on the delinquency index.
Suppose that an insurer who has a benefit payment of f(Yt) wants to hedge its losses by replicating them using $\boldsymbol{a}$ instruments of the asset A and $\boldsymbol{b}$ cash bonds of B. Thus the insurer is willing to posess a portfolio, C_t, constructed in the following manner:

$$C_t = aA(Y_t) + bB_t \tag{3.2}$$

Thus, based on the binomial tree example given above, we can represent the value of that portfolio C_t diagramatically as:



Remember the insurance company wants to replicate the corresponding losses *f(u)* and *f(d)* with this portfolio. Thus the company has to have a and b satisfying the following equations:

$$aA(u) + bB_0e^r = f(u) \ .....(i)$$

and

$$aA(d) + bB_0e^r = f(d) \ .....(ii)$$

Solving equations (i) and (ii) simultaneously, we have:

$$a = \frac{f(u) - f(d)}{A(u) - A(d)}$$

and

$$b = \frac{1}{e^r B_0}[f(u) - (\frac{f(u) - f(d)}{A(u) - A(d)})A(u)]$$

We can therefore obtain the value of the portfolio at time 0 by substituting a and b in equation (3.2). Therefore we have:

$$C_0 = e^{-r}[(\frac{A(0)e^r - A(d)}{A(u) - A(d)})f(u) + (\frac{A(u) - A(0)e^r}{A(u) - A(d)})f(d)]$$

Moreover if,

$$q_0 = \frac{A(0)e^r - A(d)}{A(u) - A(d)} \tag{3.3}$$

we have:

$$C_0 = e^{-r}[q_0 f(u) + (1 - q_0)f(d)] \tag{3.4}$$

Where q represents the probability due to A(Y_t). Thus equation (3.4) denotes another expected present value of the losses of an insurance company through the use of a technique of asset-liability matching(free arbitrage valuation). The above calculations allow the insurer to adjust the expected claims by using a portfolio which allows to replicate future losses. Based on "q" definition above,for the no arbitrage condition to hold, we have;

$$A(u)<A(0)e^r<A(d) \quad \} \ , 0<q<1$$

3.3 Generalization of the Binomial model

In order to generalize the binomia model in the last section, we need to get aquainted with the following concepts.

- **Conditional expectation**. It's a a technique for addition of information to the expected value of a stochastic process denoted by $E_\pi[Y_t|\mathcal{F}_t]$. Where π is a probability measure ($\mathbb{P}$ or $\mathbb{Q}$).

- **Filtration.** A filtration F = $(\mathcal{F}_t)_{t=0,1,2...}$ is the information available up to and including each time of a process. In this project, the filtration will represent the information of the delinquency index in time.

- **Financial Claim.** Is the function *f(.)*, that represents compensation by the insurance company in this project.

- **Adaptive and previsible processes**: A stochastic process Y_t is adaptive if given the filtration $\mathcal{F}_t$ we can determine the value of Y_s and a process is previsible if given the filtration $\mathcal{F}_t$ we can determine the value of Y_{t+1}. That is, a stochastic process Y_t is adaptive if

$$E_\pi[Y_t|\mathcal{F}_t] = Y_t$$

or previsible(capable of being predicted) if

$$E_\pi[Y_t|\mathcal{F}_{t-1}] = Y_t$$

- Martingale: An adaptive process Y_t to the filtration $\mathcal{F}i$ is a martingale under the probability measure π if $E_\pi[Y_t|\mathcal{F}_s] = Y_s$ for $t \geq s$

Based on the example presented in the previous section, the existence of the probability measure $\mathbb{Q}$ is equivalent to state that the present value of the asset that depends on the delinquency index $A(Y_t)$, is a martingale. We can base this argument from formula 3.4:

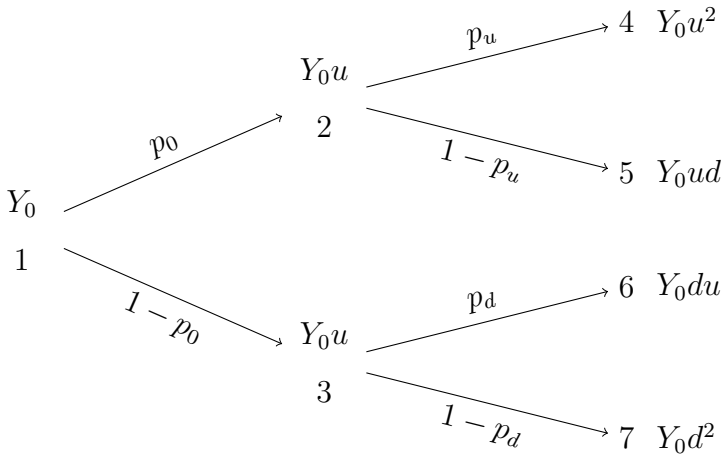
$$q_0 = \frac{A(0)e^r - A(d)}{A(u) - A(d)}$$

$$q_0 A(A(u) - A(d)) = A(0)e^r - A(d)$$

$$A(0) = q_0 e^{-r} A(u) + (1 - q_0)e^{-r} A(d)$$

$$= E_\mathbb{Q}[e^{-r} A(Y_t)|\mathcal{F}_0] = e^0 A(0)$$

We will use the binomial tree of two steps to generalize the binomial model.In this case we will do our analyzation by considering the case when N = 3, (t=0,1,2) under the probability measures $\mathbb{P}$ and $\mathbb{Q}$.
Where N is the loan term.



Based on the diagram above we can name the outcomes of the delinquency index and its proportion of upward and downward movement based on the path followed by the process, for example, *ud* shows the outcome when the process had an upward movement in the first time and then a downward movement in the second step. We can name the filration and the corresponding probabilities as follows; At time 0, the filtration of the process will be, $\mathcal{F}_0 = \{1\}$ At time 1, the filtration of the process will be $\mathcal{F}_1 = \{1, 2\}$ or $\mathcal{F}_1 = \{1, 3\}$. At time 2, the filtration is:

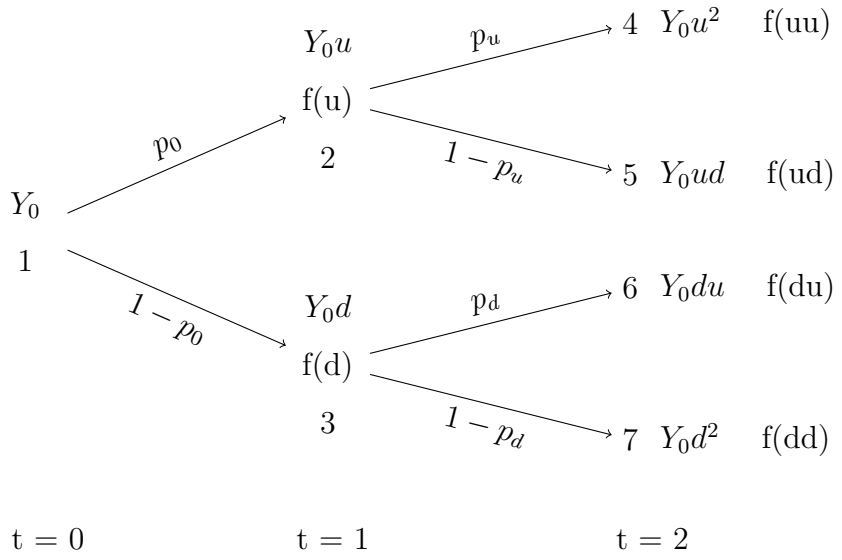
$$\mathcal{F}_2 =_{\{1,2,4\}} or \mathcal{F}_2 =_{\{1,2,5\}} or \mathcal{F}_2 =_{\{1,3,6\}} or \mathcal{F}_2 =_{\{1,3,7\}}$$

Thus the corresponding probabilities will be;

$$p_{\{1\}} = 1 : p_{\{1,2\}} = p_0 : p_{\{1,3\}} = 1 - p_0 : p_{\{1,2,4\}} = p_0 p_u : p_{\{1,2,5\}} = p_0(1 - p_u) :$$

$p_{\{1,3,6\}} = (1 - p_0)p_d : p_{\{1,3,7\}} = (1 - p_0)(1 - p_d)$

If we apply the compensation(insurance) function *f(.)* to each of the branches of the tree, we have:



In our generalization and considering the methods in the previous section, we can calculate the expected present value of the possible losses of the insurance company which issues *f(.)* as follows. At time 1, we have;

$$E_{\mathbb{P}\{1,2\}} = e^{-r} E_{\mathbb{P}}[f(Y_2)|\mathcal{F}_1 = \{1,2\}] = e^{-r}(p_u f(uu) + (1-p_u)f(ud)) \tag{3.5}$$

$$E_{\mathbb{P}\{1,3\}} = e^{-r} E_{\mathbb{P}}[f(Y_2)|\mathcal{F}_1 = \{1,3\}] = e^{-r}(p_d f(du) + (1-p_d)f(dd)) \tag{3.6}$$

Where$E_{\mathbb{P}\{1,2\}}$ is the expected present value in the tree node given by the filtration {1,2}, under the probability measure $\mathbb{P}$ and $E_{\mathbb{P}\{1,3\}}$ is the expected present value in the tree node given by the filtration {1,3}. At time 0, we can estimate the expected present value of the losses of the company at time as follows;

$$E_{\mathbb{P}\{1\}} = e^{-r} E_{\mathbb{P}}[f(Y_1)|\mathcal{F}_0 = \{1\}] = e^{-r}(p_0 E_{\mathbb{P}\{1,2\}} + (1-p_0)E_{\mathbb{P}\{1,3\}}) \tag{3.7}$$

We can now apply the method of replicating asset portfolio in order to make change of probability measure. This will enable us find another way of expressing the expected present value of the losses to the insurance company:

a) Suppose an insurance company wishes to match its losses by using the following portfolio:

$$C_t = a_t A(Y_{t+1}) + b_t B_t$$

Where $A(Y_t)$ is an asset whose value depends on the delinquency index Y_t and B_t is a cash bond that has the risk-free rate r.

b) Then the above portfolio C_t should replicate the losses of the insurance company. Thus we have;

$$a_t A(Y_{t+1})|\mathcal{F}_t) + b_t B_0 e^r = f(Y_{t+1}|\mathcal{F}_t) \tag{3.8}$$

From equation (3.8) we have;

$$a_t A(Y_{t+1}|\mathcal{F}_t^+) + b_t B_0 e^r = f(Y_{t+1}|\mathcal{F}_t^+) .....(i)$$

and

$$a_t A(Y_{t+1}|\mathcal{F}_t^-) + b_t B_0 e^r = f(Y_{t+1}|\mathcal{F}_t^-) .....(ii)$$

solving equations (i) and (ii) simultaneously,we obtain;

$$a_t = \frac{f(Y_{t+1}|\mathcal{F}_t^+) - f(Y_{t+1}|\mathcal{F}_t^-)}{A(Y_{t+1}|\mathcal{F}_t^+) - A(Y_{t+1}|\mathcal{F}_t^-)}$$

$$b_t = \frac{1}{e^r B_t}[f(Y_{t+1}|\mathcal{F}_t^+) - (\frac{f(Y_{t+1}|\mathcal{F}_t^+) - f(Y_{t+1}|\mathcal{F}_t^-)}{A(Y_{t+1}|\mathcal{F}_t^+) - A(Y_{t+1}|\mathcal{F}_t^-)})A(Y_{t+1}|\mathcal{F}_t^+)]$$

Where; $\mathcal{F}_t^+$ is the filtration at time t when in the last path of that filtration there was a rise in the value of the delinquency index, and $\mathcal{F}_t^-$ is the filtration at time t when in the last step of that filtration there was a decrease in the value of the delinquency index. If we substitute a_t and b_t in C_t we obtain:

$$C_t = e^{-r}[q_t f(Y_{t+1}|\mathcal{F}_t^+) + (1 - q_t) f(Y_{t+1}|\mathcal{F}_t^-)]$$

with

$$q_t = \frac{A(Y_t)e^r - A(Y_{t+1}|\mathcal{F}_t^-)}{A(Y_{t+1}|\mathcal{F}_t^+) - A(Y_{t+1}|\mathcal{F}_t^-)} \tag{3.9}$$

As explained above, $0<q_t<1$ because of;

$$A(Y_{t+1}|\mathcal{F}_{t+1}^+)<A(Y_t)e^r<A(Y_{t+1}|\mathcal{F}_{t+1}^-)$$

We thus can be able to change the probability measure from $\mathbb{P}$ to $\mathbb{Q}$. Formulas (3.5), (3.6) and (3.7) will therefore turn into:

$$E_{\mathbb{Q}\{1,2\}} = e^{-r}(E_{\mathbb{Q}}[f(Y_2)|\mathcal{F}_1) = \{1,2\}]) = e^{-r}(q_u f(uu) + (1 - q_u) f(ud)) \tag{3.10}$$

$$E_{\mathbb{Q}\{1,3\}} = e^{-r}(E_{\mathbb{Q}}[f(Y_2)|\mathcal{F}_1) = \{1,3\}]) = e^{-r}(q_d f(du) + (1 - q_d) f(dd)) \tag{3.11}$$

$$E_{\mathbb{Q}\{1\}} = e^{-r}(E_{\mathbb{Q}}[f(Y_2)|\mathcal{F}_0) = \{1\}]) = e^{-r}(q_0 E_{\mathbb{Q}}[f(Y_2)|\mathcal{F}_1 = \{1,2\}] + (1-q_0)E_{\mathbb{Q}}[f(Y_2)|\mathcal{F}_1) = \{1,3\}]) \tag{3.12}$$

with;

$$q_0 = \frac{A(Y_0)e^r - A(Y_1|\mathcal{F}_0^-)}{A(Y_1|\mathcal{F}_0^+) - A(Y_1|\mathcal{F}_0^-)} = \frac{A(0)e^r - A(d)}{A(u) - A(d)}$$

$$q_u = \frac{A(Y_1)e^r - A(Y_2|\mathcal{F}_1^-)}{A(Y_2|\mathcal{F}_1^+) - A(Y_2|\mathcal{F}_1^-)} = \frac{A(u)e^r - A(ud)}{A(uu) - A(ud)}$$

$$q_d = \frac{A(Y_1)e^r - A(Y_2|\mathcal{F}_1^-)}{A(Y_2|\mathcal{F}_1^+) - A(Y_2|\mathcal{F}_1^-)} = \frac{A(d)e^r - A(dd)}{A(du) - A(dd)}$$

We illustrated in the previous examples that the change in the probability measure to $\mathbb{Q}$ implied that the present value of the asset $A(Y_t$ is a martingale. The follwoing is now the generalization of the results:

$$q_t = \frac{A(Y_t)e^r - A(Y_{t+1}|\mathcal{F}_t^-)}{A(Y_{t+1}|\mathcal{F}_t^+) - A(Y_{t+1}|\mathcal{F}_t^-)}$$

$$A(Y_t) = q_t e^{-r} A(Y_{t+1}|\mathcal{F}_t^-) + (1-q_t)e^{-r}A(Y_{t+1}|\mathcal{F}_t^-)$$

$$= E_{\mathbb{Q}}[e^{-r}A(Y_{t+1})|\mathcal{F}_t] = e^0 A(Y_t)$$

This is a very significant result that will be used in the following chapters.

3.4 The Continuous model

In this section we bring to table concepts that will enable us model the delinquency index in continous time.

a) **Brownian motion:**

Let's modify the binomial tree shown in the previous sections. If we take changes in time to correspond to $\frac{1}{N}$ where N is the term of the loan (asset-based) and suppose that the borrower can increase or decrease its "delinquency index" y=$\sqrt{\frac{1}{N}}$ with probability p= $\frac{1}{2}$. Let x_i be a random variable:

$$X_i = \begin{cases} 1 \text{ with probability p} = \frac{1}{2} \\ -1 \text{ with probability 1-p} = \frac{1}{2} \end{cases}$$

Therefore the delinquency index at time t can be written down as:

$Y_t = x_1y + x_2y + x_3y + ... + x_ty = x_1\sqrt{\frac{1}{N}} + x_3\sqrt{\frac{1}{N}} + ... + x_t\sqrt{\frac{1}{N}}$

Based on the central limit theorem: Y_t ∼N(0, t).

The unconditional probability density function which follows Y_t ∼N(0, t) at a fixed time t for Wiener process (Brownian Motion process)is given by;

$$f(W_t(x)) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

The expectation is zero,

$$E(W_t) = 0$$

$$E^2[W_t] = E[W_t^2] - 0 = E[W_t^2] = t$$

Thus

$$Y_t \sim N(0, t)$$

b) **The ltô's processes;**

Suppose that there exists a real number Y_0 and two adapted process $\mu\, and\, \sigma$ such that the following relation holds for all $t \geq 0$, then, Y is an ltô process;

$$dY_t = \sigma_t dW_t + \mu_t dt \tag{3.13}$$

Where:

σ_t is a stochastic process which represents the volatility of Y, μ_t is the drift of Y. Additionally, the ltô process in equation 3.13 can be written in integral form as;

$$Y_t = Y_0 + \int_0^t \sigma_s dWs + \int_0^t \mu_s ds \tag{3.14}$$

The ltô process will be a martingale if the drift μ_t is equal to zero. The special case of ltô processes can be diffusion processes in such away that the drift and volatility are a function of the time and of the stochastic process itself as shown;

$$dY_t = \sigma(t, Y_t)dW_t + \mu(t, Y_t)dt \tag{3.15}$$

We thus consider the the markovian property of the diffussion processes that is, if for any function g:

$$E_{\mathbb{P}}[g(X_t|\mathcal{F}_s)] = E_{\mathbb{P}}[g(X_t)|X_s],\ for\, t > s$$

Another stochastic differential equations(SDEs) widely used are the Homogeneous Brownian Motion which is given by:

$$dY_t = \sigma dW_t + \mu dt$$

Where σ and μ are constants. Another very important process to be used in this work is the Geometric Brownian Motion.

Geometric Brownian Motion

It's a continuous time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian Motion with drift. Thus a stochastic process is said to follow a Geometric Brownian Motion if it satisfies the following stochastic differential equation;

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t$$

Where W_t is a Wiener process (Brownian Motion) and (μ the drift) and (σ the volatility) are constants. The expression of this diffusion is

$$\frac{dY_t}{Y_t} = \sigma dW_t + \mu dt \tag{3.16}$$

We can solve this stochastic differential equation (3.16) by applying ltô's lemma (shown below); This model is very important in the formulas developed further.

c) **ltô's lemma**

Assume that Y has a stochastic differential given by

$$dY_t = \mu_t dt + \sigma_t dW_t \tag{3.17}$$

Where μ and σ are adapted processes. Define the process Z by $Z_t = f(t, Y_t)$. Then Z has a stochastic differential given by

$$df(t, Y_t) = \{\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2}\}dt + \sigma \frac{\partial f}{\partial y} dW_t \tag{3.18}$$

Proof. Taking Taylor expansion including second order terms, we obtain;

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}dY + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial y}dtdY$$

Squaring equation (3.17) we obtain;

$$(dY)^2 = \mu^2(dt)^2 + 2\mu\sigma dt dW + \sigma^2(dW)^2 \tag{3.19}$$

Substituting equation (3.17) and (3.19) into the Taylor expansion taking into account the following conditions,

$$(dt)^2 = 0$$

$$dt \times dW = 0$$

$$(dW)^2 = dt$$

We obtain the result (3.18) □

We can obtain the solution to the Geometric Brownian Motion SDE shown above by applying Itô's lemma as follows;

$$d(\ln Y_t) = (\ln Y_t)' dY_t + \frac{1}{2}(\ln Y_t)''(dY_t)^2 \tag{3.20}$$

$$= \frac{dY_t}{Y_t} - \frac{1}{2}\frac{1}{Y_t^2}(dY_t)^2 \tag{3.21}$$

Where $(dY_t)^2$ is the quadratic variation of the SDE

$$(dY_t)^2 = \mu^2 {Y_t}^2 (dt)^2 + 2\sigma {Y_t}^2 \mu dW_t dt + \sigma^2 (Y_t)^2 (dWt)^2$$

$$Assuming;$$

$$(dt)^2 = 0$$

$$dt.dW = 0$$

$$(dW)^2 = dt$$

Thus $(dY_t)^2 = \sigma^2(Y_t)^2 dt$

Substituting the value of dY_t and $(dY_t)^2$ in equation 3.21 we obtain;

$$d(\ln Y_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

Written in the integral form,this leads to $\ln Y_t = Y_0\, exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right]$

Thus this equation can also be written as;

$$Y_t = Y_{t0} e^{\sigma(W_t - W_{t0}) + (\mu - \frac{1\sigma^2}{2})(t - t_0)} \tag{3.22}$$

3.5 Credit Insurance and reserving

In this study we are looking at credit insurance in the case of asset-based lending where a borrower borrows money to buy a car or a house. Thus the definition for credit insurance becomes; a financial tool for transferring credit risk of a credit from a financial institution to an insurance company. The financial institution has to pay a premium and the insurance company will pay a percentage of the outstanding balance of the loan plus interests if there is a default. In othis case Credit insurance will pay the benefit only when the borrower defaults in their payments and the financial institution takes over or recover the underlying car or house (collateral) for that loan .

We can explain reserving by using the concepts shown in the previous chapter such as filtration and the markovian properties. Therefore if we denote the loss reserve at time t as OCR_t, X the random variable representing the losses, and $\mathcal{F}_t$ the filtration of the delinquency index we can write

$$OCR_t = g(\mathcal{F}_t) = E(X|\mathcal{F}_t)$$

We can say that this OCR_t exhibits markovian property.We can build it further in the following sections.

3.6 Black- Scholes constructs and forecasts

Assume that the delinquency index can be modeled as a geometric Brownian motion, we will have;

$$\frac{dY_t}{Y_t} = \sigma dW_t + \mu dt \tag{3.23}$$

$$\Leftrightarrow Y_t = Y_{t_0} e^{\sigma(W_t - W_{t_0}) + (\mu - \frac{1\sigma^2}{2})(t - t_0)} \tag{3.24}$$

$$E_{\mathbb{P}}[Y_t|Y_{t_0}] = Y_{t_0}e^{\mu(t-t_0)}$$

Consider cash bond $\boldsymbol{B}_t$ as the stochastic process that give the risk-free rate, that is:

$$\frac{dB_t}{B_t} = rdt \Leftrightarrow B_t = B_{t_0}e^{r(t-t_0)} \tag{3.25}$$

Rememeber we need an asset that depends on the delinquency index whose present value is a martingale. Denote that asset as A(Y_t) and we suppose that its linearly proportional to the delinquency index as follows;

$$A(Y_t) = A_t * Y_t$$

Let's also denote D=(D_t)t ≥ 0 to be the stochastic process representing the present value of A_t:

$$D_t = B_t^{-1}A(Y_t) = B_t^{-1}A_tY_t$$

When we apply ltô's lemma to D_t we get:

$$\frac{dDt}{Dt} = \sigma dWt + (\mu - r)dt \tag{3.26}$$

Proof. Recall that $dY_t = \mu Y_t dt + \sigma Y_t dW_t$

$$\begin{aligned}
dD_t &= d(B_t^{-1}A_tY_t) \\
B_t = exp(rt)B_t^{-1} &= exp(-rt) \\
dB_t &= rexp(rt)dt \\
dB_t^{-1} &= -rexp(-rt)dt \\
dD_t = B_t^{-1}A_tdY_t + B_t^{-1}Y_tdA_t + dB_t^{-1}Y_tA_t & \\
B_t^{-1}A_t(\mu Y_tdt + \sigma Y_tdW_t) + Y_tA_t(-rB_t^{-1}) &
\end{aligned}$$

$$B_t^{-1}A_t\mu Y_tdt + B_t^{-1}A_t\sigma Y_tdW_t - Y_tA_trB_t^{-1}dt \tag{3.27}$$

But $D_t = B_t^{-1}A_tY_t$

Thus equation 3.27 becomes;

$$\frac{dD_t}{D_t} = (\mu - r)dt + \sigma dW_t \ as \ above$$

□

Based on the construction of asset portfolio strategy we were able to make change of probability measure. This implied the present value of the asset which depends on the delinquency index was a martingale. Therefore, we will make a change in probability measure from $\mathbb{P}$ to $\mathbb{Q}$ on the process D_t in order to make D_t to be a martingale. Based on the Cameron - Martin - Girsanov theorem, for each probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and if we have a previsible process $\gamma = (\gamma_t)$ then: $\tilde{W}_t = W_t + \int_0^t \gamma_s$ is also $\mathbb{Q}$ Brownian motion with $d\tilde{W}_t = dW_t + \gamma_t dt$.

Substituting in (3.26F) we have:

$\frac{dDt}{Dt} = \sigma\, d\tilde{W}_t + (\mu - r - \gamma_t\sigma)dt$

for D_t be a martingale we need $(\mu - r - \gamma_t\sigma) = 0$ since *drift* equals 0. The solution of this equation is the Market price of risk;

$$\gamma_t = \frac{\mu - r}{\sigma} \tag{3.28}$$

But D_t is a martingale;

$$\frac{dD_t}{D_t} = \sigma d\tilde{W}_t \tag{3.29}$$

Considering the concepts shown so far, the delinquency index under the probability measure $\mathbb{Q}$ is:

$$\frac{dY_t}{Y_t} = \sigma d\tilde{W}_t + rdt$$

Proof. Recall

$$\frac{dY_t}{Y_t} = \sigma dW_t + \mu dt \tag{3.30}$$

$$\begin{aligned} But\ dW_t &= d\tilde{W}_t - \gamma_t dt \\ and\ from\ \gamma_t &= \frac{\mu - r}{\sigma} \\ r &= \mu - \sigma\gamma_t \end{aligned}$$

Thus equation (3.29) becomes

$$\begin{aligned} &= \sigma(d\tilde{W}_t - \gamma_t dt) + \mu dt \\ &= \sigma d\tilde{W}_t - \sigma\gamma_t dt + \mu dt \\ &= \sigma d\tilde{W}_t + (\mu - \sigma\gamma_t)dt \\ &= \sigma d\tilde{W}_t + rdt \end{aligned}$$

as above □

Thus under the probability measure $\mathbb{Q}$ the drift of the delinquency index is the risk-free rate. Solving this stochastic differential equation, we can have estimate of the delinquency index;

$$Y_t = Y_{t_0} e^{(\sigma \tilde{W}_t - \tilde{W}_{t_0}) + (r - \frac{1\sigma^2}{2})(t - t_0)} \tag{3.31}$$

This is an important result which will enable us forecast future losses of a credit insurance which pays f(T):

$$OCR_t = E_{\mathbb{Q}}[B_u^{-1} f(T) | \mathcal{F}_{t_0}] \tag{3.32}$$

Where:

OCR_t is the reserve at time t given information of the filtration $\mathcal{F}_{t_0}$.

B_t = Is cash bond which give the risk-free rate r.

$T = t + u$ and $u = T - t$

f(T) = Is the compensation by the insurance company. T = Is the random variable that represents the time to paying the sum insured. Remember, the credit insurance pays claims when the car or house is taken over. In the case the outstanding loan balance upon default is more than the value of the collateral, the insured lender will exercise the right to sell the collateral at the outstanding loan balance. This is achieved by the lender receiving from the insurer the difference of the outstanding loan and the market value of the car or house at the time of default. The contract is therefore settled by the difference.

In this case, the total cost of the policy holder will be the premium paid and the value of the collateral upon default. In addition to the time of valuation, t we must consider another random variable u representing the time to repossession of the car or hous and thus T = t + u as denoted above.

Remember that we have **3 time functions** in equation (3.31): **T**, t and t_0. The time to repossess a car or a house is not certain, it can take more than a year. That's why we use the random variable u. Moreover, the financial institutions which insures itself with the insurance company have delays in giving out the information of the delinquency index and this gives the reason for use of t_0 which is about one month or two. Based on these reasons, we are going to forecast the future losses of credit insurance at time t,

with the information given by the financial institution at time t_0 with the filtration $\mathcal{F}_{t0}$. We can see that $t_0 < t < T$. Consider t_0 to be independent of t,we can develop equation (3.31). Consider *f(T)* to be the function that represents compensation by the insurance company(assuming that the car or the house is recovered after R months defaultable period). It will be defined as a percentage of the outstanding balance at time of recovering the car or house if the delinquency index is greater than R. Where R is the delinquency index threshold.That is:

$$f(T) = \begin{cases} \%Cov * (OB)I_T \, for \, Y_t \geqslant R \\ \\ 0 \, for \, Y_t < 0 \end{cases} = \%Cov * (OB)I_T * I_{Y_t} \geq R$$

%Cov = Is the percentage of the covered outstanding balance by the inurance company.
$(OB)I_T$ = Is the outstanding balance of the credit at time T.R is the delinquency index threshold.
I_{Y_t} = Is an indicative random variable:

$$I_{Y_t} = \begin{cases} 1 \; if \; event \; Y_t \; occurs \\ 0 \; if \; event \; Y_t \; does \; not \; occur \end{cases}$$

Then, we have;

$$OCR_t = E_{\mathbb{Q}}[B_u^{-1} f(T)|\mathcal{F}_{t_0}] = E_{\mathbb{Q}}[B_u^{-1} * \%Cov * (OB)I_T * I_{Y_t \geq R}|\mathcal{F}_{t_0}]$$

Since Y_t is modelled as Geometric Brownian motion, it conforms to a diffusion process, and it thus possess the markovian property. Consequently we can reorganize the terms in the above equation as:

$$OCR_t = E_{\mathbb{Q}}[B_u^{-1} * \%Cov * (OB)I_T * I_{Y_t \geq R}|\mathcal{F}_{t_0}] = E_{\mathbb{Q}}[B_u^{-1} * \%Cov * (OB)I_T * I_{Y_t \geq R}|Y_{t_0}] \tag{3.33}$$

$$OCR_t = \%Cov * E_{\mathbb{Q}}[B_u^{-1} * (OB)I_T|Y_{t0}] * E_{\mathbb{Q}}[I_{Y_t \geq R}|Y_{t_0}] = \%Cov * E_{\mathbb{Q}}[B_u^{-1} * (OB)I_T] * \mathbb{Q}[Y_t \geq R|Y_{t_0}] \tag{3.34}$$

Since Y_t is a Geometric Brownian Motion, we have;

$$Y_t = Y_{t_0} e^{\sigma(\tilde{W}_t - \tilde{W}_{t_0}) + (r - \frac{1\sigma^2}{2})(t - t_0)} = Y_{t_0} e^{\sigma\sqrt{t - t_0} * \epsilon + (r - \frac{1\sigma^2}{2})(t - t_0)}, \; with \, \varepsilon \sim N(0, 1)$$

$$Y_t \geq R \Leftrightarrow \ln Y_t = \ln Y_{t_0} + \sigma\sqrt{t-t_0} * \varepsilon + (r - \frac{1}{2}\sigma^2)(t-t_0) \geq \ln R$$

$$\eta = -\varepsilon \leq \frac{\ln \frac{Y_{t_0}}{R} + (r - \frac{1}{2}\sigma^2)(t-t_0)}{\sigma\sqrt{t-t_0}},\ \eta \sim N(0,1)$$

Taking the notation of the Black-Scholes we have;

$$d_2 = \frac{\ln \frac{Y_{t_0}}{R} + (r - \frac{1}{2}\sigma^2)(t-t_0)}{\sigma\sqrt{t-t_0}}$$

d_2 captures the idea of credit risk in the Merton Model. It denotes the probability of default. Formula (3.34) turns into:

$$OCR_t = \%Cov * E_{\mathbb{Q}}[B_u^{-1} * (OB)I_T] * \mathbb{Q}[\eta \leq d_2 | Y_{t_0}] = \%Cov * E_{\mathbb{Q}}[B_u^{-1} * (OB)I_{t+u}] * \Phi(d_2) \tag{3.35}$$

We thus have managed to obtain the formula to forecast loss reserves in credit insurance if we know just the percentage of outstanding balance covered by the insurer,the value of the delinquency index, a risk-free rate, and the distribution of the random variable u. We can write equation (3.37)as:

$$OCR_t = \%Cov * (\int_0^{n-t} e^{-ru}(OB)I_{t+u}p(u)du) * \Phi(d_2) \tag{3.36}$$

Where:

p(u) is the probability density function of u.

n is the loan term.

$(OB)I_{t+u}$ is the outstanding balance of the loan at time t plus interest at the rate c:

$$(OB)I_{t+u} = (OB)I_t * (1+tc)^u$$

We can simplify this calculation by taking u as a constant:

$$OCR_t = \%Cov * e^{-ru} * (OB)I_{t+u} * \Phi(d_2) \tag{3.37}$$

Since this formula was obtained under the probability measure $\mathbb{Q}$ we have a replicating portfolio P_t, which matches the future losses of the insurance company. Thus:

$$P_t = \phi_t A(Y_t) + \psi_t B_t = \phi_t A_t Y_t + \psi_t B_t \tag{3.38}$$

$$dP_t = \phi_t A_t dY_t + \psi_t dB_t \tag{3.39}$$

Where; ϕ_t denotes number of units of $A(Y_t)$.

ψ_t denotes number of units of the cash bond. Let G_t be the process representing the present value of the portfolio:

$$G_t = B_t^{-1} P_t = \phi_t D_t + \psi_t \tag{3.40}$$

Considering the following equation and applying Ito's lemma we have;

$$OCR_t = h(t,x) = E_{\mathbb{Q}}[B_u^{-1} * \%Cov * (OB) I_{t+u} * I_{Y_t \geq R} | Y_{t_0} = x]$$

$$\phi_t A_t = \frac{d}{dY_t}(\%Cov * e^{-ru} * (OB) I_{t+u} * \Phi(d_2)) = \%Cov * e^{-ru} * (OB) I_{t+u} * \frac{d}{dY_t}(\Phi(d_2))$$

$$= (\%Cov * e^{-ru} * (OB) I_{t+u} * \frac{1}{\sqrt{2\pi}} e^{\frac{d_2^2}{2}})$$

ψ_t can be obtained from (3.40). We've therefore found an alternative method for projecting losses in credit insurance products and a replicating portfolio to match them.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Analysis and Discussions

We used data simulation in this project as a means of explaining our model. The geometric brownian motion simulation for instance has been used to estimate the probability of default which has been incoporated in estimation of loss reserve in credit insurance for asset based lending companies (see appendix).

The outstanding balances of 30% of 100 loanees from car loan business with assets valued at between 1 million and 10 million were simulated and fitted to the model. The result of the future loss was estimated by considering the following assumptions:

Assumed Constants	**Values**
Risk Free Rate (Annual)	9.195%
Volatility (Monthly)	12%
Risk Free Rate (Monthly) r	0.0883%
Initial Delinquency (Monthly)	6 months
u	5 months
tc	12%
% Coverage	25%

Table 4.1: Table showing the assumed constants.

u is a random variable representing the time to recover the collateral while c is a simple interest and t is the time of valuation. Since most car loan businesses have a loan term of 5 years, we assumed this in our simulation.

Table 4.2: Estimates of the delinquency index (Y_t)

$(\tilde{W}_t - \tilde{W}_{t_0}) \sim N(0,1)$	log return $(\frac{Y_t}{Y_{t-1}})$	Estimate of (Y_t)	Time$(t - t_0)$
0.930	0.048	6.297	10
-1.162	-0.437	3.878	47
0.062	-0.163	5.096	27
1.980	0.142	6.921	15
-2.382	-0.614	3.246	52
-0.566	-0.137	5.230	11

The estimates of the delinquency index were estimated as follows:

$$\frac{Y_t}{Y_{t-1}} = (r - 1/2\sigma^2)(t - t_0) + \sigma(\tilde{W}_t - \tilde{W}_{t0})$$

$$Y_t = Y_{t_0} e^{\sigma(\tilde{W}_t - \tilde{W}_{t_0}) + (r - \frac{1\sigma^2}{2})(t - t_0)}$$

Table 4.3: Estimate of default probability d_2

Y_t Monthly	d_2	$\Phi(d_2)$	Time (t-t_0)
6.297	-0.039	0.484	10
3.878	-0.892	0.186	47
5.097	-0.535	0.296	27
6.921	0.103	0.541	15
3.246	-1.09	0.138	52
5.23	-0.51	0.302	11

The default probability d_2 was obtained using the following formula:

$$d_2 = \frac{\ln \frac{Y_{t_0}}{R} + (r - \frac{1}{2}\sigma^2)(t - t_0)}{\sigma\sqrt{t - t_0}}$$

Table 4.4: OCR_t Estimates

$\Phi(d_2)$	$(OB)I_{t+u}$	OCR_t	Time (t-t_0)
0.484	10,655,530.20	826,743.89	10
0.186	4,302,346.59	367,103.35	47
0.296	1,120,095.03	73,176.88	27
0.541	7,263,327.35	492,791.52	15
0.138	7,280,272.26	311,157.83	52
0.302	13,466,898.73	1,321,563.40	11

The OCR_t was obtained as: $OCR_t = \%Cov * e^{-ru} * (OB)I_{t+u} * \Phi(d_2)$

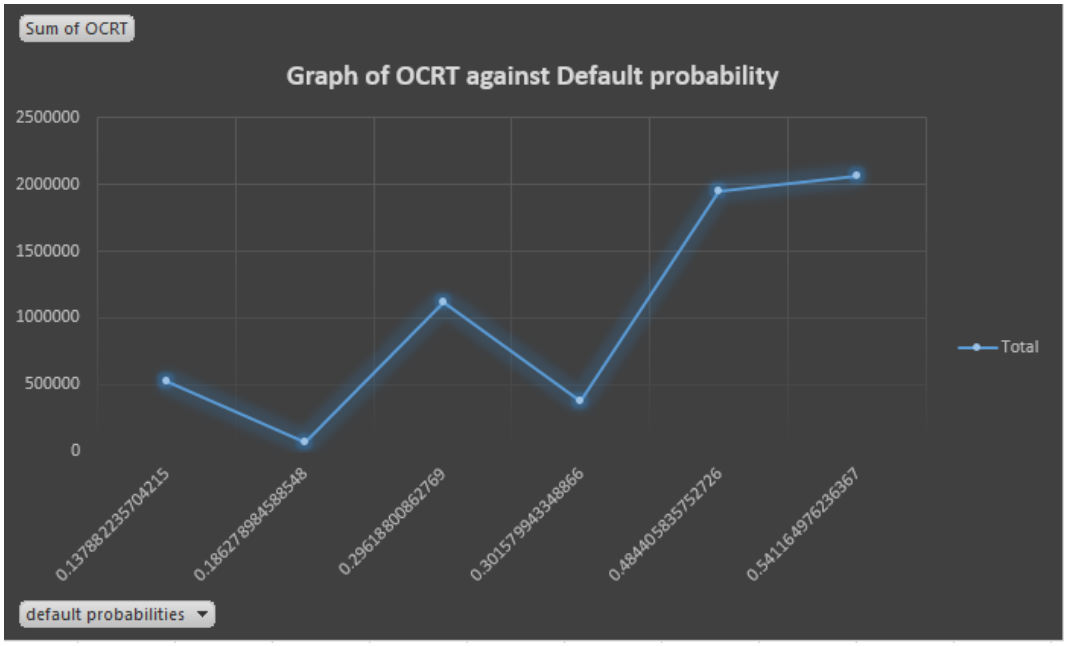


Figure 4.1: *Graph of OCR_t against Default probability.*

From table 4.4, it's clear that an increase in the value of the probability of default increases the value of the reserve estimated depending on the outstanding balance of the borrower. We can therefore deduce that a borrower with higher chances of default will make the insurance company to have longer reserve requirements. Despite this, it's also possible to have shorter reserve requirements as also seen in figure 4.1 An increase in

the loan term or the time to maturity of the payment of the insurance has the effect of reducing the probability of default by the borrower (see Table 4.3). This will result to shorter reserve requirements by the insurance company.

The number of defaultable periods(delinqunecy index) has the effect of increasing the estimate of future reserve by the insurance company. Therefore if an insurance company insures policies with a large number of defaultable periods, it will have longer reserve requirements otherwise it will have shorter reserve requiremnt (see figure 4.2)

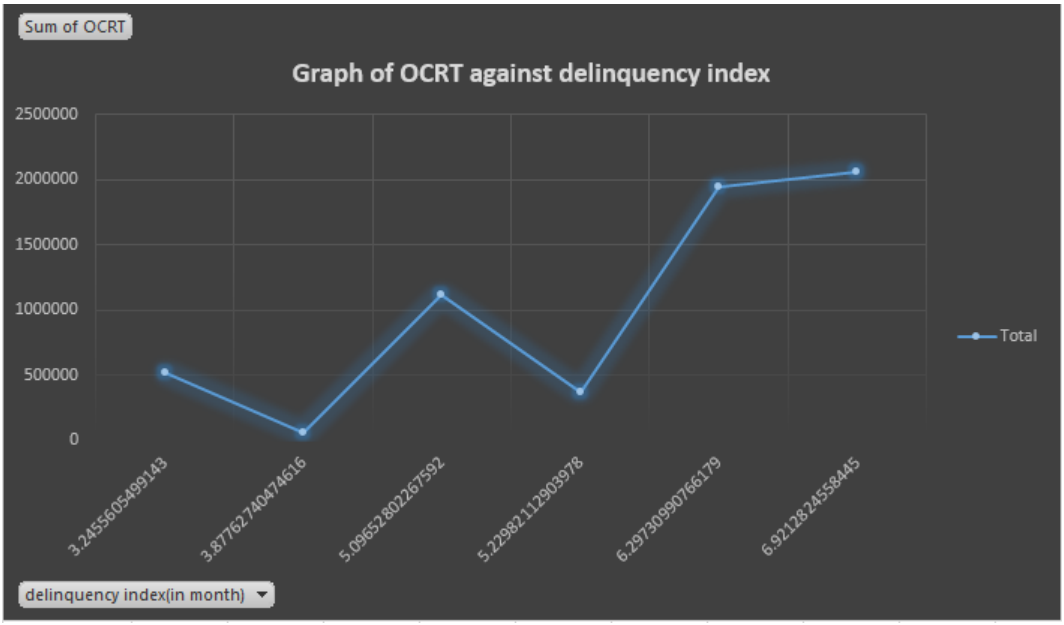


Figure 4.2: *Graph of OCR_t against delinquency index.*

A longer loan term or time to maturity of the payment of the insurance reduces the value of the delinqunecy index. This has the effect of reducing the value of the final reserve estimate.

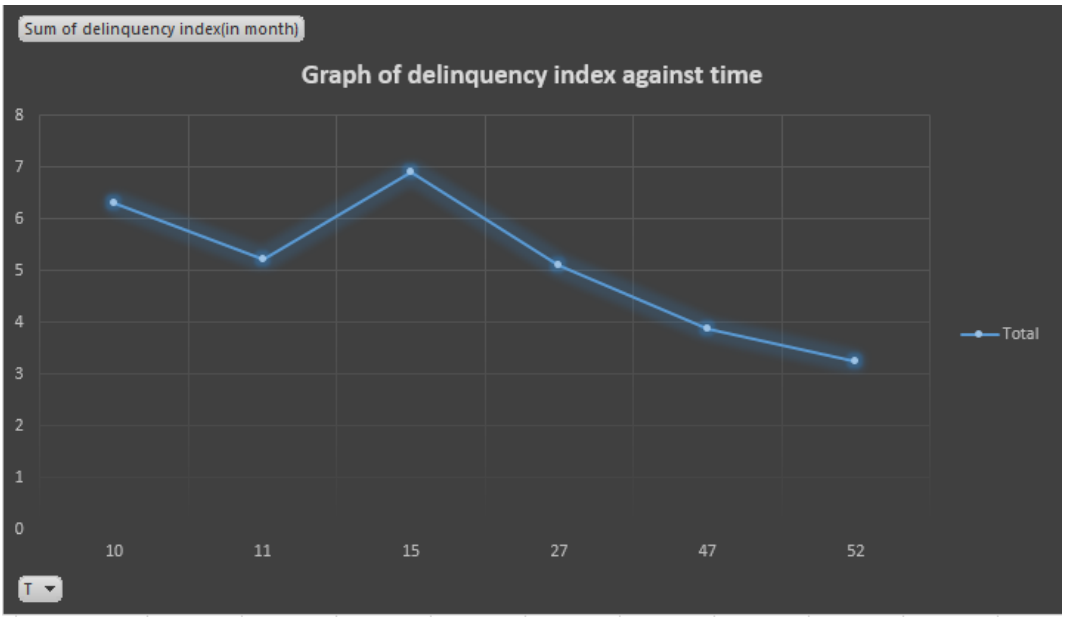


Figure 4.3: *Graph of delinquency index against time.*

The delinquency index which is modelled as Geometric Brownian Motion is controlled by trend. If we do hundreds of simulations of the delinquency index, most of the graphs will be heading towards a certain direction with some deviation. The volatility factor and the random noice of the Wiener process, will make the graphs to have different shapes in the simulations. When we change the constant risk free rate of interest(r) and volatility rate factor σ in our calculations, we will have an insight on how this inputs affect the final prediction value.We thus expect that for any given value r and σ,there is an interval of range for which the final prediction value falls into. If we find this interval range,we can have a rough idea about how the value of our delinquency index and reserve will be in future despite of the random fluctuations that affect the delinquency index.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

Based on the above calculations and results we can see that some methods of stochastic calculus can be used in the prediction of loss reserves in credit insurance for asset-based lending companies.

Our discrete-continous time model allows the adjustment of claim forecasting using the replicating of asset portfolio strategy(free arbitrage and asset liability matching point of view).

A very significant result is that this technique outputs specific formulas for forecasting loss reserves because it considers time to repossession of the collateral by the lending institution.

It's possible to have shorter reserve requirements depending on the outstanding balances, delinquency index and default probabilities.

A very relevant result of this project is that the continous model permits removal of the the Markovian approach used currently.

5.2 Recommendations

Based on the results of our project and the above conclusions, we propose the following recommendations;

- The credit insurers for asset-based lending companies already using other valuation methods to adopt our discrete-continous time model as a reasonable check for their reserve levels.

- Proper attention must be paid in the assumptions of normality that the continous model imply.
- Further research and analysis should be done to provide a wider view of the accuracy of our model.

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Appendix

Outstanding balances	delinquency index(in month)	N(0,1)	Log return	d2	Φ(d2)	Time	(OB)(It)+u(outstanding balance at time t)	OCRT
9174936	6.297309908	0.929692285	0.048363074	-0.039098733	0.484405836	10	16169372.15	1949538
800562	3.877627405	-1.162466635	-0.436535996	-0.891692137	0.186278985	47	1410863.783	65415.1
8631472	5.096528023	0.06200049	-0.163199941	-0.535396055	0.296188009	27	15211602.89	1121428
8685830	6.921282456	1.980346743	0.142841609	0.103369085	0.541164976	15	15307400.26	2061865
8701740	3.24556055	-2.381928261	-0.614471391	-1.089883487	0.137882236	52	15335439.12	526300
2821923	5.229821129	-0.565519939	-0.137382393	-0.519861822	0.301579943	11	4973192.53	373308
5671334	5.912456679	1.141516312	-0.014698043	-0.283014798	0.388582747	24	9994828.308	966692
2003234	4.673492545	0.867226594	-0.249852809	-0.672354461	0.250679033	56	3530382.779	220277
2522748	4.40078097	-0.792478758	-0.309977451	-0.750102585	0.226596461	34	4445943.957	250753
7653465	4.332389927	-0.712334444	-0.325640133	-0.764874181	0.222173235	38	13488020.39	745880
2707726	3.823280076	-0.806089637	-0.450650756	-0.895960748	0.185136864	56	4771938.396	219896
1741910	4.217587422	-0.251468381	-0.352496206	-0.787443078	0.215511269	51	3069840.601	164670
882362	4.494391794	-0.511742647	-0.288929118	-0.717290441	0.236597452	36	1555023.332	91574.8
2774827	6.044475704	0.588210658	0.007385279	-0.147084704	0.441532581	10	4890193.286	537425
4450087	7.002002847	1.708306332	0.15443676	0.306049839	0.62021665	8	7842573.814	1210685
7930079	4.593904736	-0.855909181	-0.267029102	-0.704954323	0.240419331	26	13975508.77	836308
9024943	4.718991498	-1.158702985	-0.240164358	-0.71100908	0.238539305	16	15905033.24	944330
1884667	4.478440016	-0.699372451	-0.292484694	-0.726839049	0.233662279	33	3321427.213	193171
7396987	4.012818156	-1.982880772	-0.402265693	-0.925971636	0.177230361	26	13036018.52	575059
1198343	3.589593564	-1.173670721	-0.513720486	-0.961879598	0.168055044	59	2111889.82	88338.9
4061641	4.87824888	0.223890417	-0.20697315	-0.603910295	0.272951642	37	7157999.236	486302
8451250	3.629575895	-1.028697175	-0.502643661	-0.948712726	0.17138337	60	14893990.15	635344
979286	5.824127215	1.490080706	-0.029750315	-0.345704176	0.364782523	33	1725836.538	156698
9116065	4.604454317	0.374539129	-0.264735305	-0.683827743	0.247041973	49	16065621.34	987865
4227469	6.824231634	1.599335703	0.128720284	0.17266111	0.568541093	10	7450244.833	1054293
7183821	6.119143802	0.585189299	0.019662716	-0.091031996	0.463733584	8	12660347.19	1461313
3490138	5.140859726	0.818840484	-0.154539142	-0.536716445	0.295731747	40	6150815.678	452751
1072681	4.733924561	-0.026374116	-0.237004894	-0.645053531	0.259446248	37	1890430.439	122078
6075853	4.209303969	-0.004518026	-0.354462163	-0.788846236	0.215100941	56	10707729	573283
3989544	5.097809103	-0.883905079	-0.16294861	-0.610635026	0.270720614	9	7030939.688	473766

Risk Free rate (Annual)	9.195%
Volatility(Monthly)	12%
Risk Free rate (monthly)	0.088%
Initial delinquency index (month)	6.00
u	5.00
tc	12%
%Coverage	25%

Figure 5.1: *Assumptions.*

$$(OB)I_{t+u} = (OB)I_t * (1 + tc)^u$$

$(OB)I_{t+u}$ is the outstanding balance at time t plus interest at the rate c.

$$OCR_t = \%Cov * e^{-ru} * (OB)I_{t+u} * \Phi(d_2)$$