Deficiency indices and spectra of Fourth order Differential Operators with Unbounded Coefficients on a Hilbert space

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors..

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Dedication

In memory of my late Uncle Philip May Adede who encouraged never to give up in pursuing my dreams.

Abstract

The concept of unbounded operators provides an abstract framework for dealing with differential operators and unbounded observable such as in quantum mechanics. The theory of unbounded operators was developed by John Von Neumann in the late 1920s and early 1930s in an effort to solve problems related to quantum mechanics and other physical observables. This has provided the background on which other scholars have developed their work in differential operators. Higher order differential operators as defined on Hilbert spaces have received much attention though there still lies the problem of computing the eigenvalues of these higher order operators when the coefficients are unbounded. In this thesis,using asymptotic integration, we have investigated the asymptotics of the eigensolutions and the deficiency indices of fourth order differential operators with unbounded coefficients as well as the location of absolutely continuous spectrum of self-adjoint extension operators. We have mainly endeavuored to compute eigenvalues of fourth order differential operators when the coefficients are unbounded, determine the deficiency indices of such differential operator and the location of the absolutely continuous spectrum of the self-adjoint extension operator together with their spectral multiplicity. Results obtained for deficiency indices was in the range $(2, 2) \leq defT \leq (4, 4)$ under different growth and decay conditions of coefficients. In addition, the absolutely continuous spectrum is either half or full line of spectral multiplicity 1 or 2 depending on the integrability of $p_1^{-\frac{1}{2}}$.

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Chapter 1

Introduction

1.1 Background of the study

Let

$$\tau y = w^{-1}\{y^{iv} - (p_1 y')' + p_0 y - i[(q_2 y')'' + (q_2 y'')' - (q_1 y)' - q_1 y']\} \quad (1.1)$$

where $w = w(x) > 0$ for all $x \in [0, \infty)$, be a fourth order differential equation defined on a weighted Hilbert space $\mathcal{L}^2_w([0, \infty))$ and assume that T is the corresponding differential operator generated by (1.1). Then for the spectral analysis, we solve $\tau y(x) = zy(x)$ or $Ty(x) = zy(x)$, where $z \epsilon \mathbb{C}$ is the spectral parameter and T is the minimal differential operator generated by (1.1) on $\mathcal{L}^2_w([0, \infty))$. The coefficients $p_k(x), q_j(x)$ where $k = 0, 1,\ j = 1, 2$ will be assumed to be twice differentiable with $p_1(x) \to \infty$ as $x \to \infty$. Our main interest in this research was to investigate the deficiency indices of T when the coefficients are unbounded and if its self-adjoint extension operator exists, then the location of absolutely continuous spectrum of this extension operator. Recall that an operator

T is said to be bounded if there exists a positive real number C such that for all $x \in D(T)$, $\| Tx \| \leq C \| x \|$. If this number does not exist then the operator is unbounded. Example of unbounded operators are some differential operators defined on the space of polynomials of degree n. In mathematics, more specifically functional analysis and operator theory, the concept of unbounded operators provides an abstract framework for dealing with differential operators and unbounded observables in quantum mechanics.

The domain of an operator is a linear subspace, not necessarily the whole space. In contrast to bounded operators, unbounded operators defined on a given space do not form an algebra, not even a linear space, because each one is defined on its own domain. Here, the space where T is defined, is a $\mathcal{L}^2_w([0,\infty))$ Hilbert space.

The theory of unbounded operators developed in the late 1920's and early 1930's was part of developing a rigorous mathematical framework for quantum mechanics. This theory as developed by John Von Neumann and Marshall Stone [6], is very important in this research. For instance, in [12], Naimark has used the results of Von Neumann on unbounded operators which were solved using graphs to extend his research on linear differential operators. This approach entails substantial simplifications and its applications to theory of differential equations which yield a unified approach to diverse problems arising in differential equations and their corresponding operators.

In the case of unbounded operators, the most important aspects considered are the domains and extension problems.

For the Hilbert space adjoint operator T^* of a linear operator T to exist,

T must be densely defined in $\mathcal{H}$ and $D(T) \subset D(T^*)$. It is a well known fact that a self-adjoint linear operator is symmetric, but the converse is not generally true in the unbounded case.

Generally, properties of an operator depends largely on the domain and may change under extensions and restrictions. It is shown In [9] that an unbounded linear operator satisfying the relation,

: $< Tx, y >=< x, Ty >$ cannot be defined on all of $\mathcal{H}$.

Higher order differential operators generated by (1.1) above, as defined on a Hilbert space, have received much attention, though there still lies the problem of computing the eigenvalues of these higher order differential operators when the coefficients are unbounded. Because of this, we have investigated the deficiency indices of minimal differential operators and the location of absolutely continuous spectrum of self-adjoint extension operators of the minimal differential operators generated by (1.1) on $\mathcal{L}^2_w([0,\infty))$ using asymptotic integration.

1.2 Basic Concepts

Definition 1.2.1

A linear operator $T : X \to Y$ is said to be bounded if there exists $C \geq 0$ such that $\| Tx \| \leq C \| x \|$ for all $x \in X$.

If the positive real number C does not exist, then the operator T is said to be unbounded.

Definition 1.2.2

A linear operator T from one topological vector space, X, to another one, Y, is said to be densely defined if the domain of T is a dense subset of X. For example, consider the space $C([0,1];\mathbb{R})$ of real valued continuous functions defined on the unit interval. Let $C^1([0,1];\mathbb{R})$ denote the subspace consisting of all continuously differentiable functions. Equip $C([0,1];\mathbb{R})$ with the supremum norm $\| \, . \, \|_\infty$; this makes $C([0,1];\mathbb{R})$ into a real Banach space. The differential operator D, that is, $D(f) = f'$ for $f \in C([0,1],\mathbb{R})$ is given by:

$D(f) = C^1([0,1];\mathbb{R})$. so D can only be defined on $C^1([0,1];\mathbb{R})$ and hence D is densely defined.

Definition 1.2.3

Let $\mathcal{H}$ be a Hilbert space and T be a densely defined operator from $\mathcal{H}$ into itself. If T^* is a Hilbert adjoint of T such that $T \subset T^*$, then T is called a symmetric operator, that is to say, for each x and y in the domain of T we have $< Tx, y >=< x, Ty >$

If $T = T^*$, then T is self-adjoint operator and if T is symmetric with its second adjoint T^{**} essentially self-adjoint, then $T = T^{**}$ and T is said to be essentially self-adjoint.

Definition 1.2.4

A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is said to be self-adjoint or Hermitian if $T^* = T$.

Definition 1.2.5

Let T^* and T be maximal and minimal differential operators respectively, generated by (1.1). $D(T^*)$, the domain of T^* associated to τ consist of all functions y for which the quasiderivatives as defined by Walker [15] are given by:

$$y^{[0]} = y,$$
$$y^{[1]} = y^{'},$$
$$y^{[2]} = p_2 y'' - iq_2 y',$$
$$y^{[3]} = -(y'')' + i(\tfrac{q_2}{p_2})y^{[2]} + (p_1 - \tfrac{q_2^2}{p_2})y' - iq_1 y.$$

are absolutely continuous and

$$T^* y \in \mathcal{L}^2([0,\infty), w),\ \tau y \in \mathcal{L}^2_w([0,\infty))\ \tau y = T^* y \text{ for } y \in D(T^*).$$

Precisely, this domain is given by:
$D(T^*) = \{y \in \mathcal{L}^2((0,\infty) : w) : y^{[0]}$,$y^{[1]}$,$y^{[2]}$,and $y^{[3]}$ are absolutely continuous$\}$.
$D(T^*)$ is thus the maximal possible domain in $\mathcal{L}^2_w([0,\infty))$ for which the quasiderivative makes sense. It is shown in [16] that T^* is densely defined

and closed. An operator defined by restricting the domain of the maximal operator only to those functions y with compact support is known as pre-minimal operator. It is denoted by T_1 and its domain is defined by:
$D(T_1) = \{y \in D(T^*) : y \text{ has compact support in } (0, \infty)\}$.
$T_1 y = \tau y = T^* y$ for $y \in D(T_1)$. For unbounded domains, T_1 is not necessarily closed but it is densely defined. The closure of the pre-minimal operator T_1, $\overline{T}_1$, is the minimal operator generated by (1.1) and is denoted by T. It is obvious that $T \subset T^*$. One can show, however, that $T = T^{**}$. These relations imply that T is symmetric.
Since T is a fourth order differential operator, one therefore defines a self-adjoint extension $\mathcal{H}$ of T by:
$D(\mathcal{H}) = \{y \in D(T^*) \mid (\alpha_1, \alpha_2) y(0) = 0\}$, where α_1, α_2 are 2 by 2 complex-valued matrices described by:

$$\alpha_1 \alpha_2^* + \alpha_1^* \alpha_2 = I$$

and

$$\alpha_1 \alpha_1^* - \alpha_2 \alpha_2^* = 0$$

with $\operatorname{rank}(\alpha_1, \alpha_2) = 2$.
Deficiency index can be defined as the number of linearly independent solutions that are square integrable. It is determined completely by the coefficients p_j, q_i. The deficiency index, defT, is then defined as the pair:

$$defT = (\dim N_{T^*-i}, \dim N_{T^*+i}).$$

N_{T^*-i} is the null space of $T^* - iI$ and N_{T^*+i} is the null space of $T^* + iI$. Thus N_{T^*-i} is the set of all elements such that $\tau y = iy$. If one uses nonreal complex spectral parameter z, then for $Imz > 0$, one has $\dim N_{T^*-i} = \dim N_{T^*-z}$ and $\dim N_{T^*+i} = \dim N_{T^*-\overline{z}}$. Although the definition of the deficiency indices depend on z, as a consequence of the closed symmetric nature of T, the dimension of the null spaces are independent of z provided that z remains either in the lower or upper half-planes. For $Imz > 0$, N_+ and N_- will denote $\dim N_{T^*-\overline{z}}$ and $\dim N_{T^*-z}$ respectively. N_+ and N_- may be finite or infinite. Thus $defT = (N_-, N_+)$.
For fourth order differential operator, it implies that $2 \leq N_- = N_+ \leq 4$. If $N_- = N_+$ then there exists a partial isometry V such that:

$$V : N(T^* - Z) \to N(T^* + Z),\ ImZ > 0,$$

so that H is uniquely defined. Similarly, from [9] one defines the self-adjoint extension H of T by:
$D(H) = \{y \in D(T^*) \mid (\alpha_1, \alpha_2)y(0)\}$, where α_1 and α_2 are 2×2 complex valued matrices described by:

$$\alpha_1\alpha_2^* + \alpha_1^*\alpha_2 = I \text{ and}$$

$$\alpha_1\alpha_1^* - \alpha_2\alpha_2^* = 0$$

with rank$\alpha_1, \alpha_2 = 2$

Definition 1.2.6

Let T be an operator defined on a Hilbert space $\mathcal{H}$. Let $\lambda \in \mathbb{C}$, then λ is in the spectrum of T, $\sigma(T)$, if the operator $T - \lambda I$ is not invertible. The spectrum of T is denoted by $\sigma(T)$ and is defined by:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

In addition, the complement of the spectrum, $(\mathbb{C} \setminus \sigma(T))$, is called the resolvent of the operator T and is denoted by $\rho(T)$, that is,

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

Thus we define

$$R_\lambda(T) = (T - \lambda I)^{-1},$$

as the resolvent operator of T. The sets $\sigma(T)$ and $\rho(T)$, are non intersecting.

We note that an operator $T - \lambda I$ fails to be invertible if it is neither one-to-one nor onto.

If the operator is not one-to-one, then it implies that λ is an eigenvalue of the operator T, and the set of all such $\lambda \in \mathbb{C}$ which makes $T - \lambda I$ not to be one-to-one forms the component of the spectrum known as the point (discrete) spectrum denoted by $\sigma_p(T)$. If $T - \lambda I$ is not invertible (does not have a bounded inverse) because $T - \lambda I$ is not onto then λ in this case is in the spectrum known as the continuous spectrum. The set of all such λ is denoted by $\sigma_c(T)$.

Definition 1.2.7

Asymptotic Integration.

It is a method to transform a system like (1.1) into the form $uB = (\Lambda + R)u$ where Λ is a diagonal matrix and where R is integrable. If Λ satisfies the so-called dichotomy condition, the solutions approximates like the solutions of an unperturbed system. The asymptotic integration of (1.1) basically relies on Levinson's theorem. Levinson's theorem states that the solution of a system,

$$u'(x) = \{\Lambda(x) + R(x)\}u(x),\ \Lambda(x) = diag(\Lambda_i(x))$$

look like the solutions of the unperturbed system $u' = \Lambda u$, if $R(x)$ is sufficiently small and $\Lambda(x) = diag(\Lambda_i(x))$ satisfies a dichotomy condition. In Levinson's original result, small means absolutely integrable. Levinson terms are those expressions which after all transformations turn out to be integrable in the usual sense. They do not contribute essentially to the asymptotics of the eigenfunctions. Thus they are collected separately. The notion Levinson term thus depends on the transformation as well as on the diagonal elements. This can be seen clearly in the various extensions of Levinsons theorem. Generically these terms will be denoted by R.

In our case the following z-uniform version of Levinson's results [13] suffices:

Theorem 1.2.8

Let $\Lambda(x,z)$ =diag$(\lambda_1(x,z),...,\lambda_{2n}(x,z))$ and $R(x,z)$ be $2n \times 2n$ matrices which for all x, are analytic functions of $z \in \Omega \subset \mathbb{C}$. For any unequal pair of indices i and j, $i,j \in [1,...,2n]$, assume that Λ=diag$(\lambda_1(x,z),...,\lambda_{2n}(x,z))$ satisfies the dichotomy condition uniformly in z, that is, for every unequal pair; $i,j = 1,2,...,2n$ and $a \le t \le x < \infty$.

Re$\{\lambda_i(x,z) - \lambda_j(x,z)\}$ has constant sign modulo $\mathcal{L}^1([a,\infty))$ for all $z \in \Omega$. Moreover, assume that $\|R(x,z)\| \le p(x)$ with $p \in \mathcal{L}^1([a,\infty))$.

Then

$$Y'(x,z) = [\Lambda(x,z) + R(x,z)]Y(x,z) \tag{1.2}$$

has solutions $y_k(x,z), 1 \leq k \leq 2n$, *with the asymptotic form*

$$Y_k(x,z) = (e_k + r_k(x,z)).\exp(\int_a^x \lambda_k(t,z)dt) \tag{1.3}$$

where e_k *denotes a unit vector with unity in kth position and* $r_k(x,z)$ *depends analytically on* $z \in \Omega$ *and tends to* 0 *as* $x \to \infty$.

The proof of the theorem can be obtained in [3].

Definition 1.2.9

M-matrix.

Hinton and Shaw in [9] have developed the theory of M-matrix of Hamiltonian systems that one can use to compute the spectra of H. Through a standard inversion theorem in [15], the spectral measure can be reconstructed from the M-matrix. Similarly, the M-matrix can be obtained from eigenfunctions of H. So the M-matrix is the ideal tool that connects spectral properties of H with those of its eigenfunctions.

The M-matrix generalises the m-function of Weyl Titchmarsh and thus relates the asymptotics of the eigenfunctions of higher order differential operators to the spectrum of their self-adjoint realizations.

One therefore uses the results of [15] to construct the M-matrix of the self-adjoint operator H.

Let $Y_\alpha(.,z) = (U_\alpha(.,z), V_\alpha(.,z))$ be the fundamental matrix of (3.1) with initial values

$Y_\alpha(a, z) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}$, α_1, α_2 are as defined above.
U_α, V_α are 2n by n complex-valued matrices whose every column solves $\tau u = zu$. Then $V(., z)$ satisfy the boundary conditions at a. Therefore, the columns of $Y_\alpha(., z)$ span the 2n-dimensional vector space solutions of (1.1).

In the limit point case, self-adjoint extensions are realised by fixing boundary conditions at a. Now fix the boundary conditions to the right through $\alpha = (\alpha_1, \alpha_2)$ and using the techniques of Hinton and Shaw [9],for $Imz \neq 0$, the M-matrix $M_\alpha(z) \in \mathbb{C}^{n\times n}$ is defined by:

$$\chi_\alpha(x, z) = Y_\alpha(x, z) \begin{bmatrix} I_n \\ M_\alpha(z) \end{bmatrix} \in \mathcal{L}_A^2[a, \infty).$$

$M_\alpha(z)$ is analytic for $Imz \neq 0$ and $ImM_\alpha(z)$ is positive definite in the upper half plane.The columns of $\chi_\alpha(x, z)$ form a basis for the square integrable solutions of (1.1). $M_\alpha(z)$ is thus a Herglotz function and all the properties of the classical m-function for a second order equation are satisfied for this more general M-matrix [9].

Here, H is defined by extra boundary conditions at infinity i.e
$D(H) = \{y \in D(T^*) \mid \lim_{x\to\infty} y_k(x)Jy_k^*(x) = y(0)Jy^*(0) = 0\}$; $k = 1, 2$.
It remains therefore to check on the rank of the M-matrix as well as to establish that the M-matrix is bounded. For this we use the formular [15]

$$ImM(z) = \lim_{\in\to 0} \in< y_k(x, z), y_k(x, z) > \text{ where } z = \mu + i\epsilon.$$

Here, we use one of the eigenfunctions that remain square integrable as $z \to 0^+$.

Definition 1.2.10

Inner Product Space.

Let E be a complex vector space. A mapping $< .,. >: E \times E \longrightarrow \mathbb{C}$ is called an inner product in E if for any $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$ the following conditions are satisfied:

- $< x, y >= \overline{< y, x >}$;
- $< \alpha x + \beta y, z >= \alpha < x, z > +\beta < y, z >$;
- $< x, x > \geq 0$;
- $< x, x >= 0$ implies $x = 0$

A vector space with an inner product is called an inner product space.

1.3 Statement of the problem

Higher order differential operators as defined on Hilbert spaces have received much attention though there still lies the problem of computing the eigenvalues of these operators when the coefficients are unbounded and hence because of this, the difficulties in estimating the deficiency indices of such operators. We have computed the deficiency indices of fourth order minimal differential operators and given the location of absolutely continuous spectrum of self-adjoint extension operators of the minimal differential operators generated by (1.1) on the Hilbert space $\mathcal{L}^2([0, \infty))$ using asymptotic integration.

1.4 Objective of the study

The objectives of this research were:

- To compute eigenvalues of fourth order differential operator generated by (1.1) when the coefficients are unbounded.
- To determine the deficiency indices of the minimal differential operator generated by (1.1) when the coefficients are unbounded under different asymptotic conditions.
- To locate the absolutely continuous spectrum of self-adjoint extension operator of the minimal differential operator generated by (1.1) and its spectral multiplicity.

1.5 Significance of the study

The realization of the above objectives enable mathematicians and other scholars to approximate eigenvalues of fourth order differential operators with unbounded coefficients. The results also enriches the existing literature on the deficiency indices of the minimal differential operators as well as the location of the absolutely continuous spectrum of self-adjoint extension operator and its spectral multiplicity.

1.6 Research methodology

To achieve the objectives above, we have basically applied asymptotic integration as outlined in Levinson's theorem. We first did the conversion of (1.1) into a first order system. Then we computed the eigenvalues and thereafter established the dichotomy conditions. This has been followed by determination of the deficiency indices. By application of M-matrix, we have computed spectral multiplicity. We have defined the self-adjoint extension operator of the minimal differential operator and hence located its spectrum together with other subsets of the spectrum like discrete and continuous spectrum.

Chapter 2

Literature review

In 1920, the theory of unbounded operators was found by attempts to put quantum mechanics on a rigorous mathematical foundation. The systematic development of the theory in [6,8] are results of J. Von Neumann (1929-1930, 1936) [6] and M.H. Stone (1932) [8]. When this theory was applied in differential equations a unified approach to diverse questions and their substantial simplification was yielded.

Much has been done regarding the analysis of higher order differential operators but the case of the unbounded coefficient has been of great challenge. In the late seventies and early eighties the deficiency index and partly the essential spectrum has been partially looked into [1]. The results were generated by the help of asymptotics of the eigenfunctions to the spectral properties of the underlying Hamiltonian systems.

Construction of concrete Hamiltonian with singular continuous spectrum by D. Pearson [14] came as a surprise since it exposed the spectral properties of the differential operators.

From the outcome of the asymptotic integration by Behncke and Nyamwala [5], it has been found that a differential operator of order 2n has absolutely continuous spectrum of spectral multiplicity k if there are 2k bounded and $n-k$ exponentially increasing and $n-k$ exponentially decreasing solutions.

This study entailed determination of the eigenvalues generated by (1.1) as well as their deficiency indices and spectral properties using asymptotic integration. The approach demanded some regularity and decay conditions. The regularity conditions guarantee unique solutions to the initial value problem for these operators, while the decay conditions are necessary for asymptotic integration.

We are interested in the spectral properties of the self-adjoint extension operators of minimal differential operators generated by (1.1) on the Hilbert space $\mathcal{L}_w^2([0,\infty))$ on the Hilbert space. In particular, we always assume that $w(x) > 0$ almost everywhere in $[0,\infty)$. If all the coefficient functions $w(x), p_i(x), q_j(x),\ i = 0,1,\ j = 1,2$are constant, one can of course give a complete analysis of τ. For example, by taking Fourier transforms, one sees that basically τ is unitarily equivalent to a multiplication operator. As a consequence, the operator always has absolutely continuous spectrum, but there may also be some eigenvalues [9]. It is well-known that sparse or oscillatory potentials may lead to singular continuous spectra and all sorts of spectral anomalies with SturmLiouville operators. These phenomena will most likely occur for higher order operators, too. The analysis of such properties, however, requires particular techniques like subordinacy or transfer matrices. These methods are generally not available in the higher order situation. The basic technique we will use here is asymptotic integration, which then leads to estimates of the M-matrix [12] and [13]. For this, however, regularity conditions on the coefficients which combine smoothness and decay are necessary. Such properties have been used previously by Weidmann [17] and Behncke [1] for Sturm Liouville operators. Asymptotic integration theory may be considered as a generalization of the well-known WKB-method of Schrodinger operators. In this case the asymptotics of the eigenfunctions are typically determined by an exponential factor. This in turn excludes singular continuous spectrum, which is generally connected with nonexponential decay. It also restricts the accumulation of eigenvalues. The spectral analysis of higher order operators is not only complicated by the multitude of cases, but spectral multiplicity has to be considered too. In this work we will restrict ourselves to the most typical cases which exhibit different phenomena. Upon applying the asymptotic integration, the study identifies the absolutely continuous spectrum of the differential operator with that of its limiting operator. Thus all results may also be considered as perturbation results and statements about the stability of the absolutely continuous spectrum. In this study we have noted that the coefficients p_i and q_j, where $i, j = 0, 1, 2$ will satisfy some regularity conditions. Beyond this we will only consider the cases in which there is a well defined asymptotic relation between the coefficients.

Chapter 3

ASYMPTOTIC INTEGRATION

3.1 Hamiltonian system

Weyl Titchmarsh theory for Hamiltonian systems as developed by Hinton and Shaw [9] in a series of their work was devoted to the spectral theory of Hamiltonian systems. It is important in spectral analysis to write a higher order equation as a Hamiltonian system or first order system. Hamiltonian systems are first order systems with a particular structure that allows an extension of the Weyl M-function calculus. The M-function is originally defined only off the spectrum and the spectral proportions depends on the limiting behavior of M(z) as z tends to the spectrum.

The differential operators generated by expression τ in (1.1) may give rise to self-adjoint extension operators on $\mathcal{L}^2_w([0,\infty))$. This is a classical application of von Neumann's theory of self adjoint extensions of symmetric operators. One first introduces the maximal operator T^* associated with τ as given in Definition 1.5.

In order to convert (1.1) into first order system, we introduce quasiderivatives as defined in [5,16]. Thus the appropriate quasiderivatives for (1.1) are:

$$y^{[0]} = y,$$

$$y^{[1]} = y',$$

$$y^{[2]} = y'' - iq_2 y' \text{ and}$$

$$y^{[3]} = -(y'')' + p_1 y' + i(q_2 y')' - iq_1 y.$$

Now let

$$Y = \left[\begin{array}{c} y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]} \end{array}\right]^t,$$

where t denotes the usual matrix transpose, then the first order system becomes:

$$Y' = CY$$

with $C = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$. Where $A = \begin{pmatrix} 0 & 1 \\ 0 & iq_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} p_0 - z & iq_1 \\ -iq_1 & p_1 - q_2^2 \end{pmatrix}$, and Hamiltonian system of (1.1) becomes

$$JY'(x) = (z\mathcal{A}(x) + \mathcal{B}(x))Y. \tag{3.1}$$

where $\mathcal{A} = (w, 0, 0, 0)$ and $\mathcal{B} = \begin{pmatrix} C & A^* \\ A & B \end{pmatrix}$ with C, A and B as defined above. Here, $J, A, B \in \mathbb{C}^{4\times4}$, $A(x)$, $B(x)$ are locally integrable and self-adjoint for almost every x, J is a symplectic matrix of the form: $\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$ and A has block form $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$, with $A_1 \in \mathbb{C}^{2\times2}$ positive definite almost everywhere. (3.1) can be transformed into another first order system by applying the permutation matrix $\begin{pmatrix} I_2 & 0 \\ 0 & L \end{pmatrix}$ to the solution vector Y, where $L \in \mathbb{C}^{2\times2}$ has the matrix elements $L_{ij} = \delta_{j,n+1-i}$, where $i, j = 1, 2, ..., 4$.

For this to make sense, the system in (3.1) will have to satisfy the following regularity conditions:

1. If y satisfies $Jy' - By = z_0 Ay$ for some y and some z_0 with $\| y \|_A = 0$, then $y = 0$ and if $Jy' - By = Af$ with the $\| y \|_A = 0$, then $\| f \|_A = 0$. In this case, this condition holds automatically for all $z \in \mathbb{C}$, see [5,8].

2. In order to express the higher order quasiderivetives by the lower ones, we demand that our system should satisfy the following regularity condition as well. The equation:

$$\begin{bmatrix} 0_s & 0 \\ 0 & I_{4-s} \end{bmatrix} (Jy' - By) = 0$$

can be solved uniquely for $y_{s+1}, ..., y_4$ in the terms of $y_1, ...y_s$ and formal derivatives of these first s components. For more details on this condition, see Hinton and Shaw [9].

The action of the differential operator to be defined is described by the formal differential operator τ and by the first regularity condition above we have:

$$\tau y = \begin{bmatrix} A_1^{-1}(x) & 0 \\ 0 & 0 \end{bmatrix} (Jy' - By) = A^{-1}(JY' - BY)$$

where, $A_1 \in \mathbb{C}^{2\times 2}$ and $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ with A_1 =diag$((w(x), 0))$, but $w(x) = 1$.

Therefore, one obtains the following which is another version of (3.1) but in a simplified way:

$$Y'(x, z) = C(x, z)Y(x, z), \tag{3.2}$$

where

$$\begin{bmatrix} y_0' \\ y_1' \\ y_3' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & iq_2 & 0 & 1 \\ p_0 - z & iq_1 & 0 & 0 \\ -iq_1 & p_1 - q_2^2 & -1 & iq_2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_3 \\ y_2 \end{bmatrix}$$

and $C = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ We thus employ asymptotic integration to obtain solutions of (3.2) which will be the same as the solutions of (1.1).

3.2 Asymptotic Integration

The standard results on asymptotic integration of systems of linear differential equations give sufficient conditions which imply that a system is strongly asymptotically equivalent to its principal diagonal part. These involves certain dichotomy conditions on the diagonal part as well as growth conditions on the off-diagonal perturbation terms. The main purpose of asymptotic integration is to determine the asymptotic of eigenfunctions of differential operators. The basic problem posed by the differential equation (1.1) is that it's vector solutions cannot normally be written explicitly as expressions involving the entries of the given square matrix $C(x)$. Actually this difficulty created the challenge and interest in developing a wide range of techniques for the investigation of the properties of the solutions. The most important result which has come handy in providing solutions to this problem is the Levinson's theorem [11]. In spectral theory, the matrix elements of $C(x, z)$ will generally depend also on the spectral parameter z. This therefore implies that for this work we need the spectral version of Levinson's theorem. The version required here is already given in Theorem 1.8. Therefore, in solving (3.2) we will follow the steps of asymptotic integration as required by Levinson's theorem. These are: approximation of the eigenvalues of $C(x, z)$, establishment of uniform dichotomy condition, two diagonalisations since the coefficients are twice differentiable so that (3.2) is reduced to Levinson's form .

3.3 Eigenvalues of $C(x)$

We will make the following assumptions for the rest of our work unless otherwise stated. For the growth conditions, we assume

$$p_1(x) \to \infty x \to \infty, p_0, q_1, q_2 = o(p_1) \tag{3.3}$$

and similarly for the decay condition we need that:

$$\frac{f'}{p_1} \in \mathcal{L}^2, \frac{f''}{p_1}, (\frac{f'}{p_1})^2 \in \mathcal{L}^1, \tag{3.4}$$

where $f = q_j, p_k,\ j = 1, 2\ k = 0, 1$.

Therefore, computing the eigenvalues using the characteristic polynomial of $C(x, z)$ through $\det(C - \lambda.\mathcal{I}_4)$ which we equate to $\mathcal{P}(x, \lambda, z)$ leads to:

$$\mathcal{P}(\lambda, z, x) = \lambda^4 - 2iq_2\lambda^2 - p_1\lambda^2 + 2iq_1\lambda + p_0 - z = 0. \tag{3.5}$$

This requires the computation of the zeros of the polynomial $\mathcal{P}_F(\nu, z, x)$ which are the eigenvalues of $C(x, z)$.

In order to eliminate imaginary coefficients in $\mathcal{P}_F(\nu, z, x)$, it is advantageous to replace the eigenvalue parameter λ by $-i\nu$.

Therefore, we have a Fourier polynomial $Q(\nu, z)$ of the form:

$$Q(\nu, z) = \nu^4 + 2q_2\nu^3 + p_1\nu^2 + 2q_1\nu + (p_0 - wz) = 0. \tag{3.6}$$

The closed form formula for solving the zeros of quartic polynomials exists and is via cubic roots. With coefficients that are functions, some of them unbounded, the asymptotics of the roots can as well be captured if an approximation can be arrived at. Thus we apply the approach developed by Eastham [6] and also used by Nyamwala [13] in order to approximate ν-values.

The following Lemma whose proof is similar to that of Theorem 2.2.4 [13] will simplify the approximations.

Lemma 3.3.1

Supposed that (3.6) *is expressed in the form:*
$\nu_0^2 + f_1\nu_0 + f_0 + \mathcal{R}_0(\nu_{oi}, z) = 0$, *where* $f_i = p_k$ *or* q_j, $i = 0, 1$, $k = 0, 1$, $j = 1, 2$ *with* $\mathcal{R}_0(\nu_{oi}, z) \to 0$ *as* $x \to \infty$, *then there exists a unique interval and a root* $\nu_i(x, z)$ *such that:*

$$\mid \nu_i(x, z) - \nu_{oi}(x, z) \mid \leq c(x, z),$$

where $c(x, z) = o(1)$. *Moreover, if* $\nu_{oi}(x, z)$ *is real then* $\nu_i(x, z)$ *is real and a similar argument applies for the imaginary or complex* ν_{oi} *roots.*

The proof of the Lemma above can be found in [13, Theorem 2.2.4]. We, therefore, obtain the following result which will eventually assist in approximating the roots of (3.6) .

Theorem 3.3.2

Consider (3.2), *the roots of the polynomial* (3.6) *can be approximated from:*

$$\text{(i) } \nu^2 + 2q_2\nu + p_1 + R_1(\nu, z) = 0$$
$$\text{(ii) } p_1\nu^2 + 2q_1\nu + (p_0 - z) + R_2(\nu, z) = 0$$

where $R_1(\nu, z), R_2(\nu, z) \to 0$ as $x \to \infty$. Here $R_1 = \nu^{-2}[2q_1\nu + p_0 - z]$ and $R_2(\nu, z) = [\nu^4 + 2q_2\nu^3]$.
Here the magnitude of $\nu-$roots of (i) is approximately $\mid p_1 \mid^{\frac{1}{2}}$ and that of (ii) is $\mid \frac{p_0-z}{p_1} \mid^{\frac{1}{2}}$.

PROOF. It suffices to show that $R_1(\nu, z)$, $R_2(\nu, z) \to 0$ as $x \to \infty$.
Using $\mid \nu \mid\approx \mid p_1 \mid^{\frac{1}{2}}$ in (i) then we have:

$$\begin{aligned} \mid R_1(\nu) \mid &= \mid \tfrac{2q_1}{\nu} + \tfrac{p_0-z}{\nu^2} \mid \\ &\leq 2\tfrac{|q_1|}{|\nu|} + \tfrac{p_0-z}{\nu^2} \\ &= 2\tfrac{|q_1|}{|p_1|^{\frac{1}{2}}} + \tfrac{|p_0-z|}{|p_1|} \\ &= o(1). \end{aligned}$$

Similarly, we use $\mid \nu \mid\approx \frac{|p_0|}{|p_1|}^{\frac{1}{2}}$ in (ii) with z absorbed into p_0 to get:

$$\begin{aligned} \mid R_2(\nu) \mid &= \mid \nu^4 + 2q_2\nu^3 \mid \\ &\leq \mid \tfrac{p_0-z}{p_1} \mid^2 + 2 \mid q_2 \mid \tfrac{|p_0-z|^{\frac{3}{2}}}{|p_1|^{\frac{3}{2}}} \\ &= o(1). \end{aligned}$$

Since $\mid p_1(x) \mid\to \infty$ as $x \to \infty$ and hence each term goes to zero.
□ Applying similar arguments to those in [13], we may take z in some appropriate interval such that the roots of (3.6) are distinct. Let $\epsilon > 0$ be given and pick $z \in \mathcal{K} = \{z \in \mathbb{C} \mid z - z_0 \mid\leq \eta < \epsilon\}$ where $z = z_0 + i\eta$. In such a case the ν roots and the λ roots of (3.6) and (3.5) respectively are distinct. Thus the λ-roots together with their correction terms are of

the form $\lambda \approx \lambda_0 + \frac{\delta p(\lambda,z)}{\delta\lambda}$ where λ_0 is the approximate root and assume that $q_2 = q_1 = 0$ and $\frac{\delta p(\lambda,z)}{\delta\lambda}$ is the correction term. These are therefore of the form:

$$\begin{aligned}
\nu_1 &\approx (-p_1)^{\frac{1}{2}} + \tfrac{\delta p(\lambda_1)}{\delta\lambda} \\
\nu_2 &\approx -(-p_1)^{\frac{1}{2}} + \tfrac{\delta p(\lambda_2)}{\delta\lambda} \\
\nu_3 &\approx (-\tfrac{p_0}{p_1})^{\frac{1}{2}} + \tfrac{\delta p(\lambda_3)}{\delta\lambda} \\
\nu_4 &\approx -(-\tfrac{p_0}{p_1})^{\frac{1}{2}} + \tfrac{\delta p(\lambda_4)}{\delta\lambda}
\end{aligned}$$

3.4 Dichotomy condition

The major difficulties in determining the asymptotic of the eigenfunctions and the spectral analysis of differential operators lie with the roots of the characteristics polynomial. The behavior of these polynomials near these roots, which is a necessary ingredient for the dichotomy conditions is now the key barrier to understanding the asymptotics of the eigenfunctions of these differential operators.

The uniform dichotomy condition in Levinson's Theorem guarantees a z-uniform control of the unperturbed equation $u' = \Lambda u$ which in some sense is also uniform in $Rez, 0 \leq Rez = \eta \leq \epsilon$. Since the roots of $\mathcal{P}(x, \lambda, z)$ are calculated from characteristic polynomial, the uniform dichotomy condition needed is equivalent to sign $Re\left(\lambda_j\left(x,z\right) - \lambda_i\left(x,z\right)\right)$ being constant modulo $\mathcal{L}^1$ for all unequal pair of indices i and j . In spectral theory, a z -uniform dichotomy condition is needed in general but this will only be relevant for the first cluster if $p_0 \approx w$. In this study, slightly stronger conditions will suffice. But if one considers the Fourier polynomial (3.6), then the dichotomy condition for the $\nu-$roots becomes $Im(\nu_j(x,z) - \nu_i(x,z))$ being constant modulo $\mathcal{L}^1$ for all unequal indices i and j. Thus in that case we choose $z \in \mathcal{K}$ such that $0 \leq Imz = \eta \leq \epsilon$. The following theorem whose results are in [2] greatly reduces the proof of dichotomy condition.

Theorem 3.4.1

Consider the system $u' = (\Lambda+R)u$ and assume $\lambda_i(x) = \lambda_{i0}+\lambda_{i1}(x)+\lambda_{i2}(x$ with $\lambda_{i1} = \circ(1)$ and $\lambda_{i2}(x)$ conditionally integrable, $i = 1, ..., 2n$. Sort the eigenvalues into classes $C_1, ..., C_k$ so that

(i) $\lambda_i \in C_l$ then $Re\lambda_{i0} = \alpha_l$,where α_l is a constant.

(ii)$\lambda_i \in C_l$,$\lambda_j \in C_m$,$l \neq m$ then $\mid Re\,(\lambda_{i0} - \lambda_{j0}) \mid \geq \delta > 0, l, m = 1, ..., k$.

Now let $m_{l\pm} = max_{\lambda i \in C_l}(Re\lambda_{i1}(x))\pm$, where $f(\pm)$ denotes the positive (negative) part of f ,$f = f_+ - f_-$

Let $\mid C_l \mid$ denote the number of elements in C_l. Then the system has $\mid C_l \mid$ independent solutions u associated to C_l satisfying

$K_1 exp(\alpha_l x - \int_a^x m_{l_-}(t)dt \leq \parallel u(x) \parallel$

$\leq K_2 exp(\alpha_l x + \int_a^x m_l + (t)dt$, where K_1 and K_2 are constants.

Since $\lambda_{i2}(x)$ is conditionally integrable, a simple transformation of the form $exp\left(\int_0^x \Lambda_{i2}(t)dt\right)$ eliminates these terms while preserving the $\mathcal{L}^1$ nature of the off-diagonal terms. The rest of the proof follows directly from that of [2,Theorem 2.1].

The result above implies that the non-real ν-roots leads to square integrable solutions, which decay exponentially and a corresponding set of exponentially increasing solutions. This holds regardless of dichotomy conditions. But in all cases, it suffices to check the dichotomy condition *only for the real $\nu-$roots (imaginary $\lambda-$roots). For simplicity, we will do this for ν-roots.*

Theorem 3.4.2

. *Let ν_1,ν_2,ν_3 and ν_4 be the roots of Fourier polynomial* (3.6)*, then for $z \in \mathcal{K}$, the roots ν_i $i = 1, 2, 3, 4$ roots are distinct and satisfy z-uniform dichotomy condition.*

PROOF. . Suppose that $p_1 > 0$, $\frac{p_0}{p_1} > 0$, then $\nu_{1\pm}$ and $\nu_{2\pm}$-roots are in the complex conjugate pair and by Theorem 2.3, there will be two eigensolutions which are square integrable and another two that are not square integrable irrespective of the z-uniform dichotomy condition. One thus needs no z-uniform dichotomy condition.

Suppose that $p_1 < 0$, then the dichotomy condition is required for $\nu_{1\pm}$ roots. This is done off-real axis and depends on the correction term

$$\frac{\delta p(\nu_1)}{\delta \nu} \approx 4\nu_1^3 + \frac{p_0 - z}{\nu_{1\pm}^2}.$$

It is only the term $\frac{p_0 - z}{\nu_{1\pm}^2}$ that counts. Here take $z = z_0 + i\eta$ for some $\eta > 0$. Then the correction term is given by

$$\frac{\delta p(\nu_{1\pm})}{\delta \nu} \approx -\frac{(p_0 - z_0) + i\eta}{p_1}.$$

$p_1 < 0$ implies that the correction term to ν_{1+} is negative and hence will lead to a solution that is not z-uniformly square integrable while ν_{1-} will lead to a z-uniformly square integrable solution.

For $\frac{p_0}{p_1} < 0$, then the z-parameter has an influence and therefore we have for $Imz \neq 0$, $\nu_{2\pm} \approx (\frac{(p_0 - z_0) - i\eta}{p_1})^{\frac{1}{2}}$ and this leads to $\nu_{2\pm}$ which are complex conjugate pairs.

If $p_1 < 0$, and $\frac{p_0}{p_1} < 0$, then for z-uniformly dichotomy condition between $\nu_{1\pm}$ and $\nu_{2\pm}$, then note that

$$\mid \frac{\delta p(\nu_{2\pm})}{\delta \nu} \mid \ll \mid \frac{\delta p(\nu_{1\pm})}{\delta \nu} \mid$$

and this will solve the dichotomy condition as the z- influence in the two cases are of different sizes. Finally, if $p_0 \approx w \approx 1$, then the dichotomy condition follows from results of Nyamwala [13]. $\square$

3.5 Diagonalisation

After settling the dichotomy condition, we need to transform (3.2) into Levinson's form through two diagonalisations since we had assumed that the coefficients are twice differentiable. Thus, the transforming matrix is computed using the eigenvectors of C. The transforming matrix

$$T(x, z) = (C - \lambda I)\nu_i = 0$$

is computed from the relation: $C\nu_i - \lambda\nu_i = 0$,
where $\nu_i = (x_1, x_2, x_3, x_4)^t$ and this leads to the equation:

$$\begin{bmatrix} 0-\lambda_i & 1 & 0 & 0 \\ 0 & iq_2 - \lambda_i & 0 & 1 \\ p_0 - z & iq_1 & 0 - \lambda_i & 0 \\ -iq_1 & p_1 - q_2^2 & -1 & iq_2 - \lambda_i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\nu_1 = \begin{bmatrix} 1 \\ -p_1^{\frac{1}{2}} \\ iq_1 \\ -p_1 \end{bmatrix}.$$

In a similar way, ν_2, ν_3, and ν_4 can be obtained.

So

$$\nu_2 = \begin{bmatrix} 1 \\ p_1^{\frac{1}{2}} \\ iq_1 \\ -p_1 \end{bmatrix}$$

$$\nu_3 = \begin{bmatrix} 1 \\ (\frac{p_0}{p_1})^{\frac{1}{2}} \\ p_0^{\frac{1}{2}} p_1^{\frac{1}{2}} \\ -iq_2(\frac{p_0}{p_1})^{\frac{1}{2}} \end{bmatrix}$$

$$\nu_4 = \begin{bmatrix} 1 \\ -(\frac{p_0}{p_1})^{\frac{1}{2}} \\ -p_0^{\frac{1}{2}} p_1^{\frac{1}{2}} \\ iq_2(\frac{p_0}{p_1})^{\frac{1}{2}} \end{bmatrix}$$

From these eigenvectors, $T(x, z)$, we obtain:

$$T(x,z) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ p_1^{\frac{1}{2}} & -p_1^{\frac{1}{2}} & (\frac{p_0}{p_1})^{\frac{1}{2}} & -(\frac{p_0}{p_1})^{\frac{1}{2}} \\ iq_1 & iq_1 & p_0^{\frac{1}{2}} p_1^{\frac{1}{2}} & -p_0^{\frac{1}{2}} p_1^{\frac{1}{2}} \\ -p_1 & -p_1 & -iq_2(\frac{p_0}{p_1})^{\frac{1}{2}} & iq_2(\frac{p_0}{p_1})^{\frac{1}{2}} \end{bmatrix}$$

Here $T(x,z)$ is unbounded and its determinant is approximately $\bigcirc(-4{p_0}^{\frac{1}{2}}{p_1}^2)$. Therefore,

$$T^{-1}(x,z) = \frac{1}{\det T(x,z)} \begin{bmatrix} iq_2p_0p_1^{\frac{3}{2}} & 2p_0^{\frac{1}{2}}p_1^{\frac{3}{2}} & iq_2p_0^{\frac{1}{2}}p_0 & -2p_0^{\frac{1}{2}}p_1 \\ -iq_2p_0p_1^{\frac{3}{2}} & 2p_0^{\frac{1}{2}}p_1^{\frac{3}{2}} & -iq_2p_0^{\frac{1}{2}}p_0 & 2p_0^{\frac{1}{2}}p_1 \\ -2p_0^{\frac{1}{2}}p_1 & 0 & -2p_1^{\frac{3}{2}} & 2p_0^{\frac{1}{2}}p_1 \\ 2p_0^{\frac{1}{2}}p_1^2 & 0 & -2p_1^{\frac{3}{2}} & -2p_0^{\frac{1}{2}}p_1 \end{bmatrix}$$

Hence $T^{-1}(x,z)$ is bounded.

Now we compute $T^{-1}(x,z)\ C(x,z)\ T(x,z)$ to get:

$$\frac{1}{\det T(x,z)} \begin{bmatrix} -4p_0^{\frac{1}{2}}p_1^{\frac{5}{2}} + iq_2p^{\frac{3}{2}}p_1 & & & \\ & -4p_0^{\frac{1}{2}}p^{\frac{5}{2}} - iq_2p_0^{\frac{3}{2}}p_1 & \bigcirc(\mid p_1 \mid^{-\frac{1}{2}}) & \\ & \bigcirc(\mid p_1 \mid^{-\frac{1}{2}}) & -4p_0p_1^{\frac{3}{2}} & \\ \bigcirc(\mid p_1 \mid^{-\frac{1}{2}}) & & & -4p_0p_1^{\frac{3}{2}} \end{bmatrix} \tag{3.7}$$

Thus the correction terms for the first and second eigenvalues are approximately of the form:

$$\frac{1}{4p_0^{\frac{1}{2}}p_1^2}.iq_2p_0^{\frac{3}{2}}p_1 \approx \bigcirc(p_0^{\frac{1}{2}}{p_1}^{-\frac{1}{2}})$$

Even if we write $T^{-1}(x,z)\ C(x,z)\ T(x,z)$ in the form $[\Lambda + R]$ where $\Lambda = diag(T^{-1}(x,z)C(x,z)T(x,z))$ and R is formed with the off diagonal entries of the same matrix, that is, $R_{ii} = 0$, $i = 1,2,3,4$, the form is still not in the Levinson's form and hence we need another diagonalisation. In this case we apply techniques in [13] in order to obtain the second diagonalisation.

While the general method to diagonalise $\Lambda + S$ has been described in [2], a simplified transformation will be used here, that is,

$$\nu = (I + B)\nu_1$$

with $B_{ij} = (\lambda_j - \lambda_i)^{-1} S_{ij}$ where $i \neq j$. For this, one needs

$$(\lambda_j - \lambda_i)^{-1} S_{ij} = \bigcirc(1)$$

so that one can form $(I + B)^{-1}$. The transformed system is then

$$\nu_1' = (\Lambda + S_1 + (1 + B)^{-1} R (1 + B))\nu_1$$

with $S_1 = -(I + B)^{-1}(B' - SB)$.

Chapter 4

DEFICIENCY INDICES AND SPECTRA

4.1 Deficiency Index

The deficiency index problem for self-adjoint differential operators, at least in the form that we now identify it, goes back to Hermann Weyl [15] around 1910, in one way or another the present investigation of self-adjoint boundary value problems goes back a good deal longer. The work of Weyl as well as subsequent work indicates that there may be a close connection between the index problem and the problem of describing the spectrum at least qualitatively, of the self-adjoint extensions of the minimal operators. The knowledge of the deficiency index gives quantitative information about the spectra of self-adjoint extensions and conversely. In this chapter, we have explicitly computed the deficiency index of minimal operator generated by (1.1) and located the absolutely continuous spectrum of $\mathcal{H}$, its self-adjoint extension.

Theorem 4.1.1

. *Let T be a formally symmetric differential operator of order $2n$ defined on the interval $[a,\infty)$ for which a is a regular boundary endpoint. Suppose that $defT = (n+r, n+r)$ such that $0 \leq r \leq n$ then T has self-adjoint extension operator $\mathcal{H}$ whose domain is defined by separated boundary conditions as follows:*
$D(\mathcal{H}) = \{y \in D(T^*) \mid (\alpha_1, \alpha_2)y(a) = 0,$
$\lim_{x\to\infty} w_k^*(x)Jy(x) = 0, k = 1, ..., r\}.$

The functions $w_1, ..., w_r$ are linearly independent modulo $D(T)$ at infinity and may be chosen as eigenfunctions of $T^*w_j = zw_j, z \in \mathbb{C} \setminus \mathbb{R}$. for $j, k = 1, ..., r\}$.

We therefore obtain the following results:

Theorem 4.1.2

. *Suppose that L is the minimal differential operator generated by (1.1) on $\mathcal{L}^2([0,\infty))$ and assume that (2.3) and (2.4) are satisfied.*
(i.) If $p_1 > 0$, $p_0 > 0$, then $defL = (4,4)$ if $p_1^{-\frac{1}{2}}$ is integrable and the spectrum is discrete and if $p_1^{-\frac{1}{2}}$ is not integrable, then $defL = (2,2)$ with $\sigma_{ac}(H) = [0,\infty)$ of spectral multiplicity 2.
(ii.) If $p_1 < 0$, $p_0 > 0$, then $defL = (3,3)$ if $p_1^{-\frac{1}{2}}$ is integrable and $\sigma(H)$ is discrete but if $p_1^{-\frac{1}{2}}$ is not integrable, then $defL = (2,2)$ with $\sigma_{ac}(H) = \mathbb{R}$ of spectral multiplicity 1.
(iii.) if $p_1 > 0$, $p_0 < 0$, then $defL = (3,3)$ if $p_1^{-\frac{1}{2}}$ is integrable and $\sigma(H)$ is discrete and $defL = (2,2)$, $\sigma_{ac}(H) \subseteq [\overline{c}, \infty)$ of spectral multiplicity 1 if $p_1^{-\frac{1}{2}}$ is not integrable with $\limsup p_0 = \overline{c}$.

(iv.) If $p_1 < 0$, $p_0 < 0$, then $def L = (2,2)$ and $\sigma(H)$ is discrete.

Proof. . Suppose L is the minimal differential operator generated by (1.1) on $\mathcal{L}^2([0,\infty))$ and assume that (3.3) and (3.4) are satisfied, then using quasiderivatives, (1.1) can be converted into its first order system (3.1) where $C(x,z)$ is a four by four matrix as explained in section 3.1. Since the coefficients are assumed to be twice differentiable, by application of Levinson's theorem, we need two diagonalisations in order to convert (3.2) into its Levinson form as given in the Levinson's theorem. This requires the eigenvalues and the corresponding eigenvectors of the matrix $C(x,z)$. Using $\det(C - \lambda.I_4) = 0$, we would be able to determine the eigenvalues of this matrix. Since $p_1(x)$ is allowed to be unbounded, it implies that Eastham's approximation approach [6] as outlined in section 3.3 can be used to approximate values of λ.
These are approximately given by:
$\lambda_1 \approx (p_1)^{\frac{1}{2}}$, $\lambda_2 \approx (-p_1)^{\frac{1}{2}}$, $\lambda_3 \approx (-\frac{p_0}{p_1})^{\frac{1}{2}}$ and $\lambda_4 \approx (-\frac{p_0}{p_1})^{\frac{1}{2}}$.
The z-uniform dichotomy condition now follows from Theorem 2.4.2 Therefore, using the eigenvectors of the form v_1,v_2,v_3 and v_4, the system can be diagonalised though the resultant system will not be in Levinson's forms. Because (3.4) is satisfied, a second diagonalisation is necessary. For this, we write the system after the first diagonalisation in the form.

$$V' = [\Lambda + R]V$$

where $\Lambda_i = diag(\lambda_i$+correction terms).
Here, correction terms are those perturbing terms that are added to the diagonal as a result of the first diagonalisation. In our case, these are given by $\bigcirc(p_0^{\frac{1}{2}} p_1^{-\frac{1}{2}})$ for λ_1 and λ_2 and $\bigcirc(p_1^{-\frac{3}{2}})$ for λ_3 and λ_4.

The matrix R has its main diagonal all zero's and off diagonal terms are $\bigcirc(\frac{f}{p_1})$ where $f = p_0, q_1, q_2$. Now using the Behncke's approach [2], then the second diagonalisation is done using the matrix $[I + B]$ where the matrix B determined by:

$B_{ij} = (\lambda_i - \lambda_j)^{-1}$ $i \neq j$ and

$B_{ii} = 0$

After the second diagonalisation, the system is in Levinson's form and therefore Levinson's theorem can be applied to obtain the closed form solution of (1.1) respectively (3.2). The solutions will be of the form:

$$y_k(x, z) = c_k(e_i + o(1)) \exp(\int_a^x \lambda_k(t, z)dt)$$

where c_k is the normalised eigenvector and λ_k is the corresponding eigenvalue.

The deficiency index, therefore, by Naimark's results [12] is determined by the number of eigenfunctions that are z-uniformly square integrable, that is, bounded as $x \to \infty$.

This is determined by those eigenvalues with negative real part and for those eigenvalues that are pure imaginary, this is done off the imaginary axis if they have to be z-uniformly square integrable.

(i.) Suppose $p_1 > 0$, $p_0 > 0$, then all the eigenvalues are pure imaginary and the square integrability is determined by the correction term $p_1^{-\frac{1}{2}}$. If this term is integrable, then all the eigenfunctions are z-uniformly square integrable and $defL = (4, 4)$. The minimal operator L has self-adjoint extension whose domain using Hinton and Shaw [9] results is defined by boundary conditions at left hand right hand points (limit circle case)

$$D(H) = \{y \in D(T^*)y(a)Jy^*(a) \neq 0\}$$

The spectrum of H consists of only eigenvalues and therefore is discrete spectrum. If $p_1^{-\frac{1}{2}}$ is not integrable, then off the imaginary axis, two eigenfunctions will lose their square integrability as $x \to \infty$, these are the eigenfunctions associated to λ_1 and λ_3 and hence $def L = (2, 2)$. These eigenfunctions that lose their square integrability as $Rez \to 0^+$ contributes to the absolutely continuous spectrum but since one has the freedom to pick z in the whole of $\mathbb{R}$, $\sigma_{ac}(H) = \mathbb{R}$.

Here, H is defined by extra boundary conditions at infinity i.e
$D(H) = \{y \in D(T^*) \mid \lim_{x\to\infty} y_k(x)Jy_k^*(x) = y(0)Jy^*(0) = 0\}$; $k = 1, 2$.
It remains therefore to check on the rank of the M-matrix as well as to establish that the M-matrix is bounded. For this we use the formular

$$ImM(z) = \lim_{\in\to 0} \in< y_k(x, z), y_k(x, z) >$$

Here we use one of the eigenfunctions that remain square integrable as $Rez \to 0^+$.

The correction term to λ_1 is given by:

$$\frac{w}{\partial_\lambda} p(\lambda, z, x) \approx \frac{w}{4|p_1|^{\frac{3}{2}}}$$

For $y_1(x, z)$ we have:

$$ImM(z) = \lim_{\epsilon\to 0^+} \epsilon < y_1(x, z), y_1(x, z) >= \lim_{\epsilon\to 0^+} \epsilon \int_0^\infty w(x) \mid C_1 \mid^2 \mid$$
$$e_1 + \circ(1) \mid^2 . \exp(-2 \int_0^x \mid w(x) \mid^2 . \tfrac{1}{16} \mid p_1 \mid^{-3} dt)$$

$$= \lim_{\epsilon\to 0^+} \epsilon C_2 (\int_0^\infty w(x) \mid e_1 + \circ(1) \mid^2). \exp(-2 \int_0^\infty \mid w(x) \mid^2 \mid p_1 \mid^{-3} dt) dx$$

$$\leq \lim_{\epsilon\to 0^+} \epsilon C_2 \int_0^\infty C.wx \mid e_1 + \circ(1) \mid^2 dx = C_3$$

Where $\mid C_1 \mid^2$ and since $\exp.(-2\int_0^\infty \mid w(x) \mid^2 \mid p_1 \mid^{-3} dt)$ is a bounded function, we may even assume it is a constant c. Thus $ImM(z)$ is bounded.

But since two eigenfunctions lose their square integrability, it follows that rank $M(z) = 2$ and hence the spectral multiplicity of $\sigma_{ac}(H)$ is two.

The proofs of (ii)-(iv) are similar and they follow at once only that in (iii), since only the eigenfunction associated to λ_3 contributes to absolutely continuous spectrum, one notes that p_0 is bounded and is associated to z, thus the value of p_0 affects the location of the absolutely continuous spectrum. Thus, $\sigma_{ac}(H) \subseteq [\bar{c}, \infty)$ where $\bar{c} = \limsup p_0(x)$. The spectral multiplicity in this case is 1.

$\square$. **Remark 4.3**

In Theorem 3.1.2 above, part (iv) is the classical case of limit point case and hence it follows that the spectrum is discrete [9].

Example 4.4.

The following example validates the results of Theorem 3.1.2 and for simplicity,we will assume power coefficients with the middle term tending to infinity as the variable $x \to \infty$.

Consider a fourth order differential operator generated by:

$$\tau y = y^{iv} + ax^\beta y'' + bx^\alpha y = zy$$

where $\beta > 0$ and $\alpha < 0$, $a, b \neq 0$, $a, b \in \mathbb{R}$.

Solution

The associated polynomial is of the form:

$$\lambda^4 + ax^\beta\lambda^2 + bx^\alpha - z = 0$$

This is a biquadratic and can be solved explicitly. Suppose that $\lambda^2 = \mu$ and if we absorb z into bx^α then:

$$\mu_\pm = \frac{-ax^\beta \pm \sqrt{a^2x^{2\beta} - 4bx^\alpha}}{2}$$

$$= -a\frac{x^\beta}{2} \pm (\frac{a^2x^{2\beta}}{4} - bx^\alpha)^{\frac{1}{2}}.$$

As $x \to \infty$, $x^{2\beta} \to \infty$, but $x^\alpha \to 0$, thus using binomial expansion, we can approximate the $\mu_\pm$ roots as follows:

$$\mu_\pm \approx \frac{-ax^\beta}{2} \pm \frac{ax^\beta}{2}\{1 - \frac{4bx^{\alpha-2\beta}}{a^2}\}^{\frac{1}{2}}$$

$$\approx \frac{-ax^\beta}{2} \pm \frac{ax^\beta}{2}\{1 - \frac{2bx^{\alpha-2\beta}}{a^2}\} + \bigcirc(x^{-3\beta})\frac{1}{2}(-\frac{1}{2})$$

$$\approx \frac{-ax^\beta}{2} \pm \frac{ax^\beta}{2} \mp \frac{bx^{\alpha-\beta}}{a} + \bigcirc(x^{-3\beta})$$

$$\Rightarrow \mu_+ \approx \frac{-bx^{\alpha-\beta}}{a} + \bigcirc(x^{-3\beta})$$

$$\mu_- \approx -ax^\beta + \frac{-bx^{\alpha-\beta}}{a} + \bigcirc(x^{-3\beta})$$

Here $\mu = \lambda^2$, thus

$$\lambda_{1/2} \approx \mu_+^{\frac{1}{2}} \approx (\frac{-bx^{\alpha-\beta}}{a} + \bigcirc(x^{-2.5\beta}))$$

$$\lambda_{3/4} \approx (\mu_-)^{\frac{1}{2}}$$

$$\approx (-ax^\beta)^{\frac{1}{2}} + \bigcirc(x^{-1.5\beta})$$

The correction term to the diagonals after the first diagonalisation is approximately given by:

$\frac{f'}{f} \approx \bigcirc(x^{-1})$ (off-diagonal terms). Correction to the diagonal is given by: $\bigcirc(x^{-0.5\beta})$ for $\lambda_{3/4}$ but $\bigcirc(x^{\frac{-}{\alpha}-\beta}2)$ for the $\lambda_{1/2}$.

The dichotomy condition is satisfied so we obtain the following results:

(i) $a > 0$, $b > 0$, then $def L = (4,4)$ if $\beta > 2$ and $\beta > 2+\alpha$. The spectrum of $\mathcal{H}$ is discrete.

But if $\beta < 2$ and $\beta > 2+\alpha$, then $def L = (2,2)$ and $\sigma_{ac}(\mathcal{H}) \in \mathbb{R}$ of multiplicity 1, and if $\beta > 2$ and $\beta < 2+\alpha$, then $\sigma_{ac}(\mathcal{H}) \subseteq [0,\infty)$ of multiplicity 1.

We obtain $def L = (2,2)$ if $\beta < 2$, $\beta < 2+\alpha$ with $\sigma_{ac}(\mathcal{H}) = \mathbb{R}$ of multiplicity 2

(ii) Suppose $a > 0$, $b < 0$, then $def L = (3,3)$. If $\beta > 2$ and $\beta > 2+\alpha$, then the spectrum is discrete.

But if $\beta < 2$ and $\beta > 2+\alpha$, then $def L = (2,2)$ and $\sigma_{ac}(\mathcal{H}) \subseteq [\overline{C},\infty)$ of spectral multiplicity 1 and if $\beta > 2$ and $\beta < 2+\alpha$, then

$\sigma_{ac}(\mathcal{H}) \subseteq [0,\infty)$ of multiplicity 1, we obtain $def L = (2,2)$ if $\beta < 2$ and $\beta < 2+\alpha$ with $\sigma_{ac}(\mathcal{H}) = \mathbb{R}$ of multiplicity 2.

Chapter 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In section 3.2.1, we approximated the eigenvalues of (1.1) when $p_1(x) \to \infty$ as $x \to \infty$ while the other coefficients p_0,q_1,q_2 are bounded. Theorem 3.3.2 gives the simpler version of (3.6) that approximates these eigenvalues. Theorem 4.2.2 gives the deficiency indices of T and spectra of its self-adjoint extension H together with their spectral multiplicities under various asymptotic behaviours. In particular, $(2, 2) \leq defT \leq (4, 4)$ for various signs of p_1 and p_0. Meanwhile the absolutely continuous spectrum of H, $\sigma_{ac}(H)$ is either a subset of $[0, \infty)$ or $\mathbb{R}$ with spectral multiplicity of 1 or 2 depending on the integrability of $p_1^{-\frac{1}{2}}$. The objectives as set out in chapter one are achieved by these results. These results have enriched the available literature on the spectral theory of higher order differential operators and can also be applicable in quantum mechanics where results of self-adjoint operators are very much useful. In solving these problems, we applied the techniques of asymptotic integration as outlined in Levinson's theorem which is a pertubation result.

5.2 Recommendation

Proving the dichotomy condition is still a complicated task even in the case of a fourth order differential operators, this may be due to the difficulty in approximating the eigenvalues of such operators when the coefficients are unbounded. This might be extremely difficult in higher orders, and we recommend further investigation in this direction.

References

[1] H.Behncke, *Spectral Analysis of Fourth Order Differential Operators I,*, Math. Nachr. **279**, 58-72 (2006).

[2] H. Behncke,*Asymptotically Constant Linear Systems*, Proc. Amer. Math. Soc. (4) **138**. (2010), 1387-1393.

[3] H. Behncke and C Remling,*Uniform asymptotic integration of a family of linear differential systems*, Maths. Nachr. **225** (2001), 5-17.

[4] A. Devinatz, *The deficiency index of a certain class of ordinary self-adjoint differential operators*, Advance in Math. **8** (1972), 434-473.

[5] H. Behncke and F. O Nyamwala, *Spectral Analysis of Higher order Differential Operators With Unbounded Coefficients.*Maths. Nachr **20** (2010).

[6] M.S.P. Eastham,*The Asymptotic Solution of Linear Differential systems.* London Math. Soc. Monographs New Series (Oxford University Press,Oxford) (1989).

[7] K. Erwin, *Introductory functional analysis with applications*, United States, (1978).

[8] D. Hinton, *Asymptotic behavior of the solutions of* $(ry^{(m)})^k \pm (qy) = 0$, *J. Differential equations* **4**(1968), 590-596. MR. 38 373.

[9] D.B. Hinton and J.K. Shaw,*On the Titchmarsh-Weyl $M(\lambda)$-functions for linear Hamiltonian system*, J. Diff. Eq. **80** (1981), 316-342.

[10] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980.

[11] N. Levinson, *The asymptotic nature of the solutions of linear systems of differential equations*, Duke maths. J. **15** (1948), 111-126 MR 9, 509.

[12] M.A Naimark, *Linear differential operators*, Unga, New York, 1967

[13] F.O Nyamwala,*Spetral Theory of Differential and Difference Operators in Hilbert Spaces*, Doctoral Thesis,University of Osnabruck, Germany (2010).

[14] D. Pearson, *Asymptotic theory and deficiency indices for differential equations*, **80**, (1980) 150-152.

[15] C. Remling, *Spectral analysis of higher order differential operators, I. General properties of the M-function*, J. London Math. Soc. **58** (1998)216-218.

[16] P.W Walker, *Asymptotics of the solutions to* $[(Cry'')'-py']+qy=\sigma y$, J. Differential Equations **9** (1971) 108-132.

[17] J. Weidmann, *Linear operators in Hilbert Spaces*, Graduate texts in Mathematics, Vol.**68**, Springer-Verlag, Berlin, (1989).

[18] H. Weyl, *On ordinary differential equations with singularities and the associated expansions of arbitrary functions*, Math. Anne. **68**(1910), 220-269.