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PORTFOLIO OPTIMIZATION FOR EXTENDED (P, Q)-BINOMIAL COX-ROSS-RUBINSTEIN MODEL

By

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ABSTRACT: In this paper we focus on establishing optimization conditions for the extended (p, q)-binomial Cox-Ross-Rubinstein (CRR) model, particularly in the context of managing portfolios in life insurance under varying noise conditions. We also give the convergence analysis of the model.

KEYWORDS: CRR model, Portfolio, Optimization, Convergence

INTRODECTION

When pricing options, it is important to factor in replication and volatility [1]. According to [2], [3], [4], [5], [6] and [6] a portfolio replicates the derivative security if a duplicate security can be formed by combining the underlying security with an appropriate risk-free asset whose proportions must be repeatedly adjusted over the length of the trading period, and no extra cash is required to self-finance the portfolio. For a call, replicating a portfolio requires investors to be bullish on the implicit asset's price movement [7]. However, for a put, replicating the portfolio requires taking a short position on the underlying asset [8]. That is, Suppose we have a stock A trading at $S = 100$, and the price can either go up or down by a factor of 10 in a single period such that $uS = 110$, and $dS = 90$, and it is possible to lend/ borrow from the markets at a 4% interest rate, we can set up a replicating portfolio with Δc units of the stock's call option and B units of borrowing, such that: $\Delta c.(110) + B(1.04) = 10\Delta c.(90) + B(1.04) = 0$. Solving this, will obtain $\Delta c = .50$ and $B = -43.26$ which implies that, we can replicate the call option by having a long position on .50 units of the stock and borrowing 43.26. The cost of the portfolio (C) will be given by: $(.50)(100) + (-43.26)(1) = 6.74$. Hence if $C > 6.74$, the call can be sold to buy a replicating portfolio [9]. However, if $C < 6.74$, the call can be bought and the replicating portfolio sold. For the same underlying stock A, we can set up a replicating portfolio with Δp units of the stock's call option and B units of borrowing, such that, $\Delta.p(90) + B(1.04) = 10$ and $\Delta.p(110) + B(1.04) = 0$, solving this will obtain $\Delta P = -.50$ and $+52.88$ which implies that, we can replicate the put option by having a short position on .50 units of the stock and borrowing 52.88. The cost of the portfolio (C)

will be given by, (-.50)(100) +(52.88)(1) = 2.88 implying that if $C > 6.74$, the put option can be bought to sell a replicating portfolio. However, if $C < 6.74$, the put option can be sold and the replicating portfolio bought. The portfolio will be self-financing in both options (see [10], [11], [12], [13], [14] and [15]). Portfolio replication should be done to build synthetic securities [16], which are simply ways in which traders can create the risk profile and payoff of a particular asset by using combinations of the underlying asset and different instrument options [17]. Synthetic options enable traders to hedge their positions against extreme volatility, while also minimizing the opportunity cost by enabling them to explore options with similar properties [18]. The process of call option hedging at the starting point requires investors to determine C's theoretical value. This can be achieved by allocating the replicating portfolio with an amount C [19]. This way, the delta and portfolio value will be analogous to the options [20]. After a given time interval, delta will change, and investors should repeatedly rebalance their positions [21]. As the portfolio approaches the time to maturity approaches, it will mainly be constituted with stock if the price is above K, else, its value will decrease till it reaches zero [21]. Research by [22] shows that several institutions owning significant stock portfolios seek to hedge their positions against significant market downturn risk. They can achieve this by simply buying a put that will be exercised at K. Leland and Rubinstein who introduced the idea of option-based portfolio insurance noted that insurance agencies and mutual funds could hedge their positions by simultaneously buying a stock and a put written on it. This way, the portfolio's value at maturity is always greater than the strike price even with excessive volatility [23]. For example, a trader could buy a stock on the NASDAQ (National Association of Securities Dealers Automated Quotations Stock Market) and a put option on the same allowing him to sell the index at a particular price [24]. If the put fell beneath the price, the trader can sell the put and use the profit to exercise any losses he faces from bearish market, while if the put rises the trader loses the amount paid for in the put, but continues to enjoy returns from stocks.
Creating synthetic portfolios is easy and uncomplicated [25]. However, in the real world, listed options often do not have desired strike prices and maturities. Furthermore, synthetic options can be affected by jumps in transaction costs and stock prices. Therefore, there is a need to refine existing strategies, so as to improve the performance and efficiency of option-based portfolio insurance [26]. The Cox Rubenstein model and Black Scholes formula provide insurance agencies with methods that they can use to guarantee minimum portfolio values at the end of their respective trading periods. However, the latter assumes a constant trading strategy, which makes it an impractical strategy in hedging constructed portfolios. In [27] its shown that an extension of the classical generalized CRR model is needed.

1. Materials and Methods

This section delineates the methodological framework employed in this study to derive the empirical results. The research methodology is anchored on a multifaceted approach, encompassing a spectrum of techniques that are pivotal in the comprehensive analysis and interpretation of the data [28]. These techniques encompass the utilization of Binomial ex- tensions, the exploration of Extended Fibonacci sequence generating functions, the formulation of robust mathematical models, the application of rigorous Convergence tests, and the execution of detailed Simulations. Each of these methodologies plays a quintessential role in underpinning the research findings, ensuring both the reliability and validity of the study within the context of its overarching objectives [29].

Definition 1.1 ([30]). Derivative securities known as options confer rights, but not responsibilities, on the holder to acquire or surrender financial assets within a specified timeframe and subject to particular conditions.
Remark 1.2. In the finance world, derivatives

are contracts whose valuations are dependent on underlying assets, which can be bank accounts, bonds, commodities, currencies, or stocks [31]. There are different kinds of derivatives. Options are derivatives that are traded only on single securities, such as stocks, currencies, or indexes, because their underlying assets are fixed, thus, cannot be changed [32]. A call option allows the holder to purchase a particular asset at a predetermined price. Conversely, a put option allows the holder to sell a financial asset at a predetermined price [33]. A strike price is the monetary value at which users can exercise rights to sell or purchase an option. It is different from the stock's price, which is the option's underlying stock's price [34]. There are two generally accepted techniques in which options are exercised with regard to their expiration dates: American options can be exercised during the length of the contract, and including on their expiration dates [35]. Conversely, European options are exercised only on their expiration.

Options also have intrinsic and extrinsic value [36]. The former measures the strike and stock price difference, while the latter measures the premium and intrinsic value difference [37]. The former accounts for internal price movements of the option from early exercise, while the latter accounts for both internal and external price movements after some time interval [38]. There are three main features that govern an American option's status (intrinsic value); description of asset, expiration date, and exercise price. Call options have intrinsic value if they are below the strike price; calls that enables buyers to purchase assets at values lower than market rates are more valuable than the inverse as it will be profitable upon being exercised [39]. Conversely, puts have intrinsic value if they are above the strike price; puts that enable their holders to sell assets at much higher rates than the market value is more valuable as it will be profitable upon being exercised. The exercise price is also important as they determine whether an option is profitable (in-the-money) or loss-making (out-of-the-money) [40]. Finally, the expiration date shows the option's internal value as they determine their flexibility. As a result of the high levels of flexibility offered by American options, they usually have higher valuations than European options. The following are the main factors that influence an option's extrinsic; volatility, interest rates, and rate of stock growth [41]. Volatility refers to the degree of dispersion of returns over a trading period. It is caused by buyer and seller speculation about the option's price due to uncertainty and changes in the external or internal business environment [42]. Volatility is positively correlated with the value of an option. High volatility increases the likelihood of an option being profitable, thus increasing its extrinsic value [43]. Prevailing interest rates also affects the value of options by affecting *rho* which is a variable used in many option-pricing models which measures the rate at which the price of an option contract rises or falls if the risk-free interest rate changes by 1% [44].

Generally, calls have a positive rho, which implies that rising interest rates increases their extrinsic value [45]. On the other hand, puts have a negative rho; hence, rising interest rates decreases their extrinsic value. There are two parties involved in every option trade. Parties that offer options are called option writers while parties that obtains the option are called purchasers [46]. Typically, the former faces more significant risk than the latter, who only risk losing their original premium [47]. This is because they must sell or buy the options when they are exercised, at the terms specified in the options contract. As a result, they are required to post security deposits called margins that guarantee performance.

Remark 1.3. Let us assume that someone holds a call option on a stock with strike price K and price at maturity S. The option's value is $S - K$ if $S < K$, which indicates that the option's value is zero. If, on the other hand, $S > K$, the option's value is reported. As follows, both cases can be shown: A call option's value at maturity can be found explicitly in $C(S, T) = \max(0, S - K)$. Although the outcome is different in this instance as follows, the same logic applies to put options as well; The option is put at maturity as $P(S, T) = \max(0, K - S)$. When considering an option pricing curve with different dates to expiration, [80] states that the value of a call option increases with the amount of time till expiration.

Definition 1.4 ([48], Definition 1.2)**.** Let $C(S, T)$ and $P(S, T)$ be the prices of a European put and call, whose underlying stock price is S and with a strike price K. Then, the put-call parity (the association between put and call prices), will be determined by the equation $C(S, T) - P(S, T) + dK = S$, where d is the discount factor of the underlying asset as the options approach their expiration date.

Remark 1.5. This combination is implemented to enhance speculative strategies. According to [49], combining options with stock makes it possible to determine the investment's returns.

Definition 1.6 ([50], Definition 1.3)**.** A portfolio is a collection of as- sets or financial investments that a person or an organization owns. Cash equivalents, art, real estate, mutual funds, equities, bonds, and commodities could all be included in this mix. Through the use of portfolios, customers can diversify their investments, lowering risks and optimizing profits. The author in [51] notes that stochastic processes enable investors to efficiently model the dynamics of stocks' prices. That is, given a stock, $S(t) = S_i(t)$, where $S_i(t)$ is the price of the i^{th} stock at time t. Consequently, the investors holding positions, $H_t = (H_1(t), \ldots, H_n(t))$, react according to the stochastic processes that are under their control. Supposing there is no consumption from the portfolio, no additional capital invested, and no cash dividend payout, investors can only earn money from the price fluctuations of their underlying assets, which can lead to a self-financing portfolio.

Proposition 1.9 ([52], Proposition 2.1)**.** A relative portfolio is self- financing. The primary goal of any investor is to maximize the returns on their portfolio by controlling its holdings. However, this paves way to the optimal stochastic control problem, a challenge that investors faced when trying to control the value process's movement, $V(t)$ through y_t.

Definition 1.10 ([53], Definition 1.1.6)**.** Portfolio management combines financial instruments with the right tools to generate optimum return downsizing risk within a given time horizon. One aspect of portfolio management is portfolio optimization.

Definition 1.11 ([54], Definition 1.7)**.** Portfolio Optimization is a pro- cess in which investors factor the minimization of risk and maximization of expected returns during the selection of one out of a possible set of probable portfolios.

Remark 1.12. It is based on modern portfolio theory (MPT), which states that investors want the highest possible returns for the lowest risks [55]. To achieve this, assets selected should have a low correlation with each other such that if one asset class under performs, the entire portfolio does not crash. Portfolio optimization is a 2-stage process that involves the selection of asset classes depending on relative weight, and the se- lection of particular assets and the quantity they want to include in the portfolio for optimum returns [56]. The optimum return is, in most cases uncertain. Return uncertainty can be treated with three different mathematical methods: First, mean-variance analysis, secondly utility function analysis, and lastly through arbitrage or comparison analysis.

Mean-variance analysis is a technique used in modern portfolio theory (MPT) to calculate the variance of assets against expected returns. It was advanced by Markowitz [57]. It enables analyst to pick investments that have the biggest returns at particular levels of risk. On the other hand, utility function analysis enables investment managers to calculate the optimal portfolio that meets an investor's objectives and preference. This model was advanced by authors in [58] who argued that rational investors should only select portfolios that maximize their wealth's expected utility from a set of investment alternatives (see also [59] and [60]). Finally, arbitrage analysis is a mathematical technique used to determine underpriced assets and their expected rates of returns relative to systematic risks [61]. Insurance companies, Life Insurance Companies in particular, need a deeper understanding of their portfolio and its management to help achieve portfolio efficiency, risk mitigation, capital appreciation, investment goals, asset allocation, diversification, among other reasons.

Definition 1.13 ([62], Definition 1.8**).** Diversification of a portfolio is the process of investing in different assets to minimize the portfolio's overall risk.

Remark 1.14. Investors can reduce the variance of their portfolio's returns by including many assets in the portfolio. Often, portfolios with fewer assets attract high-levels of risk, because they have more consider- able variance.

Definition 1.15 ([63], Definition 1.1.9**).** Hedging is a strategy in which investors reduce the risk of adverse price movements in an asset.

Remark 1.16. Hedging enables investors to protect themselves against risk, thus is crucial to the core functioning of financial markets. Insurance can be considered a form of hedging, since people pay premiums so that they can protect themselves against certain risks. Upon the occurrence of such risks, an individual is liable for compensation [64].

Definition 1.17 ([65], Definition 10**).** Strategy is a general direction set to achieve a desired state in the future.

Definition 1.18 ([66], Definition 1.1**).** Insurance is a contract between two parties. The first is the policyholder, the person, firm, or company confronted by risk. The other is the insurer, a person, firm, or company specializing in assuming the risk the exposure units face and making their losses collectively predictable.

Definition 1.19 ([67], Definition 12**).** A Stochastic process refers to processes and events that produce random outcomes and variables.

Remark 1.20. For non-stochastic events, the observations and outcomes at certain times can be random; however, for stochastic events, the observed value at each time is a random variable.

Definition 1.21 ([68], Definition 1.3**).** Risk refers to future uncertainty in the deviation of expected earnings. As such, it is a measure of the uncertainty that will be experienced before an investor profits from a particular investment.

Definition 1.22 ([69], Definition 14**).** Noisy observations are the error between true and observed values due to a lack of accuracy in measurement. In most cases, noise represents the risk encountered in an investment.

Definition 1.23 ([70], Definition 1.5**).** Life insurance is a contract between the insured and the insurer. The insurer agrees to pay a sum of money in exchange for a premium after agreed time or upon maturity or upon death of the insured.

Definition 1.24 ([71], Definition 1.1.16**).** (The Black-Scholes Formula) Assume that r is the interest rate and controls a security's price. The price of this security's derivative, f $(S,\ t)$, satisfies the partial differential equation. Portfolio return is the amount of money generated by a particular investment [72]. To visualize it, let's assume that we bought an asset at t_0 and sold it one year later. If X_0 and X_1 are amounts invested and amounts received respectively, then R_e is the total return. Usually, scholars use the term return to denote the total return of a portfolio.

2. Results and Discussion

This section focuses on establishing optimization conditions for the ex- tended (p, q)-binomial Cox-Ross-Rubinstein (CRR) model given below,

$\Xi^{p_r,q_r}(\theta, \xi) - \theta = \mathrm{J}$……………………..*(1)*

particularly in the context of managing portfolios in life insurance under varying noise conditions.

First, we give the convergence analysis. We provide a detailed analysis of the convergence of the $(p,\ q)$-extension model Equation 1 in this section. First, we consider the general setup. Then we consider convergence in the Skorohod space. Let $p = p_r$ and $q = q_r$ where $q_r \in [0, 1]$ and $p_r \in [q_r, 1]$ where $\lim_{r\to\infty} q_r = 1$ and $\lim_{r\to\infty} p_r = 1$. We take the limit in the increasing sense since in $\Omega^{\uparrow}{}_{\Gamma(A)}$ we talk of a class of strictly increasing continuous functions of the form $\theta : [0, 1] \rightarrow$ for which $\theta(0) = 0$ and $\theta(0) = 1$. We state the following theorem for a general setting.

Theorem 3.1. *Let* $\Omega^{\uparrow}{}_{\Gamma(A)}$ *be a Skorohod space and* $p = p_r$, $q = q_r$ *where* $\lim p_r = 1$, $\lim q_r = 1$ *for* $0 < q_r < p_r \leq 1$. *For any functions* $\theta \in \Gamma(A)[0, 1]$ *and the model in Equation 1, we have* $\lim_{r\rightarrow\infty} \| \Xi^{pr,qr} (\theta, \xi)\text{-}\theta \|_{\Gamma(A)} = 0$.

Proof. From the Theorem we see that the three conditions suffice. Indeed, we only let $p_r \rightarrow p$ and $q_r \rightarrow q$ as $r \rightarrow 0$. With the approximation property outlined in Theorem 4.15 in the general setting we now consider the convergence with respect to finite dimensional distributions. In this regard, we let the terminal time $\tau > 0$. This completes the proof.

Theorem 4.2 The extended (p,q)-Binomial CRR model Converges to Black-Scholes model

Proof. Consider the time movement from [0, 1] to 0, τ for any dimensional random vectors $p = p_m$ and $q = q_m$. Let $p \geq 1$ and $q \geq 1$. Also consider $\tau \in [0, 1]$ where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_p \leq 1$ and $0 = \tau_1 \leq \tau_2 \leq ... \leq \tau_q \leq 1$. Then we have $(\ln A_{[m\tau 1} ♩, m, \ln A_{[m\tau 2} ♩, m, ... \ln A_{[m\tau p} ♩, m)$ and $(\ln A ... \ln A_{[m\tau q} ♩, m)$. Which converges to *Black-Scholes model* Indeed by Proposition (4.1) in [33] and central limit theorem, the law of large numbers suffices. Hence, Equation 4.2.13 converges in probability since Brownian motion x_τ increaments are independent. By triangular transitions we obtain that Equation 4.2.8 holds for both $p \geq 1$ and $q \geq 1$.

Remark 4.17. It is evident that for a strike price J, the value of European option is convergent to the solution $(S_t)_{t\in[0,T]}$ of the stochastic differential equation.

In the context of optimizing portfolios with noisy observations in life insurance, the 3D convergence graph provides valuable insights.

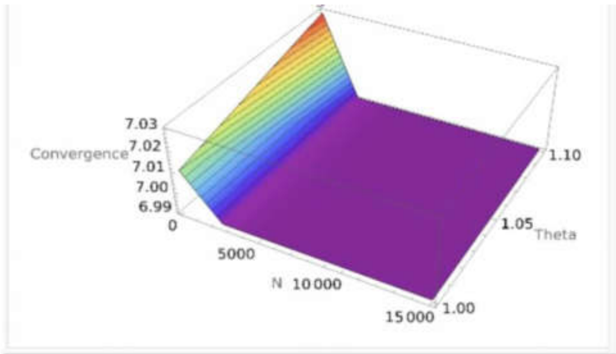

Figure 1: Convergence for η = -1

Figure 1 illustrates the convergence behavior of the extended (p, q)-binomial Cox- Ross-Rubinstein model for a specific case where $\eta = -1$. The graph is designed to visually represent how the predictions of the model approach a stable value as the number of time steps, N, increases. This visual analysis is crucial in assessing the stability and reliability of the model, especially when dealing with financial portfolios in the domain of life insurance. The 3D graph typically includes multiple surfaces, each represented by a different color, indicating the results of the model for different values of N. For example: A blue surface might represent $N = 25$, showing the model's predictions for a smaller number of time steps. A green surface could correspond to $N = 50$, illustrating the model's behavior for a medium number of time steps. A red surface might denote $N = 100$, showcasing the model's predictions for a larger number of time steps.

By observing the behavior of these surfaces and their intersections, one can assess how quickly the model converges to a stable value and understand the impact of different parameters on the rate of convergence. This analysis is pivotal for making informed decisions in the realm of life insurance and ensuring the robustness of the financial model under consideration. The second case follows a similar structure to create a 3D plot illustrating the convergence of option prices for $\eta = 1$. The only difference lies in the option prices eta 1 array, which contains the option prices corresponding to $\eta = 1$. In the context of the extended (p, q)-binomial Cox-Ross-Rubinstein model for optimizing portfolios with noisy observations in life insurance, the 3D convergence graph for $\eta = 1$ offers valuable insights. When $\eta = 1$, the model is expected to exhibit a distinct behavior compared to other η values.

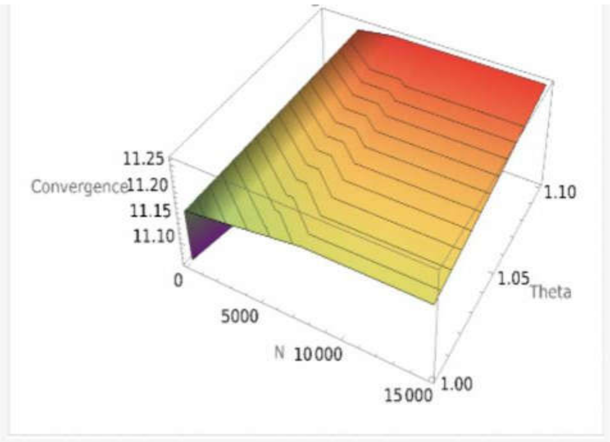

Figure 2: Convergence for $\eta = 1$

Figure 2 illustrates how the portfolio value converges to the Black- Scholes model as the number of time steps N increases. For $\eta = 1$, the convergence is observed to be relatively smoother and faster. This is indicative of the model's sensitivity to the η parameter and its impact on the portfolio optimization process.

In the 3D graph, different colors represent different values of the portfolio. For instance: - **Blue**: Represents lower portfolio values. - **Green**: Indicates intermediate portfolio values. - **Red**: Denotes higher portfolio values.

By analyzing the graph, one can observe that as N increases, the portfolio values tend to stabilize and converge towards the Black-Scholes value, demonstrating the efficacy and reliability of the extended model in the scenario where $\eta = 1$.

Table 1: Convergence Table presents numerical estimates for the graphs

η	θ	$N = 25$	$N = 50$	$N = 75$	$N = 100$	$N = 1000$	$N = 10000$	$N = 15000$	Black-Scholes
$\eta = 1$	$\theta = 1$	11.16	11.17	11.18	11.19	11.19	11.19	11.19	11.19
$\eta = 1$	$\theta = 1.1$	1.23	11.25	11.26	11.26	11.26	11.26	11.26	11.26
$\eta = 0$	$\theta = 1$	8.95	8.95	8.95	8.95	8.95	8.95	8.95	8.95
$\eta = 0$	$\theta = 1.1$	8.96	8.95	8.95	8.95	8.95	8.95	8.95	8.95
$\eta = -1$	$\theta = 1$	7.01	6.99	6.99	6.99	6.99	6.99	6.99	6.99
$\eta = -1$	$\theta = 1.1$	7.03	6.99	6.99	6.99	6.99	6.99	6.99	6.99

Table 1 shows the numerical values for the convergences for $\eta = -1$ and $\eta = 1$ illustrating how the option price converges as the number of time steps increases.

Finally we embark on the main aim of this paper. The preference and main aim of investors is to make profits when they invest regardless of presence of noisy observations or not. In this section we investigate conditions under which portfolio optimization is attained with respect to the extended CRR Model. Consider assets χ with noisy observations, we need to optimize the situation here. Let $S = 0$ be the price of χ. Now χ is given by vector α whose future is determined to pay off randomly at $t = 1$. The payoff is described by a random vectors V in same Skorohod Space (X, Ω, P) which is probabilistic. We make the following assumptions:

(i). The vectors are strictly having $+ve$ entries

(ii). χ is strictly risky.

Define portfolio F_r by $F_r = (F_o, F) \in$ R x R^{χ}. The future value of $F = F_r \cdot \alpha$. For sale of the F_r, its price regardless of the risk involved should be less or equal to initial capital. So the constrained budget becomes $F_r\alpha \leq C$ where C is the capital. Now we consider the expected utility $E_u(F_r \cdot \alpha)$. We maximize $E_u(F_r \cdot \alpha)$ over F_r under constraint $F_r\alpha \leq C$. Let Q be a χ dimensional random vector of discounted net gains

We state our optimum problem as follows:

Let $u : D \rightarrow$ R be the utility function. Maximize $E_u(F_r \cdot Q)$ over all risky portfolio F_r that satisfies $F_r \cdot Q \in D$.

Further assumptions:

(i). $D =$ R and u is bounded above

(ii). $D = [\chi, \infty)$ for some $x < 0$ and we optimize over the set of F_r such that $F_r \cdot Q \geq x$ almost surely. So $E_u(F_r \cdot Q)$ is finite.

3. Conclusion

In this paper, we have established optimization conditions for the extended (*p, q*)-binomial Cox-Ross-Rubinstein (CRR) model, particularly in the context of managing portfolios in life insurance under varying noise conditions. We also give the convergence analysis of the model.

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