MATHEMATICAL MODEL FOR NON-LINEAR STOCK PRICE ADJUSTMENT

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A Thesis submitted in partial fulfillment of the Requirements for the award of the degree of

Master of Science in Applied Mathematics

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ABSTRACT

The field of financial mathematics has drawn a lot of interest from both practitioners and academicians since the derivation of Black- Scholes model of 1970s. The celebrated option pricing formula, the so-called Black- Scholes –Merton option pricing model was developed by the use of geometric Brownian motion. But the model has a shortcoming because it assumes that the volatility is constant, when in reality it is not. To overcome this, Hull and White developed a stochastic volatility model in 1987. Recently in 2003, Onyango relaxed the geometric Brownian motion assumptions by applying the Walrasian price adjustment mechanisms, taking the supply and demand functions to respond to random fluctuation in asset trading. From the available literature, the work so far done on logistic Brownian motion does not include those assets that pay continuous dividends during the life of the option. This justified the need to develop a model that will take care of continuous dividend payments. In this study we have developed mathematical model for non-linear stock price adjustment that will be used to fit the prices of assets that pay continuous dividends and follow a non-linear trend. To achieve this, we have applied the knowledge of logistic Brownian motion and analysed the geometric Brownian motion model analytically. To verify this model, secondary data from London Stock Exchange has been used. Since dividend payments in financial markets are critical in securing investors, the result obtained from this study will help decision-makers in determining the prices of assets that pay continuous dividends that would attract more investors. We believe that this study has also contributed more knowledge to the field of mathematics of finance.

CHAPTER 1

1.1. Introduction

The uncertain movements of the values of assets over a period of time represent the dynamics of asset prices. This movement in asset price is fully described by a special Markov process –a particular type of stochastic process where only the present value of the asset price is relevant for predicting the future. The past history of the asset price and the way the present has emerged from the past are irrelevant, [8]. The fluctuation in asset price is explained using the Itô process;

$$dS(t) = \mu(S,t)dt + \sigma(S,t)dZ \quad , dZ = \varepsilon\sqrt{dt} \quad , \varepsilon \sim N(0,1) \tag{1.1.1}$$

where $S(t)$ is the price of asset, $\mu(S,t)$ is the trend, $\sigma(S,t)$ is the volatility of the underlying asset and $Z(t)$ is the standard random variable. Different assumptions about the trend and volatility give different solution for $S(t)$ in (1.1.1). For example, some benchmark models such as Bachelier model assumed that $\mu(S,t) = \mu$, $\sigma(S,t) = \sigma$ to give arithmetic Brownian motion model $dS(t) = \mu\, dt + \sigma\, dZ$ and in Black-Scholes-Merton option-pricing model, it is assumed that $\mu(S,t) = \mu S(t)$ and $\sigma(S,t) = \sigma S(t)$, ([12], [5]), to give the so-called linear geometric Brownian motion model:

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ \tag{1.1.2}$$

The stability of general price equilibrium has been studied as early as 1960s by employing Walrasian *tatonnement* (sometimes translated as 'groping', or 'tentative proceedings'). The main question in market stability is that "if markets are off the market equilibrium, will they tentatively proceed (grope) to market equilibrium? To answer this, Walras relaxed one of the assumptions in equation (1.1.2), which is that the asset price changes are associated with excess demand, [17]. If the excess demand function is given by $ED\,(S(t)) = Q_D S(t) - Q_S\,S(t)$: where $Q_D S(t)$ is demand function, $Q_S\,S(t)$ is the supply function, then the change in price is given by

$$\frac{d\,S(t)}{dt} = kED(t) \tag{1.1.3}$$

where k is referred to as the speed of market adjustment.

To accommodate stock prices that follow logistic trend, Onyango [13], further relaxed the assumptions in equation (1.1.2). In the formulation of stock price model, the Walrasian excess demand function is used and the prices of assets are driven by this excess demand function. In this model the dividend paying assets were not considered. In this study we develop a logistic Brownian motion model that is used to fit the prices of assets that pay continuous dividends, thereby confirming that the assumptions behind geometric Brownian motion used in derivation of Black- Scholes- Merton model can still be relaxed.

1.2. Literature review

The field of financial mathematics has drawn a lot of interest from both practitioners and academicians since the derivation of Black- Scholes model of 1970s. These researchers have been using the knowledge of Brownian motion that was first brought up by Scottish botanist Robert Brown in 1827 and was further studied in 1905 by Albert Einstein, [6].

The application of the knowledge of Brownian motion to the field of finance was discussed by Bachelier in (1900) in his thesis, [3].

In 1923, Norbert Weiner [18] made further rigorous mathematical development to Brownian motion hence Weiner- Bachelier process. Although Bachelier [3] introduced

Brownian motion to model stock price in his thesis, its modern applications to financial markets began in the late 1960s and 1970s.

Samuelson [15] developed a geometric Brownian motion also known as economic exponential model that is an alternative to Bachelier's arithmetic Brownian motion model. One advantage of the geometric Brownian motion is that the asset price at time t, S (t) is the value of an exponential function and non-negative at all times.

In early 1970s Fischer Black, Myron Scholes [5] and Robert Merton [12] made a major breakthrough in pricing of stock options by developing the so-called Black –Scholes-Merton model. To derive this celebrated option pricing formula, they applied geometric Brownian motion. This model has had a huge influence in the way traders set prices and hedge their options and has been fundamental to the growth of financial engineering from the 1980s to date. It has also become very popular among both researchers and traders to date. This is because of the fact that in many cases the normal distribution allows explicit computation of derivative prices if the exclusion of arbitrage through the risk neutral valuation formula is applied.

One of the shortcomings of this model is that it assumes that the volatility is constant but in reality, the volatility is not constant. Models with non-constant volatility have been developed by among others, [9]. Onyango [13] developed a time varying volatility model by considering a moving-window method, based on parameter estimate for an assumed geometric Brownian motion. The geometric Brownian motion assumption has also been relaxed by applying the Walrasian excess demand function of Walrasian Price adjustment mechanisms, taking the supply and demand functions to respond to random fluctuation in asset trading, [13]. This culminated into the development of logistic Brownian motion model for asset prices.

The work so far done on the logistic Brownian motion does not include those assets that pay continuous dividend during the life of the option. There is, therefore a need to develop a model that will take care of continuous dividend payments. Thus we develop a model that contains continuous dividend payments.

1.3. Statement of the problem

Dividend payments on stock in the financial market are critical in securing investors. Most investments do pay dividends yearly, half yearly, quarterly or continuously through out the year or period of investment. None payments of dividends and fluctuations of market prices make the investors to consider alternative avenues of investing their capital. Thus the inclusion of dividends is critical for the analysis of option pricing. In the linear Brownian motion, the Black – Scholes model has been modified to include payments of continuous dividends, [8]. In this study we have attempted to develop a logistic Brownian motion model as opposed to the linear geometric Brownian motion model to be used to fit the prices of shares with continuous dividend payments.

1.4. Objective of the study

The aim of this study was to develop a mathematical model that can be used to fit stock prices that follow logistic trends and pay continuous dividends.

1.5. Significance of the study

The result obtained will help decision-makers in predicting or determining the future prices of their stock depending on how dividend payments move.

It will also contribute to knowledge in the field of mathematics of finance.

1.6. Methodology

To develop this model, we have used the following:

i) application of logistic Brownian motion

ii) analysis of geometric Brownian motion model

iii) analysis of secondary data from London stock exchange, that is the British Airways Historical stock price data.

In the next chapter, we are going to look at some of the basic mathematical concepts that will be applied to develop this model.

CHAPTER 2

Stochastic processes.

Any variable whose value changes over time is said to follow a stochastic process. Stochastic processes are classified as discrete time or continuous time. Discrete time stochastic process is one where the value of the variable can change only at a certain fixed points in time, where as a continuous time stochastic process is one where changes in the value can take place at any time. Stochastic processes can also be classified as continuous variable or discrete variable. In a continuous variable process, the value of the underlying variable can take any value within a certain range, where as in a discrete variable process, only certain discrete values are possible.

2.1. Markov property

This is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged are irrelevant. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past. The future price only depends on the present information.

2.2. Weiner process

A Weiner process is a particular type of a Markov stochastic process with a mean change of zero and a variance rate of 1.0. It has been used in physics to describe the motion of particles that are subjected to a large number of small molecular shocks and is sometimes referred to as Brownian motion.

Expressed formally, a variable Z follows a Wiener process if it has the following two properties:

(i) the change in ΔZ during a small period of time Δt is, $\Delta Z = \varepsilon\sqrt{\Delta t}$, where ε is a random variable drawn from a standard normal distribution with mean 0 and standard deviation 1, i.e. $\varepsilon \sim N(0,1)$.

(ii) the values of ΔZ for any two different short intervals of time Δt are independent.

From property (i), ΔZ itself has a normal distribution with mean 0 and standard deviation $\sqrt{\Delta t}$. The second property implies that Z follows a Markov process.

Consider an increase in the value of Z during a relatively long period of time, T. This can be denoted by $Z(T) - Z(0)$, and let μ be the sum of the increases in Z on small intervals of length Δt where, $\mu = \frac{T}{\Delta t}$, then

$$Z(T) - Z(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}\, , \Delta t = T - T_0 \tag{2.2.1}$$

where the ε_i $(i = 1,2,3...,N)$ are random variables drawn from $N(0,1)$. From property (ii) of Weiner process, the $\varepsilon_{i,s}$ are independent of each other. It follows from equation

(2.2.1) that $Z(T)-Z(0)$ is normally distributed with mean 0, and standard deviation $\sqrt{T}$, i.e. $Z(T)-Z(0) \sim N(0,\sqrt{T-T_0})$

2.3. Brownian motion

This is perhaps the most interesting class of stochastic process used in financial economics. At first it was used to describe the random movements of an erratic jiggling particle. As time passes the particle slowly drifts through space, gradually wandering away from its starting point.

Brownian motion can be defined in 1-dimension, 2-dimension, or in any higher dimension depending for example in finance, on the number of traded assets taken to represent a given sector of the market. Examples include prices followed by prices of derivatives whose pay offs depends on the future values of several, possibly correlated, underlying assets. Some of the examples are interest rate derivatives, spread, and external barrier options, relative price and sector index, relative price and relative price index, and market index, [4]. We have different types of Brownian motion and some of them are given below:

2.3.1. A standard one-dimension Brownian motion

A standard one-dimensional Brownian motion $Z(t)$, $t \in [0,T]$ is a continuous-time process with the following properties:

- $Z(0)=0$ (with probability one)
- $Z(t)$ is continuous

- $Z(t)$ is memory-less in the sense, if sampled at times $0 \leq t_0 < t_1 < \ldots < t_n$ and $n \geq 0$, then the changes of values $\Delta Z(t_1) = Z(t_1) - Z(t_0), \ldots \Delta Z$, then $(t_n) = Z(t_n) - Z(t_{n-1})$ are statistically independent.
- The increments $\Delta Z(t_i)$ for $i = 1, 2, \ldots n$ are normally distributed with mean 0 and standard deviation $\sqrt{\Delta t_i}$.

More particularly, if $Z(t)$ satisfies the first three properties above for any $t \in [0,T]$, then the change $dZ(t)$, over small interval dt, is a Weiner process and can be expressed as $dZ(t) = \varepsilon_i \sqrt{dt}$ where ε_i are independent and identically normally distributed, i.e. $\varepsilon_i \sim N(0,1)$, with mean given by $Exp< dZ(t) >=0$, and variance given by

$$\text{var}(dZ(t)) = \text{var}(\varepsilon_i \sqrt{dt}) = dt\,\text{var}(\varepsilon_i) = dt, \text{ since } \text{var}(\varepsilon_i) = 1.$$

- For any $0 \leq s \leq t$, the increment $Z(t) - Z(s)$ has a normal distribution with mean zero and variance $(t - s)$, i.e. $Z(t) - Z(s) \sim N(0, \sqrt{t-s})$.

2.3.2. One- dimensional generalised Weiner process

A 1- dimensional generalised Weiner process has X(t) determined by a stochastic differential equation of the form

$$dX(t) = \mu\, dt + \sigma\, dZ(t), \quad X(0) = X_0 \qquad (2.3.1)$$

where μ (drift rate) and σ (standard deviation) are constants, and $Z(t)$ is a standard Weiner process. The mean of $dX(t)$ is given by $\mu\, dt$ and its variance is given by $\sigma^2 dt$.

Over a very short interval of $[t, t+\Delta t]$ the expected change is approximately $\mu \Delta t$ and the variance of the change is approximately $\sigma^2 \Delta t$. More precisely, the increase over any interval $[t_{i-1}, t_i]$ is given in the integral form

$$X_i(t) - X_{i-1}(t) = \mu \int_{t_{i-1}}^{t_i} ds + \sigma \int_{t_{i-1}}^{t_i} dZ(s) \tag{2.3.2}$$

In particular, for interval [0,*t*), equation (2.3.2) becomes

$$X(t) = X_0 + \mu \int_0^t ds + \int_0^t dZ(s) \tag{2.3.3}$$

The solution of equation (2.3.2) is given as

$X_i(t_i) = X_{i-1}(t_{i-1}) + \mu(t_i - t_{i-1}) + \sigma(Z(t_i) - Z(t_{i-1}))$ and the solution to equation (2.3.3) is given as:

$$X(t) = X_0 + \mu t + \sigma Z(t) \tag{2.3.4}$$

with $Z(0) = Z_0 = 0$ and $X(0) = X_0$.

The variable X(*t*) is normally distributed with mean and variance given as $X_0 + \mu t$ and $\sigma^2 dt$ respectively, that is $X(t) \sim N(X_0 + \mu t, \sigma\sqrt{t})$.

2.4. Itô processes

In the study of stochastic calculus, Itô processes generalise Brownian motion by taking parameters μ and σ to be the functions of the underlying variable X(t) and time t, therefore, from equation (2.3.1) $\mu = \mu_0(X(t),t)$ and $\sigma = \sigma_0(X(t),t)$ The process X(t) is then generated by the stochastic differential equation of the form:

$$dX(t) = \mu_0(X(t),t)\,dt + \sigma_0(X(t),t)\,dZ(t) \tag{2.4.1}$$

Here the process parameters are the instantaneous drift rate $\mu_0(X(t),t)$, which measures the expected rate of change of X(t) and the instantaneous variance $\sigma_0(X(t),t)$, which measures the amount of random diffusion. Both are liable to change over time, to a first order approximation, this means that for small increments,

$$Exp < X(t+\Delta t) - X(t) >= \mu_0(X(t),t)\Delta t \text{ and}$$

$$\mathrm{var} < X(t+\Delta t) - X(t) >= \sigma^2(X(t),t)\Delta t$$

$$\text{i.e. } X(t+\Delta t) - X(t) \sim N(\mu_0(X(t),t)\Delta t,\ \sigma_0(X(t),t)\sqrt{\Delta t})$$

To be more precise, the change over any interval $[t_{i-1}, t_i]$, is given in the integral form as;

$$X(t_i) = X(t_{i-1}) + \mu_0 \int_{t_{i-1}}^{t_i} (X(s),s)ds + \sigma_0 \int_{t_{i-1}}^{t_i} (X(s),s)dZ(s) \qquad (2.4.2)$$

More exactly over the interval [0, t], X(t) in the integral form is:

$$X_i(t) = X_0(t) + \mu_0 \int_0^t (X(s),s)ds + \sigma_0 \int_0^t (X(s),s)dZ(s) \qquad X_0 > 0 \qquad (2.4.3)$$

In equation (2.4.1), the expected drift $\mu_0(X(t),t)dt$ and standard deviation $\sigma_0(X(t),t)\sqrt{dt}$ are functions of the level X(t) itself and time t.

2.5. Geometric Brownian motion (gBm)

This is a particular class of Itô process commonly used to model stock returns. It is always referred to as "*log-normal diffusions*", geometric Weiner process or " *economic exponential models*", [15]. In the derivation of the so-called Black- Scholes- Merton option pricing model, Black and Scholes [5], and Merton [12] assume that the stock returns S(t) follows a geometric Brownian motion. The general expression of geometric Brownian motion is given by

$$dX(t) = \mu X(t)dt + \sigma X(t)dZ(t), \qquad \mu > 0, \ \sigma > 0 \tag{2.5.1}$$

where μ represents the annual mean of returns dX(t)/ X(t) and σ represents the annual volatility of returns.

Over a short interval $[t, t+\Delta t]$, the expected change is approximately $\sigma^2 X^2 \Delta t$. More precisely, the change over any interval $[t_{i-1}, t_i]$ is given in the integral form as:

$$X(t_i) - X(t_{i-1}) = \mu \int_{t_{i-1}}^{t_i} X(s)ds + \sigma \int_{t_{i-1}}^{t_i} X(s)dZ(s) \tag{2.5.2}$$

In particular for interval [0, t) equation (2.5.1) is expressed as an integral equation using integral of the type of equation (2.4.2):

$$X_i(t) = X_0 + \mu \int_0^t X(s)ds + \sigma \int_0^t X(s)dZ(s) \tag{2.5.3}$$

where $X_0 > 0$. Equation (2.5.1) can also be expressed as a law of returns give

$$dX(t)/X(t) = \mu dt + \sigma\, dZ(t), \qquad X(0) = X_0 \tag{2.5.4}$$

which is the relative (percentage) change of the process over the infinitesimally short interval $[t, t+\Delta t]$. If X(t) is the price of a traded asset, then $dX(t)/X(t)$ is the rate of return on the asset over the next instant and the solution is given by

$$\log(X(t)/X_0) = \int_0^t \mu\, ds + \int_0^t \sigma dZ(s) \tag{2.5.5}$$

Equation (2.5.4) cannot be solved directly since it is not linear in X(t) and contains parts that involves Weiner process, Z(t). This can be solved by an analogy of chain rule of ordinary calculus. The integral form of equation (2.5.5) indicates that the logarithmic stock return executes the relatively simple generalised Weiner process now called "geometric Brownian motion" that was first introduced in finance by Samuelson in 1965. It has an advantage over the generalised Weiner process introduced by Bachelier in that $X(t) = \exp(S(t))$ can never assume negative values. In the next sub-topic we state the so-called Itô's lemma.

2.6. Itô's lemma

Consider a random variable X(t) that follows a diffusion process (2.4.1), with a predictable rate of return $\mu(X(t),t)dt$ and instantaneous rate of variance $\sigma^2(X(t),t)dt$ that is

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dZ \quad (2.6.1)$$

If $\sigma = 0$, then we have a deterministic case

$$dX(t) = \mu(X(t),t)dt \quad (2.6.2)$$

But when $\sigma \neq 0$, X(t) has sample paths that are differentiable nowhere. We thus use equation (2.4.1), which does not require us to divide by *dt*. In the integrals, use of ordinary calculus to integrate for values of X(t) cannot be applied in equations (2.5.2) and (2.5.4), since these integrals contain parts which involve Weiner process , Z(t). Therefore, there is a need for some set rules that will enable us to solve such differential equations, hence the use of Itô's multiplication table.

The Itô multiplication rule only considers quantities of order *dt* and ignores quantities of order dt^n, $n > 0$, and are taken to be zero. Therefore, we have the Itô multiplication rule:

$dZ(t)dt = dtdZ(t) = dt\varepsilon\sqrt{dt} = \varepsilon(dt)^{\frac{3}{2}}$ since $dZ(t) = \varepsilon\sqrt{dt}$.

Taking the limit as $dt \to 0$, $dt^2 \to 0$, $dt^{\frac{3}{2}} \to 0$, and any other higher order of dt will tend to zero. Hence, we have the multiplication table:

$\times$	dt	dZ
dt	0	0
dZ	0	dt

Table. 3.6.1 Itô multiplication

When dt is small then,

$$dZ^2(t) = Exp< \varepsilon^2 dt >= dt\, Exp< \varepsilon^2 > = dt$$

since $Exp < \varepsilon^2 >= 1$ and $dX^2(t) = \sigma^2 dt$. For further information, see [8], (Chapter 10).

If a random variable X(t) follows a diffusion process in equation (2.4.1), then by use of Itô's lemma we compute the stochastic differential equation of smooth function G(x(t),t), which depends on X(t) and t.

As earlier discussed, if the diffusivity coefficient $\sigma(X(t),t) = 0,$ then $dX(t) = \mu(X(t),t)dt$ and ordinary calculus gives us the answer: thus the change in G(X(t),t) over an infinitesimal time interval is the total derivative G(X(t),t) given as:

$$dG(X(t),t) = \frac{\partial G}{\partial X} dX(t) + \frac{\partial G}{\partial X} \mu(X(t),t)dt \qquad (2.6.3)$$

If G(X(t), t) is not linear, however, then the solution can be found by use of Itô's lemma. Let X(t) be an Itô process with drift $\mu(X(t),t)$ and diffusion $\sigma(X(t),t)$. Let G(X(t),t) be twice continuously differentiable function in X and once in t, then by Taylor series expansion and ignoring terms of higher orders, the stochastic differential of G(X(t),t) is given by:

$$dG(X(t),t) = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial X}dX(t) + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}dX^2(t) \tag{2.6.4}$$

Equation (2.6.4) means that the change in G(X(t),t) is a function of change in X(t).

If $dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dZ(t)$, and putting $\mu = \mu(X(t),t)$, $\sigma = \sigma(X(t),t)$ then by Itô's multiplication rules, $dX^2(t) = \sigma^2 dt$ and substituting for dX(t) and $dX^2(t)$ in (2.6.4), we get

$$dG(X(t),t) = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial X}\left[\mu\, dt + \sigma dZ(t)\right] + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2 dt \tag{2.6.5}$$

which simplifies to :

$$dG(X(t),t) = \left(\frac{\partial G}{\partial t} + \mu\frac{\partial G}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial X^2}\right)dt + \sigma\frac{\partial G}{\partial X}dZ(t) \tag{2.6.6}$$

Thus G(X(t),t) is an Itô process with diffusion coefficient $\sigma\frac{\partial G}{\partial X}$ and drift rate

$$\left(\frac{\partial G}{\partial t} + \mu\frac{\partial G}{\partial X} + \sigma^2\frac{1}{2}\frac{\partial^2 G}{\partial X^2}\right)$$

The integral form of (2.6.6) is given as

$$G(X,t) = G(X,0) + \int_0^{t_i}\left(\frac{\partial G}{\partial s} + \mu\frac{\partial G}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2 G}{\partial X^2}\right)ds + \int_0^{t_i}\sigma\frac{\partial G}{\partial X}dZ(s) \tag{2.6.7}$$

If G is a function of X only, then equation (2.6.6) becomes

$$G(X(t),t) = \left(\mu\frac{\partial G}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial X^2}\right)dt + \sigma\frac{\partial G}{\partial X}dZ(t) \tag{2.6.8}$$

and the corresponding integral form is;

$$G(X) = G(X_0) + \int_0^{t_i} \left(\mu \frac{\partial G}{\partial X} + \frac{\sigma^2}{2} \frac{\partial^2 G}{\partial X^2} \right) ds + \int_0^{t_i} \sigma \frac{\partial G}{\partial X} dZ(s) \qquad (2.6.9)$$

The above result is summarized below as Itô's lemma.

Lemma 1: Itô's lemma

Let X(t) be an Itô process given by

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dZ(t)$$

and let G(X(t), t) be a function that is twice differentiable with respect to X and once with respect to t and of the random process X(t), then if we let $Y(t) = G(X(t),t)$, then

$Y(t) = G(X(t),t)$ is again an Itô' process given by:

$$dY(t) = dG(X(t),t) = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial X} dX(t) + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial X^2} dX^2(t) \qquad (2.6.10)$$

or , after substituting for dX(t) and dX^2(t) on the right hand side of equation (2.6.10), we get

$$dG(X,t) = [(\mu(X(t),t)\frac{\partial G}{\partial X} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2((X(t),t)\frac{\partial^2 G}{\partial X^2})]dt + \sigma(X(t),t)\frac{\partial G}{\partial X} dZ(t)$$

(2.6.11)

Here the change in G is a function of X and t. Thus as X and t change, they induce change in G. Note that the values of dX(t) and dX^2(t) have been defined on page 15.

It is convenient to express equation (2.6.11) in the integral form as:

$$G(X(t),t) = G(X(0),0) + \int_0^{t_i} \left(\mu(X(s),s)\frac{\partial G}{\partial X} + \frac{\partial G}{\partial s} + \frac{\sigma^2}{2}(X(s),s)\frac{\partial^2 G}{\partial X^2} \right) ds + \int_0^{t_i} \sigma(X(s),s)\frac{\partial G}{\partial X} dZ(s)$$

(2.6.12)

Itô's lemma can be extended to more than one market to get a multi-dimensional Itô's lemma.

2.6.1. Multi dimensional Itô's lemma

Let $X_1, X_2, \ldots, X_n$, be random variables with drifts $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Let $G(X_1, X_2, \ldots, X_n)$ be continuously differentiable function , twice in X and once in t, then $G(X_1, X_2, \ldots, X_n, t)$ is an Itô process governed by the stochastic process:

$$dG = \frac{\partial G}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial G}{\partial X_i}dX_i + \frac{1}{2}\sum\frac{\partial^2 G}{\partial X_1 \partial X_j}dX_i dX_j = \left[\frac{\partial G}{\partial t} + \sum_{i=1}^{n}\mu_i\frac{\partial G}{\partial X_i} + \frac{1}{2}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_i\frac{\partial^2 G}{\partial X_i \partial X_j}\right]dt$$

$$+\sum_{i=1}^{n}\sigma_i\frac{\partial G}{\partial X_i}dZ_i \qquad (3.6.13)$$

where $dZ_i = \varepsilon_i\sqrt{dt}$, $\varepsilon_i \sim iidN(0,1)$ and ρ ij is the coefficient of correlation between Z_i and Z_j, and $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1$. Here the multi dimensional Itô rule is given as:

$dtdZ_i = dZ_idt = dt.dt = 0$ and $dZ_idZ_j = \rho_{ij}$, $dZ_idZ_i = dt$ and if Z_i are independent Weiner proceses then $dZ_idZ_j = 0$, [13].

3.6.2. Two-dimensional Itô's lemma

Suppose $X_1(t)$ and $X_2(t)$ are Itô processes governed by:

$dX_1 = \mu_1 dt + \sigma_1 dZ_1$ and $dX_2 = \mu_2 dt + \sigma_2 dZ_2$ respectively.

We assume that Weiner processes Z_1 and Z_2 are independent (or dependent). If $G(X_1, X_2, t)$ is a twice differentiable function of t, then by equation (2.6.11) $G(X_1, X_2, t)$ is an Itô process driven by:

$$dG(X_1,X_2,t) = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial X_1}dX_1 + \frac{\partial G}{\partial X_2}dX_2 + \frac{1}{2}\left[\frac{\partial^2 G}{\partial X_1^2}(dX_1)^2 + \frac{\partial^2 G}{\partial X_2^2}(dX_2)^2 + 2\frac{\partial^2 G}{\partial X_1 \partial X_2}dX_1 dX_2\right] \quad (2.6.14)$$

If $X_1(t)$ and $X_2(t)$ are independent then $dZ_1 dZ_2 = 0$ if not correlated, otherwise, if they are correlated, then $dZ_1 dZ_2 = \rho\, dt$, where ρ is the coefficient of correlation.

If X_1 and X_2 are dependent variables, then by Itô multiplication table (2.6.1), we have:

$(dX_1)^2 = \sigma_1^2 dt$, $(dX_2)^2 = \sigma_2^2 dt$, $dX_1 dX_2 = \rho\sigma_1\sigma_2 dt$, substituting in equation (2.6.14) gives

$$dG(X_1,X_2,t) = \left[\frac{\partial G}{\partial t} + \frac{\partial G}{dX_1}\mu_1 + \frac{\partial G}{\partial X_2}\mu_2 + \frac{1}{2}\left(\frac{\partial^2 G}{\partial X_1^2}\sigma_1^2 + \frac{\partial^2 G}{\partial X_2^2}\sigma_2^2 + 2\frac{\partial^2 G}{\partial X_1 \partial X_2}\rho\sigma_1\sigma_2\right)\right]dt$$

$$+\sigma_1 \frac{\partial G}{\partial X_1}dZ_1 + \sigma_2 \frac{\partial G}{\partial X_2}dZ_2 \quad (2.6.15)$$

If X_1 and X_2 are independent, $\rho = 0$, then,

$$dG(X_1,X_2,t) = \left[\frac{\partial G}{\partial t} + \frac{\partial G}{dX_1}\mu_1 + \frac{\partial G}{\partial X_2}\mu_2 + \frac{1}{2}\left(\frac{\partial^2 G}{\partial X_1^2}\sigma_1^2 + \frac{\partial^2 G}{\partial X_2^2}\sigma_2^2\right)\right]dt + \sigma_1 \frac{\partial G}{\partial X_1}dZ_1$$

$$+\sigma_2 \frac{\partial G}{\partial X_2}dZ_2 \quad (2.6.16)$$

If X_1 and X_2 are both Itô's processes as is the case in most financial markets, that is, $Z_1 = Z_2 = Z$ and $\rho = 1$, then equation (2.6.14) becomes:

$$dG(X_1,X_2,t) = \left[\frac{\partial G}{\partial t} + \frac{\partial G}{dX_1}\mu_1 + \frac{\partial G}{\partial X_2}\mu_2 + \frac{1}{2}\left(\frac{\partial^2 G}{\partial X_1^2}\sigma_1^2 + \frac{\partial^2 G}{\partial X_2^2}\sigma_2^2 + 2\frac{\partial^2 G}{\partial X_1 \partial X_2}\sigma_1\sigma_2\right)\right]dt$$

$$+\left(\sigma_1 \frac{\partial G}{\partial X_1} + \sigma_2 \frac{\partial G}{\partial X_2}\right)dZ \quad (2.6.17)$$

2.7. Applications of Itô's lemma

In this section, we apply Itô's lemma in some areas of interest in financial sector.

2.7.1. Application to Geometric Brownian motion

This type of stochastic process is used to model stock markets and is one of the assumptions used by Black and Scholes [5] to derive the celebrated Black- Scholes option-pricing model. Here X(t) is replaced by the price of the underlying, S(t) so that $\mu = \mu S(t)$ and $\sigma = \sigma S(t)$, from equation (2.5.1), we have

$$d\,S(t) = \mu\, S(t)dt + \sigma S(t)\, dZ \tag{2.7.1}$$

Here the drift rate and standard deviation change proportionally to $S(t)$. Dividing both sides of (3.7.1) by $S(t)$, we get,

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dZ \tag{2.7.2}$$

From which the drift and diffusion change of percentage change of $S(t)$ $dS(t)/S(t)$ still have time homogeneous parameters.

It can be said that $S(t)$ follows geometric Brownian motion if the percentage change in $S(t)$, $\left(i.e. \dfrac{dS(t)}{S(t)} \right)$ follows a generalised geometric Brownian motion.

In order to get a solution to (2.7.2), we let G(S, t) be a function that is twice differentiable with respect to S and once with respect to t. Suppose $G(S,t) = \log S(t)$, then differentiating G(S, t) twice with respect to S and once with respect to t then

$\dfrac{\partial G}{\partial S} = \dfrac{1}{S}$, $\dfrac{\partial^2 G}{\partial S^2} = -\dfrac{1}{S^2}$, $\dfrac{\partial^3 G}{\partial S^2 \partial t} = 0$, and putting $\mu = \mu S(t)$, $\sigma = \sigma S(t)$, and replacing X(t) by S(t) in equation (2.6.10), and simplifying, we get,

$$dG(S,t) = d(\log S(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dZ \quad ,dZ = \varepsilon\sqrt{dt}\,, \tag{2.7.3}$$

From (2.7.3), $d(\ln S(t))$ follows a generalised Weiner process in which drift rate is $\left(\mu - \frac{\sigma^2}{2}\right)$, and σ as the diffusion coefficient. Both drift rate and diffusion coefficient are constants.

Over an interval $[t_{i-1}, t_i]$, a generalised solution to equation (2.7.3) is given as:

$$\log S(t_i) = \log S(t_{i-1}) + \left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\,\varepsilon_i \sqrt{(t_i - t_{i-1})} \tag{2.7.4}$$

A strong solution is given as,

$$S(t_i) = S(t_{i-1})\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\,\varepsilon_i\sqrt{(t_i - t_{i-1})}\right] \tag{2.7.5}$$

with mean $Exp < dS(t_i) >= S^2(t_{i-1})\exp[\mu(t_i - t_{i-1})]$ and variance,

$\text{var} < S(t_i) >= S^2(t_i)\exp(2\mu(t_i - t_{i-1}))[\exp(\sigma^2(t_i - t_{i-1})) - 1]$

For the interval [0, t], equation (2.7.4) is written as:

$$\log S(t) = \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\,\varepsilon\sqrt{t} \quad [16] \tag{2.7.6}$$

where $S(0) = S_0 > 0$, Thus $\log S(t)$ is normally distributed for any time t, with mean given by:

$\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance given by $\sigma^2 t$. That is,

$$\log S(t) \sim N\left(\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma\sqrt{t}\right) \tag{2.7.7}$$

The change in logarithm of the stock price during time [0, T] is given by the following equation:

$$\log S(T) - \log S_0 = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} \qquad (2.7.8)$$

The corresponding distribution of equation (2.7.8) is given by:

$$\log S(t) - \log S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right),$$

and the distribution of logarithm of percentage change in price is:

$$\log\left(S(t)/S_0\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right) \qquad (2.7.9)$$

From equation (2.7.8), we have a strong solution given by:

$$S(T) = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{T}\right] \qquad (2.7.10)$$

From (2.7.10), it can be shown that S (T) is log-normally distributed with expected mean value and variance given by $S_0 \exp(\mu T)$ and $S^2 \exp(2\mu T)\left[\exp\left(\sigma^2 T\right) - 1\right]$ respectively.

The distribution of S (t) is then given as

$$S(t) \sim \log - normal\left(S_0 \exp(\mu T), S_0 \sqrt{\left(\exp(2\mu T)\left[\exp(\sigma^2 T) - 1\right]\right)}\right) \qquad (2.7.11)$$

It should be noted that if $\sigma = 0$, then equation (2.7.10) becomes $S(t) = S_0 \exp(\mu T)$, thus $S(t)$ grows exponentially with expectation $S_0 \exp(\mu T)$, and variance zero, [19].

Suppose S_1 and S_2 are two Itô's processes with stochastic differential equations:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dZ_1 \qquad (2.7.12)$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dZ_2 \qquad (2.7.13)$$

Then the following lemma follows:

2.7.2. Itô's lemma for sum of Itô's processes

We let $G(S_1, S_2, t) = S_1 + S_2$, (2.7.14)

$$\text{then, } \frac{\partial G}{\partial t} = 0, \ \frac{\partial G}{\partial S_1} = \frac{\partial G}{\partial S_2} = 1, \ \frac{\partial^2 G}{\partial S_1^2} = \frac{\partial^2 G}{S_2^{\ 2}} = \frac{\partial^2 G}{\partial S_2 \partial S_1} = \frac{\partial^2 G}{\partial S_1 \partial S_2} = 0$$

Hence by Itô's lemma, equation (2.6.14), becomes

$$dG(S_1, S_2, t) = \mu_2 dt + \sigma_1 dZ_1 + \mu_2 dt + \sigma_2 dZ_2 \qquad (2.7.15)$$

$$\text{or } dG(S_1, S_2, t) = (\mu_1 + \mu_2)\, dt + \sigma_1 dZ_1 + \sigma_2 dZ_2 \qquad (2.7.16)$$

If S_1,S_2 are both affected by the same Weiner process, Z, then (2.7.16) becomes:

$$dG(S_1 + S_2) = (\mu_1 + \mu_2)\, dt + (\sigma_1 + \sigma_2) dZ \qquad (2.7.17)$$

Thus G is an Itô's process with drift $(\mu_1 + \mu_2)\, dt$ and variance $(\sigma_1 + \sigma_2)^2 dt$

2.7.3. Itô's lemma for a logarithm of product of Itô's process

Suppose the sum logarithm of two Ito's processes is given by:

$$G(S_1, S_2, t) = \log(S_1) + \log(S_2) \tag{2.7.18}$$

Then the partial derivatives of G with respect to S_1, S_2 and t are:

$$\frac{\partial G}{\partial t} = 0, \quad \frac{\partial G}{\partial S_1} = \frac{1}{S_1}, \quad \frac{\partial G}{\partial S_2} = \frac{1}{S_2}, \quad \frac{\partial^2 G}{\partial S_1^{\,2}} = -\frac{1}{S_1^{\,2}}, \quad \frac{\partial^2 G}{\partial S_2^{\,2}} = -\frac{1}{S_2^{\,2}}$$

$$\frac{\partial^2 G}{\partial S_1 \partial S_2} = \frac{\partial^2 G}{\partial S_2 \partial S_1} = 0.$$

using these partial derivatives and applying Ito's lemma in two dimension, equation (2.6.15), then the change in G becomes:

$$dG(S_1, S_2, t) = \left[\frac{\mu_1}{S_1} + \frac{\mu_2}{S_2} + \frac{1}{2}\left(-\frac{\sigma^2}{S_1^{\,2}} - \frac{\sigma_2^{\,2}}{S_2^{\,2}}\right)\right] dt + \frac{\sigma_1}{S_1} dZ_1 + \frac{\sigma_2}{S_2} dZ_2 \tag{2.7.19}$$

It can be concluded that if S_1 and S_2 are Itô's processes then $G(S_1, S_2, t)$ is also an Itô's process as shown in equation (2.7.19).

2.7.4. Itô's lemma for the reciprocal of an Itô process

Suppose S in Itô's process, then if $G(S,t) = \frac{1}{S}$, then $\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial S} = -\frac{1}{S^2}$, $\frac{\partial^2 G}{\partial S^2} = \frac{2}{S^3}$,

hence by Itô's lemma we have:

$$dG(S,t) = \left(\frac{-\mu}{S^2} + \frac{\sigma^2}{S^3}\right) dt - \frac{\sigma}{S^2} dZ \tag{2.7.20}$$

which is simplified to:

$$dG(S,t) = \frac{1}{S^2}\left[\left(-\mu + \frac{\sigma^2}{S}\right)dt - \sigma\, dZ\right] \qquad (2.7.21)$$

that is, $$d\left(\frac{1}{s}\right) = \frac{1}{S^2}\left[\left(-\mu + \frac{\sigma^2}{S}\right)dt - \sigma\, dZ\right] \qquad (2.7.22)$$

Thus, the inverse of Itô process is also an Itô process.

2.7.5. Itô's lemma for a ratio of two Itô processes

Let $$G(S_1,S_2,t) = S_1/S_2 \qquad (2.7.23)$$

Then $\frac{\partial G}{\partial t} = 0,\ \frac{\partial G}{\partial S_1} = \frac{1}{S_2},\ \frac{\partial G}{\partial S_2} = \frac{-S_1}{S_2^{\,2}}$.

$\frac{\partial^2 G}{\partial S_1^{\,2}} = 0,\ \frac{\partial^2 G}{\partial S_2^{\,2}} = \frac{2S_1}{S_2^{\,3}},\ \frac{\partial^2 G}{\partial S_1 \partial S_2} = \frac{\partial^2 G}{\partial S_2 \partial S_1} = \frac{-1}{S_2^{\,2}}$

Hence by Itô's lemma we have,

$$dG = \left(\frac{\mu_1}{S_1} - \frac{\mu_2 S_1}{S_2^{\,2}} + \frac{\sigma_2^{\,2} S_1}{S_2^{\,2}} - \frac{\rho\sigma_1\sigma_2}{S_2^{\,2}}\right)dt + \frac{\sigma_1}{S_2}dZ_1 - \frac{\sigma_2 S_1}{S_2^{\,2}}dZ_2 \qquad (2.7.24)$$

If the processes are affected by the same Weiner process, Z, (ρ =1), then equation (2.7.5) becomes:

$$dG = \left(\frac{\mu_1}{S_1} - \frac{\mu_2}{S_2^{\,2}} + \frac{\sigma_2^{\,2} S_1}{S_2^{\,3}}\sigma_1\sigma_2\right)dt + \left(\frac{\sigma_1}{S_2} - \frac{\sigma_2 S_1}{S_2^{\,2}}\right)dZ \qquad (2.7.25)$$

Thus the ratio of two Itô processes is also an Itô process as shown in equations (2.7.23), (2.7.24), and (2.7.25).

2.7.6.Application of Itôs lemma in financial market

Here, we make use of Itôs lemma to derive the dynamics of a derivative whose dynamics depend on the dynamics of the underlying assets. Suppose a stock price, $S(t)$ evolves according to the Itô process

$$dS(t) = \mu_1 S(t)\,dt + \sigma_1 S(t)V(t)\,dZ_1(t) \tag{2.7.26}$$

where $V(t)$ is itself the price of an asset that evolves according to:

$$dV(t) = \mu_2 V(t)\,dt + \sigma_2 V(t)\,dZ_2(t) \tag{2.7.27}$$

Here $Z_1(t)$ and $Z_2(t)$ are standard Weiner processes and $dZ_1(t)dZ_2(t) = \rho\,dt$ for some constant ρ .

Let the price of the derivative whose pay-off depends on stock price levels of S and V be denoted by

$G(S, V, t)$ and by Itô multiplication table we have:

$$dS(t).dV(t) = \rho\,\sigma_1\sigma_2 S(t)\,V(t)^2\,dt$$

Substituting in multi-dimensional Itô's lemma, equation (2.6.12), we get:

$$dG(S, V, t) = \left[\frac{\partial G}{\partial t} + \mu_1 S\frac{\partial G}{\partial S} + \mu_2 V\frac{\partial G}{\partial V} + \frac{1}{2}\left[\sigma_1^2 S^2 V^2\frac{\partial^2 G}{\partial S^2} + \sigma_2^2 V^2\frac{\partial^2 G}{\partial V^2} + \rho\sigma_1\sigma_2 V^2 S\frac{\partial^2 G}{\partial S\partial V}\right]\right]dt + \sigma_1 VS\frac{\partial G}{\partial S}dZ_1 + \sigma_2 V\frac{\partial G}{\partial V}dZ_2 \tag{2.7.28}$$

This is also an Itô's process.

CHAPTER 3

In this chapter we look at different types of financial markets, different types and styles of options, time value of an asset and finally the Black-Scholes model.

3.1. Financial markets

The security market is an economic market in which buyers and sellers of corporate and government securities are brought together. The securities (receipts for individual's savings available to the users) fall into two categories:

- Certificate of indebt ness (these are known as bonds or notes)
- Certificate of ownership (these are stocks or equity).

Some of the security exchanges are stock markets, capital and money markets and future markets.

3.1.1. Stocks Markets

Stock markets are interested in securities with nationwide market. The market decides on the security to list or trade. Incases where the securities are not listed or traded in an organized exchange, dealers will buy and sell them in what is called *over- the- counter market*.

3.1.2. Capital and Money Markets

Capital market is a market where money is loaned and borrowed for more than one year. Corporate and government bonds are examples. A market where money is loaned for a period of less than one year is money market.

3.1.3. Future markets

Futures are contracts to buy or sell a commodity at some point in the future at rates decided on at the present. The markets where futures are traded is known as future markets.

3.1.4. Stock Price

Common stocks are ownership shares in corporations. When business is low, corporate earnings fall. If business is bad enough, the companies may go bankrupt. People do not want to own stock in companies that might go bankrupt, and so they sell their shares pushing the stock price down. However, no one can sell the stock unless someone is buying. The effort to sell pushes the stock price down until someone is willing to buy. Thus the stock price logically goes down during recession when the business turns bad. If the sellers anticipate recession in the future they may push the stock down. If they anticipate a revival of the business they push the stock price up. The stock price movements are random and adjust to the new information as it comes available. During this adjustment period, the price moves up and down around some trend line that reflects current market equilibrium.

3.2. An option

An option gives its holder the right, but not the obligation to buy or sell a certain amount of an asset by a certain date and for a specific price. The price in the contract is known as the exercise or strike price and the date in the contract is the maturity or expiry date.

3.2.1. Types of options

There are two types of options:

a) Call options: this gives the holder the right, but not the obligation, to buy the underlying asset by a certain date for a certain price.

b) Put option: this gives the holder the right, but not the obligation, to sell the underlying asset by a certain date for a certain price.

3.2.2. Styles of options

The style of option refers to when an option is exercisable. Two of the most common styles of options are:

a) European styles of options; these are contracts to be exercised only at maturity

b) American styles of options; these are contracts to be exercised at any time prior to maturity. In this study we are going to look at the European styles of options.

3.3. Volatility

The volatility of the stock price is denoted by σ. It determines how uncertain we are about the future movement of stock price.

3.4. Time value of an asset.

Let us consider the price, S_t, of an asset whose value appreciates at the rate r, with change in time dt. Then we have

$$dS_t = S_t r\, dt \qquad (3.4.1)$$

Equation (3.4.1) is solved by first separating the variables to give

$$\ln S_t = rt + C \qquad (3.4.2)$$

where C is any arbitrary constant.

Considering time $t = 0$, and take $S_t = S_0$, this is called the initial asset price. Therefore at time, $t = 0, C = \ln S_0$. Substituting this value of C in equation (3.4.2), we get

$$\ln S_t = rt + \ln S_0$$

Solving for S_t, we have

$$S_t = S_0 e^{rt} \qquad (3.4.3)$$

Equation (3.4.3) is referred to as the *economic exponential model*. From this equation, we see that when $r > 0$, the price, S_t, increases. When $r = 0$, the price, $S_t = S_0$, that is there no change from the initial price. When $r < 0$, the price, S_t, decreases but it will never be equal to zero.

Figure (3.1) depicts the three graphs for S_t

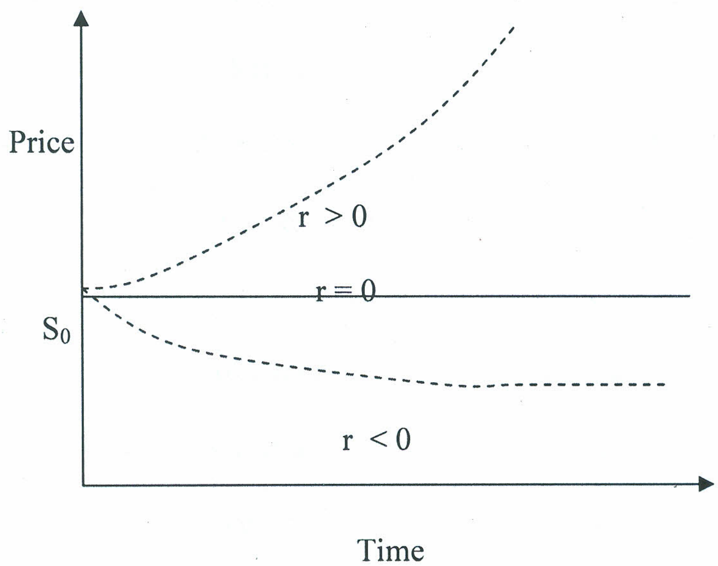

Fig. 3.1. Exponential model.

3.5. Black- Scholes model

Fischer Black, Myron Scholes and Robert Merton are the three economists, who in 1973 made a major breakthrough in the pricing of stock options by developing the *Black-Scholes model*. This model has had a major influence in the way traders price and hedge options and also contributed heavily to growth and success of financial engineering in 1980s and 1990s. In this sub-section we will look at how the black-Scholes equation is derived, solved and used for valuing both European call and put options

3.5.1. Derivation of the Black- Scholes equation

The Black- Scholes- Merton differential equation is an equation that has to be satisfied by the price S(t) of any derivative dependent on non-dividend paying stock. To derive the Black- Scholes equation the following assumptions were made:

- No dividends are paid out on the underlying stock during the life of the option.
- The price of the underlying asset follow a geometric Brownian motion
- The option can only be exercised at the expiry (European option)
- Efficient market (market movements are predictable)
- Commissions and other fees are non- existent (no transaction cost)
- Interest rate do not change over life of the option (constant and are known)
- Stock returns have lognormal property.

3.5.2. Lognormal Property of the stock

A variable whose natural logarithm is normally distributed is said to have a lognormal distribution. In equation (2.7.3), we showed that if the stock price S(t), follows a geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ$$

Then by applying Itô's lemma we get

$$d(\ln S(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dZ \qquad (3.5.1)$$

From equation (3.5.1), the variable lnS(t) follows a generalized Weiner process with mean $\left(\mu - \frac{\sigma^2}{2}\right)dt$ and standard deviation $\sigma\sqrt{dt}$. The change in lnS(t) between time 0 and T is normally distributed, that is

$$\ln S_T - \ln S_0 \sim N\left[\left(\mu - \sigma^2/2\right)T, \sigma\sqrt{T}\right]$$

or

$$\ln \frac{S_T}{S_0} \sim N\left[\left(\mu - \sigma^2/2\right)T, \sigma\sqrt{T}\right] \qquad (3.5.2).$$

and also

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu - \sigma^2/2\right)T, \sigma\sqrt{T}\right] \qquad (3.5.3)$$

where S_T is the stock price at future time T, S_0 is the stock price at time 0.

To derive the Black- Scholes equation, we let the value of a European call option to be $C_t(S_t,t)$ that is a function of asset price, S_t and time, t and is twice differentiable with respect to S and once with respect to t. This assumption allows us to apply Itô's lemma, on $C_t(S_t,t) = C$ to get

$$dC(S_t,t) = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS_t^2 \qquad (3.5.4)$$

Substituting the values of dS and dS^2 into equation (3.5.4), we get

$$dC(S_t,t)=\left(\frac{\partial C}{\partial t}+\mu S_t\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S_t{}^2\frac{\partial^2 C}{\partial S^2}\right)dt+\sigma S_t\frac{\partial C}{\partial S}dZ$$

(3.5.5) If we consider a portfolio consisting of a European call option and has α stocks, we obtain

$$dC(S_t,t)=\left(\frac{\partial C}{\partial t}+\mu S_t\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S_t{}^2\frac{\partial^2 C}{\partial S^2}+\alpha\mu S_t\right)dt+ \sigma S_t\left(\frac{\partial C}{\partial S}+\alpha\right)dZ \qquad (3.5.6)$$

if we let α to be $-\frac{\partial C}{\partial S}$,and substituting in equation (3.5.6), we get

$$dC(S_t,t)=\left(\frac{\partial C}{\partial t}\quad+\frac{1}{2}\sigma^2 S_t{}^2\frac{\partial^2 C}{\partial S^2}+\alpha\mu S_t\right)dt \qquad (3.5.7)$$

The choice of α means that we are carrying out a *Delta- hedging*; (that is eliminating risks completely) giving us a portfolio that is deterministic; that is, it has no random component. Since a risk- free portfolio must grow at a risk- free rate, we conclude that the drift of $(C+\alpha S)$ must be equal to $r(C+\alpha S)$. Therefore we have

$$\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}=r\left(C-S\frac{\partial C}{\partial S}\right) \qquad (3.5.8)$$

since $\alpha=-\frac{\partial C}{\partial S}$

Rearranging equation (3.5.8), we get

$$\frac{\partial C}{\partial S}+rS\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}=rC \qquad (3.5.9)$$

Equation (3.5.9) is the so-called Black- Scholes equation. A unique solution can be found when the following boundary conditions are applied:

- The value of the option must be at expiry i.e. t=T, then we have C(S,T) = max (S – K,0) for European call option and C(S,T) = max (K – S,0) for a European put option.
- C(S,T) = f(S), this is applied on the European contingent claim i.e. derivative that pays off a function f, of S at time T.

For the solution of Black-Scholes equation, see [19].

3.1.3. Black- Scholes –Merton formula for European call and put options

The value of a European call option is given by

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

with N(x) denoting cumulative normal distribution, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-S^2/2}\, dS$, and

$$d_1 = \frac{\log\left(S/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log\left(S/K\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Also the value of a Put option on a non- dividend paying stock is given as $P(S,t) = Ke^{-rT}N(-d_2) - S_0N(-d_1)$, where d_1 and d_2 are as in the call option above, [19].

CHAPTER 4

Deterministic Walrasian price -adjustment model

In this chapter, we are going to look at how the forces of supply and demand determines the prices of shares, that is the effect of Walrasian excess demand on market prices.

4.1. The laws of supply and demand

The supply and demand analysis explains how prices are established in the market through competition among buyers and sellers. Prices are the tools by which the invisible hand- the market- coordinates individual's desires and limits how much the consumers are willing to purchase (demand) and how much the producers are willing to offer (supply). Through the theory of demand and supply, a free market can successfully move toward the market clearing or equilibrium.

An equilibrium point is where the forces of demand are equal to the forces of supply. At this point the market is said to be ' *stable*' that is there is neither deficit nor surplus. *Market demand* refers to a schedule of quantities of goods that will be bought per unit of time at various prices: and the specific amount that will be demanded per unit of time is the *quantity demanded.* Likewise, various quantities offered for sale at various prices is the *market supply*: and the specific quantities offered for sale at specific prices is the *quantity supplied*

4.1.1. The law of demand

The law of demand states that the *quantity demanded* of a good or service is negatively related to its price, *ceteris paribus*, [2], in other words, holding all else constant, consumers will purchase more of a good or service at a lower cost than at a higher price. As the price rises, *ceteris paribus*, consumers will demand a smaller quantity of a good or service.

4.1.2. The law of supply

The law of supply states that, in general other factors being constant, the higher the price of a good, the greater the quantity of that good the sellers are willing and able to make available over a given period of time. It is based on the assumption that the sellers seek to maximize net gains from their activities.

4.2. Market equilibrium- equilibrium price

An equilibrium prevails when economic forces balance so that the economic variables neither increase nor decrease. Market equilibrium is attained when the price of a good adjusts so that the quantity buyers will buy at that price is equal to quantity the sellers will sell, that is, $Q_D(S(t)) = Q_S(S(t))$. At this point, forces of supply and demand are balanced, thus there is no tendency for the market price or quantity to change over a given period of time. The equilibrium price acts to ration the goods so that everyone who wants to buy or sell will do so successfully, the market is then said to be *clear*.

A shortage exists in the market when the quantity demanded of a good exceeds quantity supplied over a given period of time. Similarly, a surplus exists when the quantity supplied exceeds the quantity demanded over a given period. In a free market, when the prices are raised, the quantity demanded decrease but the quantity supplied increase (Fig. 4.2). A point where both demand curve and supply curve cross is the *equilibrium point.* The price at this point is the equilibrium price, denoted by S* and the quantity is the equilibrium quantity, denoted by Q*. At the equilibrium point, (S*,Q*), there is neither shortage nor surplus. But below this point there is shortage and above the equilibrium point there is surplus. These are depicted in (Fig. 4.2).

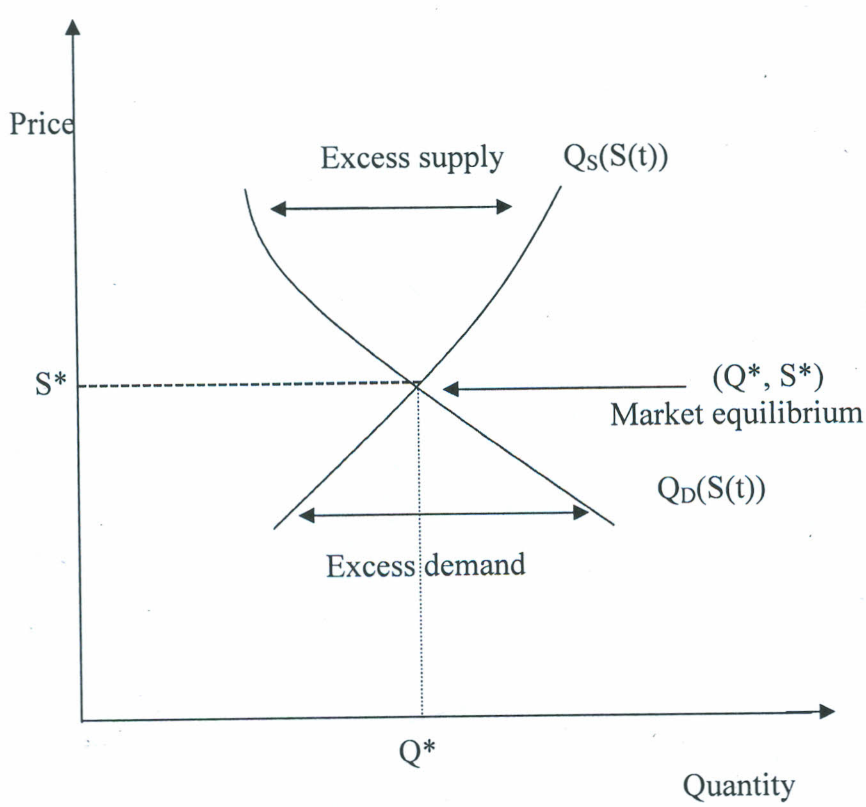

Fig 4.2. Demand and supply curves

4.3. Price Adjustment Mechanism

A positive shift in demand with constant supply will move both the equilibrium price, S* and equilibrium quantity, Q* upwards. A negative shift in demand with constant supply will decrease both the equilibrium price S* and equilibrium quantity Q* downwards. Similarly, a decrease in supply with constant demand will increase the equilibrium price, S* and decrease equilibrium quantity Q*. Fig. 4.3.1 and 4.3.2 depict shifts in demand and supply respectively.

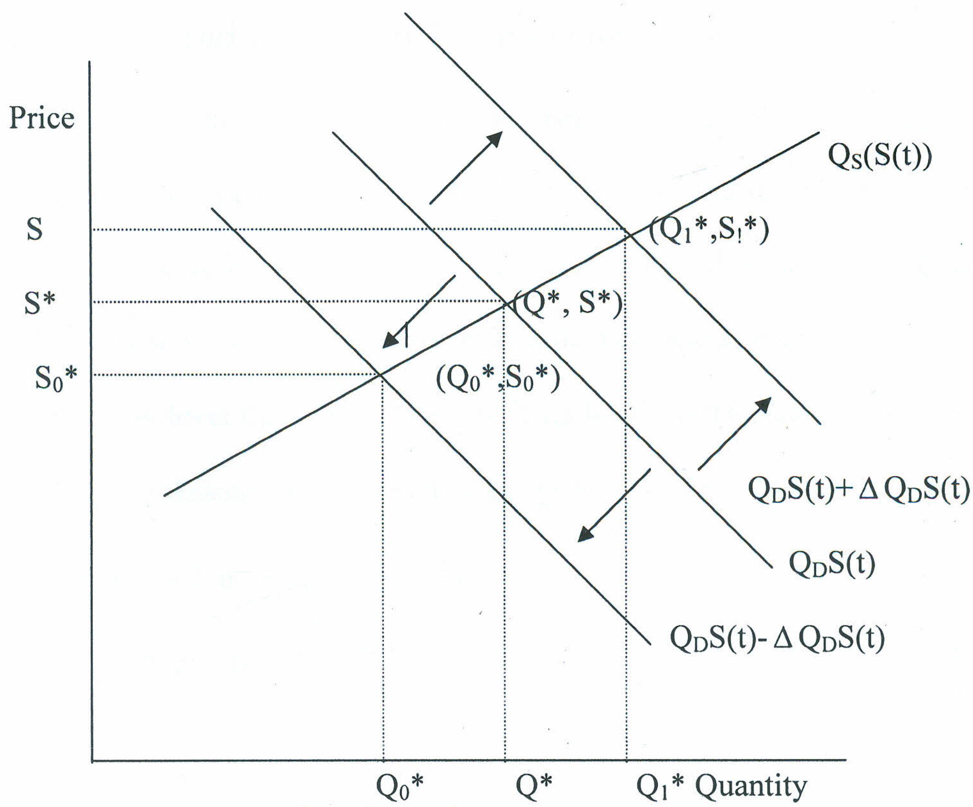

Fig. 4.3.1. Shift in demand curve

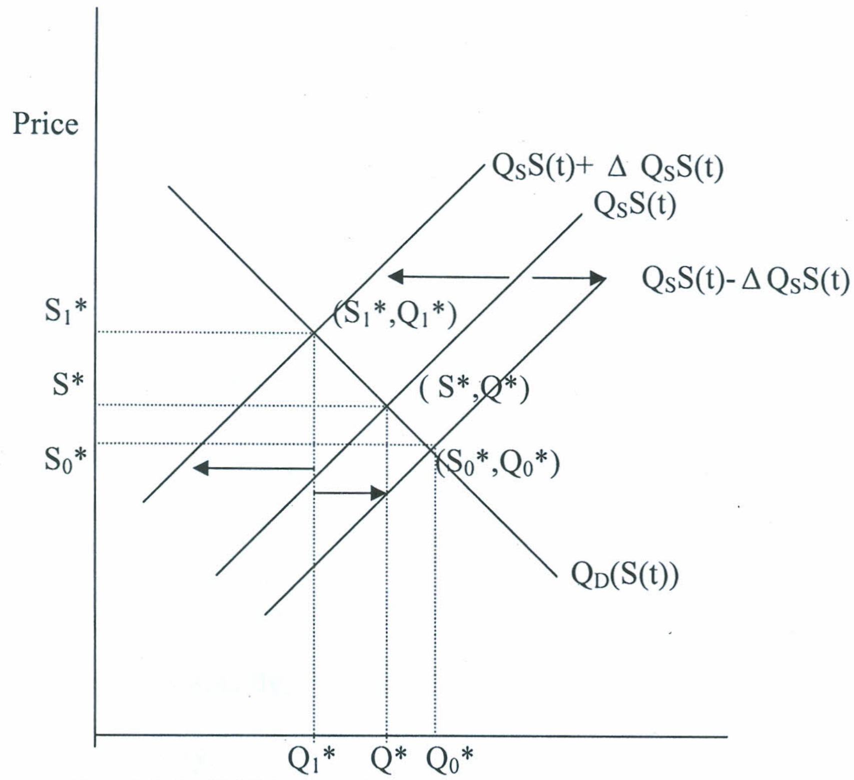

Fig. 4.3.2. Shift in supply curve

4.4. Excess demand and excess supply functions

Excess demand occurs when there is more demand than supply. At this point, the price of the security is below the equilibrium price. When the price of the security is above the equilibrium price, there is excess supply: the quantity supplied exceeds the quantity demanded. Market traders do control both the excess demand and supply by adjusting the market prices to a new level that clears them off. This is called the *clearing price* or the *equilibrium price*. We denote supply function by $Q_S S(t)$ and the demand function by $Q_D S(t)$. Then the excess demand function given by:

$$EDS(t) = Q_D S(t) - Q_S S(t) \tag{4.4.1}$$

We note that when $EDS(t) > 0$, when demand is greater than supply and when $EDS(t) < 0$, when supply is greater than demand. The market equilibrium also referred to as Walrasian equilibrium is attained when $EDS(t) = 0$ or $Q_D S(t)) = Q_S S(t)$.

4.5. The Walrasian Price Adjustment

An early model of equilibrium price adjustment was proposed by [17]. In this scheme, equilibrium prices are a goal toward which the market tends to settle. Changes in price are motivated by information from the market about the *degree of excess* demand at any particular price. Mathematically, Walrasian price adjustment specifies that the change in price over time is given by:

$$\frac{dS(t)}{dt} = k[Q_D S(t) - Q_S S(t)] = kEDS(t), \qquad (4.5.1)$$

where $EDS(t)$ represents excess demand at price S(t) and $k > 0$ is the speed of adjustment. Price will increase if there is positive excess demand (shortage) and decrease if there is negative excess demand (surplus). Such a mechanism is known as "tatonnement" (groping) process. For any price above the equilibrium price (S*), the tatonnement process lowers the price. Similarly, for prices less than the equilibrium price (S*), the process raises the price. The actual rate of change is proportional to level of excess demand and the factor of proportionality is *k*. In the next sub-section we are going to look at different systems of financial dynamisms.

4.5.1 Discrete financial dynamical system

For this we consider a sequence of prices $S_1, S_2, \ldots, S_n$. In the discrete dynamic system, the Walrasian adjustment process, price changes $(S_{i+1} - S_i)$ are proportional to excess demand:

$$S_{i+1} - S_i = k_i(Q_D S(t_i) - Q_S S(t_i)) = k_i EDS(t_i), \quad i=1,2,\ldots n. \qquad (4.5.2)$$

where k_i is a positive constant for the ith price and $ED(S(t_i))$ is the excess demand. When $ED(S(t_i)) > 0$, prices will go up and when $ED(S(t_i)) < 0$, the prices will go down. This model is used to compute each price based on demand and supply functions and the price at each preceding period. From equation (4.5.2), we can note that the value of k and the absolute value of the excess demand determine whether the price will fall or rise. The behavior of buyers and sellers will determine the value of k

4.5.2 Continuous financial dynamical system

In this system, trading assumed to take place continuously and the adjustment of prices is continually done as per the unit time of period. The prices of assets are represented by a continuous function $f(S(t))$, where excess demand (4.4.1) is expressed as

$$f(S(t)) = Q_D(S(t)) - Q_S(S(t)) = ED(S(t)) \text{ [13].} \qquad (4.5.3)$$

In this case, we consider excess demand with the following properties:

i) Continuous: $f(S(t))$ is continuous

ii) Walras' law: for any price vector, S(t), $S(t)f(S(t)) = 0$ for all S(t), that is, value of each individual excess demand is zero.

iii) Homogeneity of degree zero: $f(\lambda S(t)) = f(S(t))$ for all $\lambda > 0$ and all S(t)

The Walrasian -Samuelson adjustment mechanism of the jth asset is given by

$$\frac{dS_j(t)}{dt} = \begin{cases} h_j[f_j(S(t))], & j = 1,2,..,n \\ 0 & if \;\; S_j(t) = 0 \;\; and \;\; f_j(S(t)) = 0 \end{cases} \tag{4.5.4}$$

where S_j is the price of the jth asset, $f_j(S(t))$ is the total excess demand function for the jth asset and h_j is any (fixed) monotonic increasing differentiable real-valued function. Equations (4.5.4) were first suggested by Paul Samuelson, [15] for prices that move at the same time in response to the excess demand in them. From [13], for an economy in which assets are isolated from substitution, the Walrasian price adjustment is expressed as

$$\frac{dS_j(t)}{dt} = k_j(Q_{jD}(S_j(t)) - Q_{jS}(S_j(t))) \;, \tag{4.5.5}$$

where k_j is a positive adjustment coefficient and is interpreted as the "speed of adjustment" of the market to changes in supply and demand, [1]. It should also be noted that:

- if $Q_D(S(t)) > Q_S(S(t))$ then dS(t)/dt > 0 and so S(t) increases so that the balance between supply and demand can be achieved,
- if $Q_D(S(t)) = Q_S(S(t))$ then dS(t)/dt = 0, so S(t) is held constant at the equilibrium level,
- if $Q_D(S(t)) < Q_S(S(t))$ the dS(t)/dt < 0, so S(t) decreases in order to achieve the balance between supply and demand.

4.6. Deterministic Walrasian price-adjustment model

From equation (4.5.5) it can be shown that the fractional rate of increase of asset price, $\frac{1}{S(t)}\frac{dS(t)}{dt}$ is proportional to the excess demand function, [13], hence we have

$$\frac{1}{S(t)}\frac{dS(t)}{dt} \propto (Q_D S(t) - Q_S S(t)) \tag{4.6.1}$$

If S(t) is the price of asset at time t and S^* be the price at equilibrium. The general expression of a linear demand function is given as

$$Q_D(S(t)) = -aS(t) + b, \tag{4.6.2}$$

for some appropriate parameters a and b. Similarly, a linear supply function is expressed as

$$Q_S(S(t)) = cS(t) - d, \tag{4.6.3}$$

for some appropriate parameters c and d, [11].

As depicted in Fig.4.6.1, we linearise the demand and supply curves about the equilibrium, equations (4.6.2) and (4.6.3), respectively, become:

$$Q_D(S(t)) = \alpha(S^* - S(t)) \text{ and } \quad Q_S(S(t)) = -\beta(S^* - S(t)) \tag{4.6.4}$$

where α is the demand elasticity and β is the supply elasticity. Monotonicity, equation (2.7.4), requires that $Q_D(S(t))$ is a decreasing linear function of S(t), $\alpha > 0$ and $Q_S(S(t))$ is an increasing linear function of S(t), $\beta < 0$. The excess demand function is given as:

$$ED(S(t)) = Q_D(S(t)) - Q_S(S(t)) = (\alpha + \beta)(S^* - S(t)). \tag{4.6.5}$$

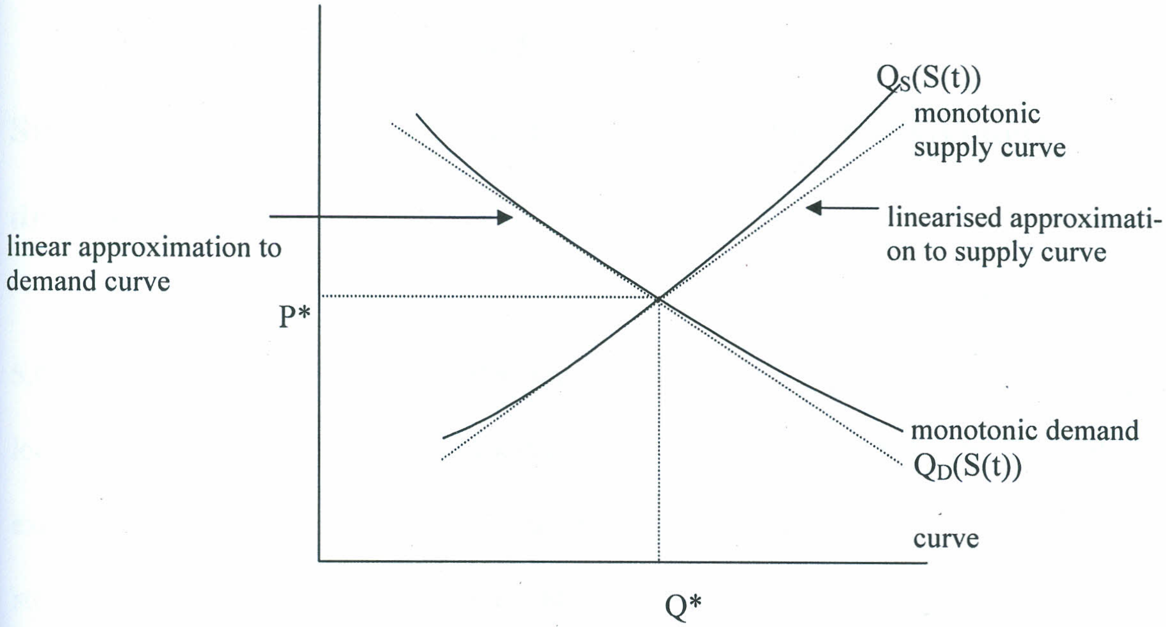

Fig. 4.6.1 Linearising the demand and supply curves

From the proportional rate of increase (4.6.1) and the excess demand function (4.6.5) we have:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = h(\alpha+\beta)(S^* - S(t)), \tag{4.6.6}$$

where h is the constant of proportionality, or

$$\frac{dS(t)}{dt} = kS(t)(S^* - S(t)), \quad k = h(\alpha+\beta) \tag{4.6.7}$$

or

$$dS(t) = kS(t)(S^* - S(t))dt, \tag{4.6.8}$$

where k is the constant of growth rate. If $S^* = S(t)$ in equation (4.6.7), the term in parenthesis becomes zero, so the asset price growth becomes zero when the asset price hits S^* regardless of the initial asset price, say, S_0. The asset price growth rate is also equal to zero when $S(t) = 0$. Equation (4.6.8) is a deterministic logistic equation in stock prices $S(t)$ with a limiting constant S^*, and is also referred to in the literature as the Verhulst logistic equation or Verhulst–Pearl logistic equation, [13].

CHAPTER 5

Stochastic logistic stock price adjustment model with continuous dividend payment.

5.0. This chapter discusses both deterministic and stochastic price models. We start by looking at the deterministic price models shares prices that do not pay dividend then extend to those that pay dividend. Then finally we look at the effect of 'noise', that is stochastic model with continuous dividend payments.

5.1. Deterministic Logistic Model with no dividend

As stated in Chapter 4, equation (4.6.7), if we take an asset whose price is S_t, invested at the risk-free rate of interest, μ and earns no dividend during its life. If S^* is the market equilibrium price of the asset then the change in price with time will be given as

$$dS_t = \mu S_t (S^* - S_t) dt, \quad S^* \neq S_t. \qquad (5.1.1)$$

Separating variables in equation (5.1.1) and solving, we get

$$\ln\left|\frac{S_t}{S^* - S_t}\right| = \mu t S^* + \ln\left|\frac{S_0}{S^* - S_0}\right| \qquad (5.1.2)$$

This can be written as

$$\left|\frac{S_t(S^* - S_0)}{S_0(S^* - S_t)}\right| = e^{\mu S^* t}$$

On solving for S(t) we get

$$S_t = \frac{S_0 S^*}{S_0 + (S^* - S_0)e^{-\mu S^* t}} \tag{5.1.3}$$

From equation (5.1.3), we see that when $t \to 0$, $S_t \to S_0$, and when $t \to \infty$, $S_t \to S^*$.

When we consider any infinitesimal change in time i.e. $t_{i+1} - t_i$, then, the general expression for the asset price at t_{i+1} is given by

$$S_{t_{i+1}} = \frac{S_{t_i} S^*}{S_{t_i} + (S^* - S_{t_i})e^{-\mu S^*(t_{i+1} - t_i)}}$$

A Typical curve for equation (5.1.3) is given by Fig .5.1 below

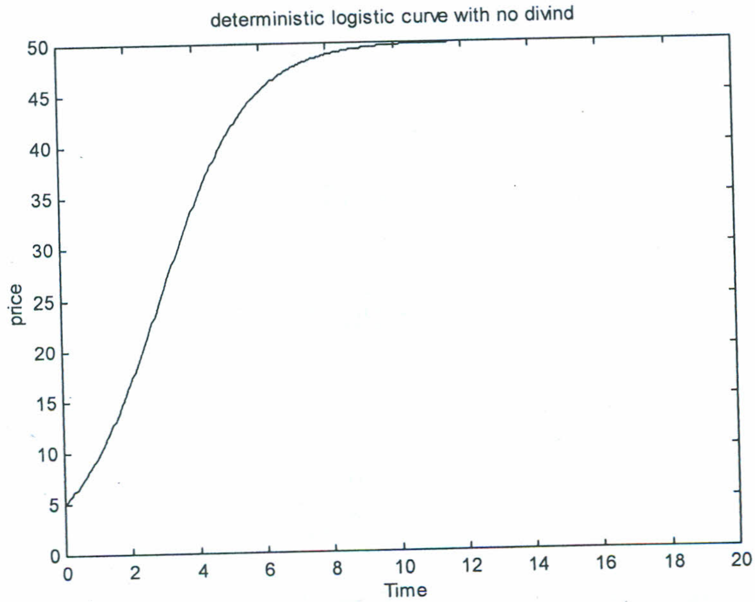

Fig. (5.1) shows aTypical logistic curve without dividend, with $\mu = 0.015, S^* = 50, S_0 = 5$.

5.2. Deterministic logistic model with continuous dividend payments

In this section, we are going to consider two cases:

5.2.1. Case 1,

In this case we look at the situation where the dividend to be paid to an investor depends on the price of the asset, S_t, and is not restricted to a given value. If an asset whose price is S_t is invested at the free interest rate μ and pays dividend at the rate q, then the total dividend to be paid after time period, dt, is given by $qS_t dt$. Thus the total change in price of the asset following a logistic trend over a given period of time, dt, is given by

$$dS_t = \mu S_t(S^* - S_t)dt - qS_t dt, \quad S^* \neq S_t. \tag{5.2.1}$$

Separating the variables in equation (5.2.1), we get

$$\frac{dS_t}{S_t[\mu(S^* - S_t) - q]} = dt \tag{5.2.2}$$

Using partial fractions in equation (5.2.2) and solving for A and B, we have

$$\frac{A}{S_t} + \frac{B}{\mu(S^* - S_t) - q} = \frac{1}{S_t[\mu(S^* - S_t) - q]} \tag{5.2.3}$$

then

$$A = \frac{1}{\mu S^* - q}, \quad B = \frac{\mu}{\mu S^* - q}.$$

Substituting the values of A and B into equation (5.2.3) and integrating both sides we get

$$\ln\left|\frac{S_t}{\mu(S^* - S_t) - q}\right| = (\mu S^* - q)t + C \tag{5.2.4}$$

Putting t=0, $S(t) = S_0$ we have

$$C = \ln\left|\frac{S_0}{\mu(S^*-S_0)-q}\right|$$

Hence equation (5.2.4) becomes

$$\ln\left|\frac{S_t[\mu(S^*-S_0)-q]}{S_0[\mu(S^*-S_t)-q]}\right| = (\mu S^*-q)t \quad (5.2.5)$$

Taking the exponential of (5.2.5) and then solving for the value of S_t, we get

$$S_t = \frac{S_0(\mu S^*-q)}{\mu S_0+[\mu(S^*-S_o)-q]e^{-(\mu S^*-q)t}} \quad (5.2.6)$$

5.2.2. Case 2.

Here we consider a situation where the company has decided to pay dividend on an asset whose price does not exceed its equilibrium price, S*. If the conditions for investment remain as in case 1 above, then the change in price is given as

$$dS_t = \mu S_t(S^*-S_t)dt - qS_t(S^*-S_t)dt \quad , S^* \neq S_t. \quad (5.2.8)$$

To solve this equation, we separate the variables

$$\frac{dS_t}{S_t(S^*-S_t)} = (\mu-q)\,dt \quad (5.2.9)$$

Integrating both sides of equation (5.2.9), by method of partial fractions we have

$$\int\frac{dS_t}{S_t} + \int\frac{dS_t}{S^*-S_t} = S^*\int(\mu-q)\,dt$$

which results to

$$\ln\left|\frac{S_t}{S^*-S_t}\right| = S^*(\mu-q)t+C \quad (5.2.10)$$

At the initial time, t=0, $S(t)=S_0$ the value of C = $\ln\left|\frac{S_0}{S^*-S_0}\right|$

Substituting the value of C into equation (5.2.10), we get

$$\ln\left|\frac{S_t(S^*-S_0)}{S_0(S^*-S_t)}\right| = S^*(\mu-q)t \tag{5.2.11}$$

lving for S_t from equation (5.2.11), we get

$$S_t = \frac{S_0 S^*}{S_0 + (S^*-S_0)e^{-S^*(\mu-q)t}} \tag{5.2.12}$$

om equation (5.2.12), we see that when $\mu = q$, $S_t = S_0$, this shows that if the interest te is the same as the dividend rate paid, then at the end of investment period, t, the alue of asset remains constant.

typical curve for equation (5.2.12) is by the figure below

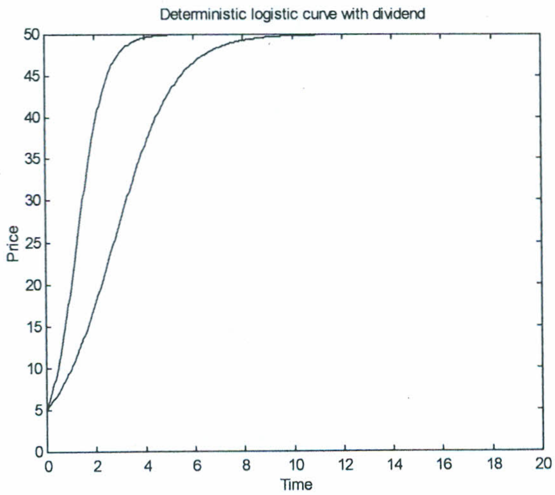

Fig. (5.2) *typical deterministic curve with continuous dividend, with* $\mu = 0.025, q = 0.009, S^* = 50, S_0 = 5,$ *for continuous line and* $\mu = 0.04, q = 0.005, S^* = 50, S_0 = 5,$ *for dotted line.*

The broken line increases faster than the continuous line since the free rate of interest has gone up and rate of dividend payment has been reduced.

Since investment is subjected to stochastic effect, '*noise*', we therefore incorporate the effect of this 'noise' into our deterministic models, equations (5.2.1) and (5.2.8, to come up with stochastic logistic models.

5.3. Stochastic logistic model without dividend payments

This model has been developed by considering an asset, S_t, invested at the interest rate of μ but earns no dividend, taking $\sigma\, dZ$, to be the element of 'noise', that is, the interfering forces, [13], then the change in price of the asset is given by

$$dS_t = \mu\, S_t(S^* - S_t)\, dt + \sigma\, S_t(S^* - S_t)\, dZ, \quad S^* \neq S_t\,. \tag{5.3.1}$$

If we let G to be a function that is twice differentiable in S_t and once in t, then applying Itô's lemma, equation (2.6.10) into equation (5.3.1), we get

$$\begin{aligned} dG(S_t, t) = {} & \frac{\partial G}{\partial t} + \mu\, S_t(S^* - S_t)\frac{\partial G}{\partial S_t} + \sigma\, S_t(S^* - S_t)\frac{\partial G}{\partial Z} \\ & + \frac{1}{2}\sigma^2 S_t^{\,2}(S^* - S_t)^2 \frac{\partial^2 G}{\partial S_t^{\,2}} \end{aligned} \tag{5.3.2}$$

To simplify equation (5.3.2), we let

$$G = \ln\left|\frac{S_t}{S^* - S_t}\right|,\quad \frac{\partial G}{\partial t} = 0,\quad \frac{\partial G}{\partial S_t} = \frac{S^*}{S_t(S^* - S_t)},\quad \frac{\partial^2 G}{\partial S^2_t} = \frac{2S_tS^* - S^{*2}}{S_t^{\,2}(S^* - S_t)^2}$$

Substituting these values into equation (5.3.2), we get

$$dG(S_t, t) = \left[\mu + \frac{1}{2}\sigma^2\left(2S_tS^* - S^{*2}\right)\right]dt + \sigma\, dZ \tag{5.3.3}$$

Equation (5.3.3) gives the distribution of $dG(S_t, t)$ as

$$dG(S_t, t) \sim N\left((\mu + \frac{1}{2}\sigma^2(2S_tS^* - S^{*2}))dt, \sigma\sqrt{dt}\right)$$

To solve equation (5.3.1), we re-write it as

$$\frac{dS_t}{S_t(S^* - S_t)} = \mu\, dt + \sigma\, dZ \tag{5.3.4}$$

If we let

$$G(t)=\ln\left|\frac{S_t}{S^*-S_t}\right|, \quad then \quad \frac{\partial G}{\partial S_t} = \frac{S^*}{S_t(S^*-S_t)}, \text{ this gives}$$

$$dS_t = \frac{S_t(S^*-S_t)\,dG(t)}{S^*}$$

tting the value of dS_t into equation (5.3.4), we get

$$dG(t) = \mu S^* dt + \sigma S^* dZ, \qquad (5.3.5)$$

is is a generalised Weiner process of type (2.3.1). Integrating equation (5.3.5), we get

$$G(t) = \mu S^* t + \sigma S^* Z(t) + C \qquad (5.3.6)$$

Explicit solution of (5.3.6) is given as

$$G(t) = G(0) + \mu S^*(t-t_0) + \sigma S^* Z(t), \quad Z(0)=0 \qquad (5.3.7)$$

Equation (5.3.7) is a generalized Brownian motion and its distribution is given as

$$G(t) \sim N\left[G(0) + \mu S^*(t-t_0),\ \sigma S^* Z(t)\right]$$

Since $G(t) = \ln\left|\frac{S_t}{S^*-S_t}\right|$ *and* $G(0) = \ln\left|\frac{S_0}{S^*-S_0}\right|$, equation (5.3.7) simplifies to

$$\ln\left|\frac{S_t(S^*-S_0)}{S_0(S^*-S_t)}\right| = \mu S^*(t-t_0) + \sigma S^* Z(t) \qquad (5.3.8)$$

Taking the exponential and solving for S_t, we get

$$S_t = \frac{S_0 S^*}{S_0 + (S^*-S_0)\left[e^{-(\mu S^*(t-t_0)+\sigma S^* Z(t))}\right]} \qquad (5.3.9)$$

This is the so-called Verhulst – logistic stochastic equation for asset prices, [13].

From equation (5.3.4), when $\sigma = 0$, i.e. no noise, we get the deterministic equation (5.1.2). In equation (5.3.9), when $t \to 0, S_t \to S_0$, and when $t \to \infty, S_t \to S^*$, and this is the so-called market clearing price.

5.4. Stochastic logistic model with continuous dividend payments

In this case our asset, S_t, is invested at the rate of μ and earns a continuous dividend rate of q. The amount of assets for which the dividend is paid must be less than S*. The effect of 'noise' is then given by $\sigma\, dZ$. Then the change in the price of our asset will be given as

$$dS_t = \mu\, S_t(S^* - S_t)\, dt + \sigma\, S_t(S^* - S_t) dZ - qS_t(S^* - S_t)\, dt \qquad (5.4.1)$$

where μ is called the drift rate of the stock and it expresses the trend of the stock movement. σ is called the volatility of the underlying asset and it expresses how much the stock wobbles up and down or how risky it is to invest.

If we let G(S,t) be a function that is twice differentiable in S_t and once in t, then applying Itô's lemma, equation (2.6.10) into equation (5.4.1), we get

$$\begin{aligned} dG(S_t,t) = & \frac{\partial G}{\partial t} + (\mu - q) S_t(S^* - S_t)\frac{\partial G}{\partial S_t} + \sigma\, S_t(S^* - S_t)\frac{\partial G}{\partial Z} \\ & + \frac{1}{2}\sigma^2 S_t^2 (S^* - S_t)^2 \frac{\partial^2 G}{\partial S_t^2} \end{aligned} \qquad (5.4.2)$$

To simplify equation (5.4.2), we let

$$G = \ln\left|\frac{S_t}{S^*-S_t}\right|, \quad \frac{\partial G}{\partial t}=0, \quad \frac{\partial G}{\partial S_t}=\frac{S^*}{S_t(S^*-S_t)}, \quad \frac{\partial^2 G}{\partial S^2{}_t}=\frac{2S_tS^*-S^{*2}}{S_t^2(S^*-S_t)^2}$$

Substituting these values into equation (5.3.2), we get

$$dG(S_t,t)=\left[(\mu-q)+\frac{1}{2}\sigma^2\left(2S_tS^*-S^{*2}\right)\right]dt+\sigma\, dZ \tag{5.4.3}$$

The distribution of equation (5.4.3) is

$$dG(S_t,t) \sim N\left(((\mu-q)+\frac{1}{2}\sigma^2(2S_tS^*-S^{*2}))dt, \sigma\sqrt{dt}\right)$$

Equation (5.4.1), can be re-written as

$$\frac{dS_t}{S_t(S^*-S_t)}=(\mu-q)\,dt+\sigma\, dZ \tag{5.4.4}$$

Putting

$$G(S,t)=\ln\left|\frac{S_t}{S^*-S_t}\right|, \quad then \quad \frac{\partial G(S,t)}{\partial S_t}=\frac{S^*}{S_t(S^*-S_t)}, \text{ this gives}$$

$$dS_t=\frac{S_t(S^*-S_t)\,dG(S,t)}{S^*}$$

Putting the value of dS_t into equation (5.4.4), we get

$$dG(S,t)=(\mu-q)S^*dt+\sigma S^* dZ, \tag{5.4.5}$$

this is a generalised Weiner process of type (2.3.1). Integrating equation (5.4.5), we get

$$G(S_t,t)=(\mu-q)S^*t+\sigma S^*Z(t)+C \tag{5.4.6}$$

Explicit solution of (5.5.6) is given as

$$G(S_t,t)=G(S_0,0)+(\mu-q)S^*(t-t_0)+\sigma S^*Z(t), \quad Z(0)=0 \tag{5.4.7}$$

The distribution of (5.4.7) is given as

$$G(S_t,t) \sim N[G(S_0,0)+(\mu-q)S^*(t-t_0),\ \sigma S^* Z(t)]$$

Since $G(S_t,t) = \ln\left|\frac{S_t}{S^*-S_t}\right|$, $G(S_0,0) = \ln\left|\frac{S_0}{S^*-S_0}\right|$, then equation (5.4.7) becomes

$$\ln\left|\frac{S_t(S^*-S_0)}{S_0(S^*-S_t)}\right| = (\mu-q)\,S^*(t-t_0)+\sigma\,S^*Z(t) \qquad (5.4.8)$$

Taking the exponential of (5.4.8) and then solving for solving for S_t, we get

$$S_t = \frac{S_0 S^*}{S_0+(S^*-S_0)(e^{-((\mu-q)S^*(t-t_0)+(\sigma S^*Z(t))})} \qquad (5.4.9)$$

In equation (5.4.9), when $t \to 0, S_t \to S_0$, and when $t \to \infty, S_t \to S^*$, and this is the market clearing price.

5.5. Model verification

To verify the model we have developed, equation (5.4.9), we analyse British Airways daily share price quotes traded in the U.K stock markets between 4/10/04 and 7/3/05 data in question is given in appendix 1.

Using the values from appendix 2, we calculate the value of μ by applying the formula $\mu = \frac{1}{N-1}\sum \log\left(\frac{S_{i+1}}{Si}\right)$, where N is the total number of observations, as:

$$\mu = \frac{1}{103}(0.14175224)\times\frac{252}{104}=0.00333472.$$

The value of σ is given by

$$\sigma = \sqrt{\frac{1}{N-1}(R_i-\mu)^2},$$

Substituting the values from appendix 2, we get

$$\sigma = \sqrt{\frac{1}{103}(0.006314)\times\frac{252}{104}} = 0.0189715,$$

q, which is the rate of dividend payment has been declared as 0.0012316.

Substituting the values of σ , μ and q into our model equation (5.4.9) and plotting the trend curve using Mat lab we get the stochastic logistic curve with continuous dividend, (fig 5.3) below:

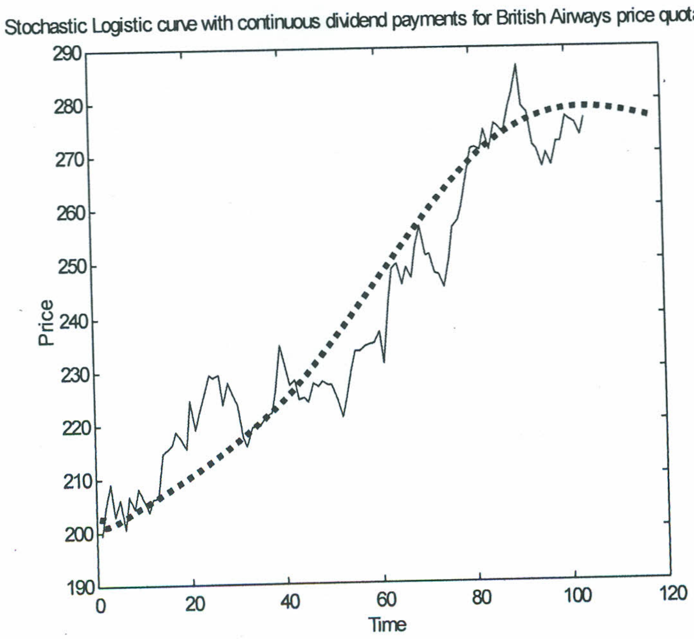

Fig (5.3).Stochastic logistic trend curve for British Airways price quotations with continuous dividend payments,
$\sigma = 0.02$, $\mu = 0.003$, $q=0.001$, $S^*=286.5$, $S_0=199.5$

m fig. (5.3), the continuous line shows the trend of British Airways price quotes while dotted line shows the trend that fits the stochastic logistic model with continuous ments we have developed.

Chapter 6

Conclusion

In this section we highlight the result we have obtained as per our objective. The main objective of this study was to develop a mathematical model that can be used to fit stock prices which follow logistic trend and pay continuous dividend.

This objective has been attained by first looking at the logistic models both with and without dividend payments and simulating them to get their trends (Figures 5.1 and 5.2). We have also developed stochastic logistic model without dividend payments (equation 5.3.9). We then finally developed stochastic logistic model with dividend payments equation (5.4.9). We have verified this model by analysing real market data (appendix 1) to find variables such as volatility of the underlying assets, risk free rate of interest and the rate of dividends yield. Application of these variables on the model shows the trend that fits the prices that follow stochastic logistic trends and pay continuous dividends as shown in fig. (5.3).

To develop this model, we restricted the price at which dividend was supposed to be paid and was not to exceed the market equilibrium price, equation (5.4.1). We therefore suggest to the future researchers to see if the same can be developed if this price is not restricted.

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