# ON NORMS OF A DERIVATION

BY

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# Abstract

The study of derivations still remains an area of interest to mathematicians today. Of special attention has been the study of norms of inner derivations. Most of the work in this area is based on Stampfli's result of 1970, where he established the equality between the norm of an inner derivation and twice the distance between an element of an algebra to the centre of that algebra, specifically for a primitive C*-algebra with an identity. This result has been extended by other mathematicians to other algebras, like Von Neumann, Calkin, W* - algebras among others. In this study, we've continued to investigate Stampfli's result. In particular, we've used the approach of tensor product to establish the equality for the algebra of bounded linear operators on a Hilbert space. Further, we have explored the norm of inner derivations on norm ideals and established the relationships between norms of inner derivations restricted to algebras, norm ideals and the quotient algebra. On the other hand, an interesting relationship between the diameter of the numerical range and the norm of inner derivation has been established. Moreover, their applications to hyponormal and S - universal operators have been investigated. The methodology has been majorly based on the previous works of Stampfli, Fialkow, Kyle, Barraa and Boumazgour, among others. We also revisited related theories from operator algebra and analysis in general. In the operator - algebraic formulation of quantum theory, these results are useful to theoretical physicists and applied mathematicians alike. For pure mathematicians, we hope this will provide a motivation for further research in the development of the area.

# Chapter 1

# MATHEMATICAL BACKGROUND

## 1.1 Introduction

Rings of operators, renamed Von Neumann algebras by J. Dixmier, were first introduced by J. Von Neumann [41] in 1929 with a grand aim of giving a sound foundation to mathematical sciences of infinite nature. J. Von Neumann and his collaborator F. J. Murray laid down the foundation for this field of mathematics, called operator algebras in a series of papers, [27, 28, 29, 42], during the period of 1930's and early in the 1940's. The theory of operator algebras is concerned with algebras of bounded linear operators on a Hilbert space, closed under the weak operator topology. Since then there has appeared a large volume of literature, and a great deal of progress has been achieved by many mathematicians. Many important results and powerful techniques were added to the theory. This has led to the emergence of various related fields of mathematics, and a number of topics in this subject have branched out to independent fields, for instance, the study of derivations.

Derivations form a topic in operator algebra that was vastly studied in the 1960's. In fact there was excellent expositions of the theory of inner derivations by that time. Surprisingly, the study of norms of derivations, especially norms of inner derivations had not been explored until in 1966 when Sakai [34] used the norm of an inner derivation to prove one of the richest results on inner derivations, that is; every inner derivation is bounded. These results attracted the attention of a mathematician by the name Joseph G. Stampfli [40] who worked on the area of norms of inner derivations and in 1970 produced a paper of several results with the same title. Stampfli's paper [40] forms the basis of the study of norms of inner derivations and even of elementary operators in general and is among the first serious studies on this topic. Since then, this study has attracted a lot of attention of pure mathematicians who have written papers with excellent results on this area of study, [1, 5, 17, 24, 40]. We would like to point out that the available literature in this area is still scarce. Hence, we continue to investigate certain aspects of norms of derivations and their applications.

In chapter one, we establish the background information to this study which enables us to state the problems with a lot of ease. We also present terminologies and symbols in addition to some definitions regarding norms of derivations.

Chapter two is concerned with norms of derivations. We present exhaustively the algebraic properties of derivations, then concentrate on inner derivations. Lastly, we establish the Stampfli's equality for the algebra of bounded linear operators on a Hilbert space using the approach of tensor products.

In chapter three, we investigate norms of inner derivations on norm ideals. We revisit the concept of S - universality and give several results with respect to it. Further, we explore the interesting relationships between inner derivations and the numerical range and give results of their applications to S - universality. Moreover, applications of norms of inner derivations to hyponormal operators have been investigated.

Finally, in the last chapter we give a summary of our work and suitable recommendations.

## 1.2 Background Information

In this section, we categorize the terms used in the subsequent chapters into algebras (see subsection 1.2.2), operators and functionals (see subsection 1.2.1). We further provide in a much simplified way, the definitions of these terms.

### 1.2.1 Operators and Functionals

**Definition 1.2.1.** A set $V$ is called a **vector space** over a field $\mathbb{K}$ if it forms an abelian group under vector addition and it has a scalar multiplication satisfying the following axioms; $\forall\ a,\ b\ \in\ V,\ \ \alpha,\ \beta\ \in\ \mathbb{K}$,

**(i)** $(\alpha+\beta)a = \alpha a + \beta a$

**(ii)** $\alpha(a+b) = \alpha a + \alpha b$

**(iii)** $\alpha(\beta a) = (\alpha\beta)a$

**(iv)** $1.a = a$

**Definition 1.2.2.** Given a vector space $V$ over a field $\mathbb{K}$, a subset $W$ of $V$ is called a vector **subspace** if $W$ is a vector space over $\mathbb{K}$ and under the operations already defined on $V$.

**Definition 1.2.3.** **Functionals** are mappings from a vector space to the space of scalars.

**Operators** are mappings from one vector space to another vector space.

**Definition 1.2.4.** Let $X$ and $Y$ be linear spaces over $\mathbb{K}$. Then a function $T : X \to Y$ is called a **linear operator** if and only if $\forall\ x,\ y \in X$ and $\alpha, \beta \in \mathbb{K}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

**Definition 1.2.5.** Let $X$ be a linear space over $\mathbb{K}$. A functional $f : X \longrightarrow \mathbb{K}$ is **linear** if it is a linear operator.

**Definition 1.2.6.** Let $X$ be a vector space and $X^*$ the set of all linear functionals on $X$, then $X^*$ is called the **dual space** of $X$.

**Definition 1.2.7.** Let $V$ be a vector space over $\mathbb{K}$. A function $\|,\| : V \longrightarrow \mathbb{R}$ is called a **norm** if it satisfies the following properties; $\forall\ a, b \in V$ and $\forall\ \lambda \in \mathbb{K}$

1. $\|a\| \geq 0$,
2. $\|a\| = 0$ iff $a = 0$,
3. $\|\lambda a\| = |\lambda|\|a\|$,
4. $\|a + b\| \leq \|a\| + \|b\|$.

**Definition 1.2.8.** Let $X$, $Y$ be normed linear spaces. A linear operator $T : X \to Y$ is said to be **bounded** if and only if there exists a constant $M > 0$ such that $\|Tx\| \leq M\|x\|$ for $x \in X$.

**Definition 1.2.9.** A bounded operator $A : X \longrightarrow Y$ between normed linear spaces $X$ and $Y$ is said to be a **contraction** if its operator norm $\|A\| \leq 1$.

**Definition 1.2.10.** A **basis** $S$ for a vector space $V$ is a nonempty set of linearly independent vectors that span $V$.

**Definition 1.2.11.** Let $V$ be a vector space over $\mathbb{K}$. A mapping $\langle .,. \rangle : V \times V \to \mathbb{K}$ is called an **inner product** if $\forall\ x,\ x',\ y \in V$ and $\alpha \in \mathbb{K}$, the following conditions are satisfied:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(ii) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$,

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,

(iv) $\overline{\langle x, y \rangle} = \langle y, x \rangle$, where $\overline{\langle x, y \rangle}$ is the conjugate of the complex number $\langle y, x \rangle$.

The pair $(V, \langle .,. \rangle)$ is called an **inner product space**.

The function $\|,\| : V \longrightarrow \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x, x \rangle}\ \forall x \in V$ is a norm called the norm generated by the inner product $\langle , \rangle$. A norm in an inner product space will be understood to be this norm.

**Definition 1.2.12.** Let $(V, \langle .,. \rangle)$ be an inner product space. Then $\forall\ x,\ y \in V$, $x$ and $y$ are said to be **orthonormal** if $\langle x, y \rangle = 0$ and $\|x\| = \|y\| = 1$.

**Definition 1.2.13.** A **Hilbert space** is an inner product space which is a Banach space with respect to the norm generated by its inner product function.

**Definition 1.2.14.** An **algebra** is a vector space $V$ over a field $\mathbb{K}$ together with a mapping $(a, b) \longmapsto ab$ of $V \times V \longrightarrow V$ that satisfies the following axioms; for all $a, b, c \in V$, $\alpha \in \mathbb{K}$,

**(i)** $a(bc) = (ab)c$,

**(ii)** $a(b + c) = ab + ac$, $(a + b)c = ac + bc$,

**(iii)** $(\alpha a)b = \alpha(ab) = a(\alpha b)$.

**Example 1.** *Let $H$ be a Hilbert space and $B(H)$ the set of all bounded linear operators on $H$. Then $B(H)$ is an algebra when multiplication is defined pointwise.*

**Definition 1.2.15.** If $T \in B(H, K)$, where $H$ and $K$ are Hilbert spaces, then the linear operator $T^* \in B(K, H)$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ $\forall\, x \in H$ and $\forall\, y \in K$ is called the **(Hilbert space) Adjoint** of $T$.

**Definition 1.2.16.** An operator $T \in B(H)$ is said to be;

- **Self - adjoint** if $T^* = T$.
- **Positive** if $\langle Tx, x\rangle \geq 0$ for all nonzero $x \in H$.
- **Normal** if $T^*T = TT^*$.
- **Unitary** if $T^*T = TT^* = I$.
- **Subnormal** if there exists a Hilbert space $K$ such that $H$ is a closed linear subspace of $K$ and a normal operator $N \in B(K)$ such that $Nx = Tx$ for all $x \in H$.
- **Hyponormal** if $T^*T \geq TT^*$.

**Definition 1.2.17.** If $H$ is a Hilbert space, then an operator $T \in B(H)$ is a **finite rank** operator if the dimension of the range of $T$ is finite, and a **compact** operator if for every bounded sequence $\{x_n\}$ in $H$, the sequence $\{Tx_n\}$ contains a convergent subsequence.

**Definition 1.2.18.** The **singular values** of a compact operator $A \in B(H)$ are defined as the eigenvalues of the operator $\sqrt{A^*A}$ (where $A^*$ denotes the adjoint of $A$ and the square root is taken in the operator sense). The **sequence of singular values** are nonnegative real numbers, usually listed in decreasing order $S_1(A)$, $S_2(A)$,.... The largest singular value $S_1(A)$ is equal to the operator norm of $A$.

**Definition 1.2.19.** Let $D = (\lambda_{jk})$ $(j,\, k = 1, ..., n)$ be an $n$-rowed square matrix. Then the sum of its eigenvalues equals to the **trace** of $D$, that is, the sum of the elements of the principal diagonal: $tr\,(D) = \lambda_{11} + \cdots + \lambda_{nn}$.

**Definition 1.2.20.** Let $M$ be a closed linear subspace of a Hilbert space $H$. Then $M^{\perp} = \{y \in H : \langle x, y\rangle = 0 \ \forall x \in \ M\}$ and $H = M \bigoplus M^{\perp}$; that is, any $h \in H$ has a unique decomposition as $h = x + y$ with $x \in M$, $y \in M^{\perp}$. The **orthogonal projection** $P$ onto $M$ is defined by $Ph = x$ where $h = x + y$ is the decomposition above.

**Definition 1.2.21.** The **Hilbert - Schmidt class** operators $C_2(H)$ is a Hilbert space when equipped with the inner product $\langle X, Y\rangle = tr(XY^*)$, $(X, Y \in C_2(H))$ where $tr$ stands for the usual trace functional and $Y^*$ denotes the adjoint of Y.

**Definition 1.2.22.** Let $K$ be a non - empty bounded subset of the plane. The **diameter** of $K$ is defined by $\operatorname{diam}(K) = \sup\{|\alpha - \beta| : \alpha, \beta \in K\}$.

**Definition 1.2.23.** The **numerical range** of an operator $A \in B(H)$ is defined by $W(A) = \{\langle Ax, x\rangle : x \in H, \ \|x\| = 1\}$ and the **numerical radius** of $A$ by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. The **spectrum** of an operator $A$, $\sigma(A)$, consists of those complex numbers $\lambda$ such that $A - \lambda I$ is not invertible while the **spectral radius** of $A$ denoted by $r(A)$ is defined by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. **Approximate point spectrum** of $A$, $\sigma_{ap}(A)$, consists of those complex numbers $\lambda$ for which there exists a unit sequence $\{x_n\}_n \subseteq H$ such that $\lim_n \|(A - \lambda)x_n\| = 0$.

**Definition 1.2.24.** Let $\mathfrak{A}$ be an algebra and $A \in \mathfrak{A}$. Then the mappings $R_A$ and $L_A$ of $\mathfrak{A}$ into $\mathfrak{A}$ defined by $L_A(X) = AX$, and $R_A(X) = XA$, $X \in \mathfrak{A}$ are called **left multiplication** and **right multiplication** respectively.

### 1.2.2 Algebras

**Definition 1.2.25.** A **subalgebra** of an algebra $\mathfrak{A}$ is a vector subspace $W$ such that $\forall\, b, b' \in W$, we have $bb' \in W$.

**Definition 1.2.26.** A left (respectively, right) **ideal** in an algebra $\mathfrak{A}$ is a vector subspace $\mathfrak{J}$ of $\mathfrak{A}$ such that, $\forall\, a \in \mathfrak{A}$ and $b \in \mathfrak{J}$, we have $ab \in \mathfrak{J}$ (respectively, $ba \in \mathfrak{J}$).

An ideal in $\mathfrak{A}$ is a vector subspace that is simultaneously a left and a right ideal in $\mathfrak{A}$

An ideal $\mathfrak{J}$ is **modular** if there is an element $u$ in $\mathfrak{A}$ such that $a - au$ and $a - ua$ are in $\mathfrak{J}$ for all $a \in \mathfrak{A}$.

If $\mathfrak{J}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{J}$ is an algebra with the multiplication given by $(a + \mathfrak{J})(b + \mathfrak{J}) = ab + \mathfrak{J}$. This is called a **quotient algebra.**

A **maximal** ideal in $\mathfrak{A}$ is a proper ideal that is not contained in any other proper ideal in $\mathfrak{A}$.

If $L$ is a modular maximal left ideal in an algebra $\mathfrak{A}$, we call the ideal $\mathfrak{J} = \{a \in \mathfrak{A} : a\mathfrak{A} \subseteq L\}$ **primitive** ideal of $\mathfrak{A}$ associated to $L$.

**Definition 1.2.27.** If $S$ is a subset of an algebra $\mathfrak{A}$, the **centre** of $S$ is the set

$$Z(S) = \{x \in \mathfrak{A} : xs = sx \; \forall \; s \in S\} .$$

**Definition 1.2.28.** A norm $\|.\|$ on an algebra $\mathfrak{A}$ is said to be **submultiplicative** if it satisfies $\|ab\| \leq \|a\|\|b\| \; \forall \; a, b \in V$.

**Definition 1.2.29.** An algebra $\mathfrak{A}$ with a norm which is submultiplicative is a **normed algebra**.

**Definition 1.2.30.** A normed algebra $\mathfrak{A}$ which admits a unit $e$ such that $ae = ea = a$ and $\|e\| = 1$ is called a **unital normed algebra.**

**Definition 1.2.31.** A complete normed algebra is called a **Banach algebra**.

**Example 2.** *If $S$ is a set, $l^{\infty}(S)$ (the set of all bounded complex valued functions on the set $S$) is a unital Banach algebra if $\forall \; x \in S, \;\; \alpha \in \mathbb{K}$ and $f, \; g \in l^{\infty}(S)$, the operations are defined as follows:*

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (fg)(x) &= f(x)g(x) \\ (\alpha f)(x) &= \alpha f(x). \end{aligned}$$

*And the norm defined as*

$$\|f\|_{\infty} = \sup_{x \in S} |f(x)|.$$

**Definition 1.2.32.** An algebra $\mathfrak{A}$ is called **commutative** (*abelian*) if $ab = ba$, $\forall\, a,\, b \in \mathfrak{A}$. It is *non-abelian* if the product is non-commutative.

**Definition 1.2.33.** Let $\mathfrak{A}$ be an algebra. A mapping from $\mathfrak{A}$ to $\mathfrak{A}$ defined by $a \longmapsto a^*$ is called an **involution** on $\mathfrak{A}$ if it satisfies the following four properties: $\forall\, a, b \in \mathfrak{A},\ \lambda\ \in \mathbb{K}$;

1. $(a+b)^* = a^* + b^*$
2. $(\lambda a)^* = \bar{\lambda} a^*$
3. $(ab)^* = b^* a^*$
4. $a^{**} = a$.

**Definition 1.2.34.** An algebra with an involution is called an **involutive algebra** or * - algebra.

**Definition 1.2.35.** A Banach algebra $\mathfrak{A}$ with an involution $a \longmapsto a^*$ that satisfies

$$\|a\| = \|a^*\|$$

$\forall\, a \in\ \mathfrak{A}$ is known as **Banach * - algebra**.

**Definition 1.2.36.** A Banach * - algebra $\mathfrak{A}$ such that

$$\|aa^*\| = \|a\|^2$$

$\forall\, a \in \mathfrak{A}$ is called a **C* - algebra**.

**Example 3.** *We consider $B(H)$, the set of all bounded linear operators on a Hilbert space $H$ as a $C^*$-algebra.*

**Definition 1.2.37.** A **Von Neumann algebra** $\mathfrak{A}$ is a strongly closed *-subalgebra of the algebra $B(H)$ of bounded operators on a Hilbert space, $H$.

**Definition 1.2.38.** A **W*- algebra** is a C*- algebra $\mathfrak{A}$ for which there is a Banach space $\mathfrak{A}_*$ such that its dual is $\mathfrak{A}$. Then the space $\mathfrak{A}_*$ is uniquely defined and is called the pre-dual of $\mathfrak{A}$.

**Definition 1.2.39.** Let $H$ be a Hilbert space and $B(H)$ be a space of bounded linear operators on $H$. We further define $K(H)$ ( a modular bi-ideal of B(H) ) to be a set of compact linear operators on $H$. Then the quotient space $B(H)/K(H)$, which is an algebra, is known as **Calkin algebra.**

**Definition 1.2.40.** A **derivation** on an algebra $\mathfrak{A}$ is a linear map from $\mathfrak{A}$ to $\mathfrak{A}$ satisfying $\Delta(AB) = \Delta(A)B + A\Delta(B)$, $\forall\ A, B \in \mathfrak{A}$.
Fix $A \in \mathfrak{A}$ and define a mapping from $\mathfrak{A}$ to $\mathfrak{A}$ defined by $\Delta_A(B) = AB - BA$. Then $\Delta_A$ is called **inner derivation.**
The **norm of an inner derivation** in this case is defined as:

$$\|\Delta_A|\mathfrak{A}\| = \sup\{\|\Delta_A(B)\| : B \in \mathfrak{A}, \|B\| = 1\}.$$

A simple application of the triangle inequality shows that

$$\|\Delta_A|\mathfrak{A}\| \leq 2d(A, Z(\mathfrak{A})),$$

where $d(A, Z(\mathfrak{A}))$ denotes the distance from $A$ to $Z(\mathfrak{A})$, the centre of $\mathfrak{A}$.

## 1.3 Elementary theory of tensor products

In this section, we revisit some elementary theory of tensor products. A good account of the theory of tensor products can be found in [25] , [26].

### 1.3.1 Tensor products of vector spaces

**Definition 1.3.1.** Let $X$ be a non-empty set and $\mathbb{K}$ be the field of real or complex numbers. Let $\mathbb{K}_X$ be the set of all finite linear combinations of elements of $X$ such that $\mathbb{K}_X = \{\sum_{i=1}^{n} \alpha_i x_i : \ x_i \in X, \alpha_i \in \mathbb{K}\}$ where the operations are as $\alpha x_i + \beta x_i = (\alpha + \beta)x_i$ and $\alpha(\beta x_i) = (\alpha\beta)x_i$. Then the vector space $\mathbb{K}_X$ over $\mathbb{K}$ is called the **free vector space.**

The term free is used to connote the fact that there is no relationship between the elements of $X$.

**Definition 1.3.2.** Let $X$ and $Y$ be two vector spaces over $\mathbb{K}$, and let $T$ be the subspace of the free vector space $\mathbb{K}_{X\times Y}$ generated by all the vectors of the form $\alpha(x, y)+\beta(x', y)-(\alpha x+\beta x', y)$ and $\alpha(x, y)+\beta(x, y')-(x, \alpha y+\beta y')$ $\forall\ \alpha,\ \beta \in \mathbb{K}$ and $x,\ x' \in X,\ \ y, y' \in Y$. Then the quotient space $\mathbb{K}_{X\times Y}/T$ is called the **tensor product** of $X$ and $Y$ and is denoted by $X \otimes Y$. An element of $X \otimes Y$ has the form $\sum \alpha_i(x_i, y_i) + T$. The coset $(x, y) + T$ is denoted by $x \otimes y$ and therefore any element $\mu$ of $X \otimes Y$ has the form $\mu = \Sigma_i x_i \otimes y_i$.

Let $X$ and $Y$ be vector spaces over $\mathbb{K}$. A function $f : X \times Y \to \mathbb{K}$ is

**bilinear** if it is linear in both variables separately, that is,

$$f(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 f(x_1, y) + \alpha_2 f(x_2, y)$$

and

$$f(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 f(x, y_1) + \beta_2 f(x, y_2)$$

for all $x$, $x_1$, $x_2 \in X$ and $y$, $y_1$, $y_2 \in Y$.

We write $B(X, Y; \mathbb{K})$ to denote the set of all bilinear functions from $X \times Y$ to $\mathbb{K}$. A bilinear function $f : X \times Y \to \mathbb{K}$ with values in the base field is called a **bilinear form** on $X \times Y$. One important use of tensor products is that they turn bilinear maps into linear maps as we can see in Lemma 1.3.3.

**Example 4.** *Let $f$ be a mapping from a cartesian product space to the tensor product space i.e $f : X \times Y \to X \otimes Y$. Then $f$ is a bilinear map.*

*Proof.* Let $x$, $x_1$, $x_2 \in X$ and $y$, $y_1$, $y_2 \in Y$. Also let $\alpha, \beta \in \mathbb{K}$. To show that $f$ is bilinear, it suffices to show that it is linear in each vector space $X$ and $Y$ separately. To show linearity in $X$, let $f(x, y) = x \otimes y$. Then,

$$\begin{aligned} f(\alpha x_1 + \beta x_2, y) &= (\alpha x_1 + \beta x_2) \otimes y \\ &= (\alpha x_1 \otimes y) + (\beta x_2 \otimes y) \\ &= \alpha(x_1 \otimes y) + \beta(x_2 \otimes y) \\ &= \alpha f(x_1, y) + \beta f(x_2, y). \end{aligned}$$

Hence $f$ is linear in $X$.

To show linearity in Y,

$$\begin{aligned} f(x, \alpha y_1 + \beta y_2) &= x \otimes (\alpha y_1 + \beta y_2) \\ &= (x \otimes \alpha y_1) + (x \otimes \beta y_2) \\ &= \alpha(x \otimes y_1) + \beta(x \otimes y_2) \\ &= \alpha f(x, y_1) + \beta f(x, y_2). \end{aligned}$$

Hence $f$ is linear in $Y$ and therefore, $f$ is a bilinear map. □

The tensor product, $X \otimes Y$, of the vector spaces $X$ and $Y$ can be constructed as a space of linear functionals on $B(X \times Y)$ in the following way; for $x \in X$, $y \in Y$ we denote by $x \otimes y$ the functional given by evaluation at the point $(x, y)$. In other words,

$$(x \otimes y)(f) = \langle f, x \otimes y \rangle = f(x, y)$$

for the bilinear form $f$ on $X \times Y$. The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)^*$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form $u = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i$ where $n$ is a natural number, $\alpha_i \in \mathbb{K}$, $x_i \in X$ and $y_i \in Y$.

**Lemma 1.3.3.** *([26], p. 184) If $f : U \times V \longrightarrow W$ is a bilinear map, where $U$, $V$ and $W$ are vector spaces, then there is a unique linear map $f' : U \otimes V \longrightarrow W$.*

### 1.3.2 Tensor products of Hilbert spaces and operators

Let $H_1$ and $H_2$ be two Hilbert spaces. Let $H_0$ denote the algebraic tensor product of $H_1$ and $H_2$. A general element $\xi$ of $H_0$ is of the form;
$\xi = \sum_{i=1}^{n} \xi_{1,i} \otimes \xi_{2,i}, \ \ \xi_{1,i} \in H_1, \ \ \xi_{2,i} \in H_2, \ 1 \le i \le n$. In $H_0$, we define an inner product $\langle .,. \rangle$ by

$$\langle \xi, \eta \rangle = \sum_{i=1} \sum_{j=1} \langle \xi_{1,i}, \eta_{1,j} \rangle \langle \xi_{2,i}, \eta_{2,j} \rangle$$

for $\xi = \sum_{i=1}^{n} \xi_{1,i} \otimes \xi_{2,i} \in H_0$ and $\eta = \sum_{j=1}^{m} \eta_{1,j} \otimes \eta_{2,j} \in H_0$.

**Definition 1.3.4.** The completion $H$ of $H_0$ is called the tensor product of $H_1$ and $H_2$, and denoted by $H_1 \otimes H_2$.

Take arbitrary $x_1 \in B(H_1)$ and $x_2 \in B(H_2)$. We then get an operator $x_0$ on $H_0$ defined by

$$x_0(\xi_1 \otimes \xi_2) = (x_1 \xi_1) \otimes (x_2 \xi_2),$$

which is denoted by $x_1 \otimes x_2$. So that $x_1 \otimes x_2$ is extended to a bounded operator on $H_1 \otimes H_2$, which is also denoted by $x_1 \otimes x_2$ and called the tensor product of $x_1$ and $x_2$.
The following five properties hold [25];

**(i)** $(\lambda x_1 + \mu y_1) \otimes x_2 = \lambda(x_1 \otimes x_2) + \mu(y_1 \otimes x_2)$

**(ii)** $x_1 \otimes (\lambda x_2 + \mu y_2) = \lambda(x_1 \otimes x_2) + \mu(x_1 \otimes y_2)$

**(iii)** $(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 y_1 \otimes x_2 y_2$

**(iv)** $(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*$

**(v)** $\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|$.

### 1.3.3 Tensor product of Banach spaces

This is a review of some general properties of tensor products of Banach spaces.

Suppose $E_1$ and $E_2$ are two complex Banach spaces. Let $E_1 \otimes E_2$ be the algebraic tensor product of $E_1$ and $E_2$ over the complex number field. If a norm $\|.\|$ on $E_1 \otimes E_2$ satisfies the condition

$$\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|, \; x_1 \in E_1, \; x_2 \in E_2,$$

then it is called a **cross - norm** of $E_1 \otimes E_2$.

In general, a cross - norm of $E_1 \otimes E_2$ is not a priori determined. We have to specify which cross - norm is considered on $E_1 \otimes E_2$.

The completion of $E_1 \otimes E_2$ under a cross - norm $\beta$ is denoted $E_1 \otimes_\beta E_2$. We now define two important cross - norms of $E_1 \otimes E_2$. For each $x \in E_1 \otimes E_2$,

$$\|x\|_\lambda = \sup\Big\{|\sum_{i=1}^{n} f(x_{1,i})g(x_{2,i})| : f \in E_1^*, \|f\| \leq 1; g \in E_2^*, \|g\| \leq 1\Big\},$$

where $x = \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i}$, and

$$\|x\|_\gamma = \inf\Big\{\sum_{i=1}^{n} \|x_{1,i}\|\|x_{2,i}\| : \; x = \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i}\Big\}.$$

Being greatest, it is known that $\gamma$ majorizes all other cross - norms: so it is called the **greatest** (or projective) cross - norm.
The completion $E_1 \otimes_\gamma E_2$ is called the **projective tensor product** of $E_1$ and $E_2$.
On the other hand, $\lambda$ is called the **injective cross - norm**, and $E_1 \otimes_\lambda E_2$ is called the **injective tensor product.**

### 1.3.4 Tensor product of C* - algebra

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C* - algebras. The algebraic tensor product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ turns out to be an involutive algebra over the complex number field $\mathbb{C}$ in the natural fashion;

$$\begin{aligned}(x_1 \otimes x_2)(y_1 \otimes y_2) &= x_1 y_1 \otimes x_2 y_2 \\ (x_1 \otimes x_2)^* &= x_1^* \otimes x_2^*,\ x_1\ , y_1 \in \mathfrak{A}_1,\ x_2,\ y_2 \in \mathfrak{A}_2.\end{aligned}$$

If a norm $\beta$ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ satisfies the C* - condition;

$$\begin{aligned}\|xy\|_\beta &\leq \|x\|_\beta \|y\|_\beta \\ \|x^* x\|_\beta &= \|x\|^2,\ x,\ y \in \mathfrak{A}_1 \otimes \mathfrak{A}_2,\end{aligned}$$

then it is called a **C* - norm** of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.
The completion $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$ of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ under any C* - norm $\beta$ is a **C* - algebra**. However, it is not a priori clear that a C* - norm is a cross - norm.

The projective C* - cross - norm $\|.\|_{\max}$ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is given by

$$\|x\|_{\max} = \sup\left\{\|\pi(x)\| : \pi \text{ runs through all representations of } \mathfrak{A}_1 \otimes \mathfrak{A}_2\right\}.$$

The completion $\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2$ of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ under $\|.\|_{\max}$ is called the **projective C* - tensor product** of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Given C* - algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the injective C* - cross - norm $\|x\|_{\min}$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is defined by

$$\|x\|_{\min} = \sup\left\{\|(\pi_1 \otimes \pi_2)(x)\|, \quad x \in \mathfrak{A}_1 \otimes \mathfrak{A}_2\right\},$$

where $\pi_1$ and $\pi_2$ run over all representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. The completion $\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$ is called the **injective C* - tensor product** of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. In general, the projective C* - cross - norm and the injective C* - cross - norm are different.

## 1.4 Literature review

A derivation on an algebra $\mathfrak{A}$ is a linear map from $\mathfrak{A}$ to $\mathfrak{A}$ satisfying $\Delta(AB) = \Delta(A)B + A\Delta(B)$, $\forall\ A, B \in \mathfrak{A}$. Fixing $A \in \mathfrak{A}$, then a mapping from $\mathfrak{A}$ to $\mathfrak{A}$ defined by $\Delta_A(B) = AB - BA$ is called inner derivation. The norm of an inner derivation in this case is defined as:

$$\|\Delta_A|\mathfrak{A}\| = \sup\left\{\|\Delta_A(B)\| : B \in \mathfrak{A}, \|B\| = 1\right\}.$$

When $\mathfrak{A}$ is a Banach algebra, it is clear that each inner derivation $\Delta_A$ is a bounded map on $\mathfrak{A}$. In fact, a simple application of the triangle inequality and submultiplicity of the norm shows that

$$\|\Delta_A|\mathfrak{A}\| \leq 2d(A, Z(\mathfrak{A})), \tag{1.4.1}$$

where $d(A, Z(\mathfrak{A}))$ denotes the distance from $A$ to $Z(\mathfrak{A})$, the centre of $\mathfrak{A}$. In a case where $\mathfrak{A}$ is a C*-algebra, Sakai [34], showed that every derivation on $\mathfrak{A}$ is bounded.

The inequality (1.4.1) has received considerable attention, mainly devoted to showing that equality holds for various algebras.

One of the first studies on the norm of an inner derivation is Stampfli's paper of 1970 [40], in which case $B(H)$ is treated and the maximal numerical range of an operator $A \in B(H)$ is introduced.

It was preceded by Kadison, Lance and Ringrose [22] who established the expected formula, i.e that the equality holds for all elements of a Von Neumann algebra, and at the same time, gave a first example that strict inequality is possible in the Stampfli's result.

Apostol and Zsido [2] showed that equality holds for all elements of a Von Neumann algebra not necessarily when they are self - adjoint while Kyle [24] established the result for a uniformly convex Banach space.

Stampfli [40] showed that the equality holds when $\mathfrak{A}$ is a primitive C*-algebra with identity and in particular when $\mathfrak{A}$ is the algebra of the bounded operators on a Banach space.

Stampfli's approach was followed up by Fong [33] who used the essential numerical range to obtain an analogue of Stampfli's formula for the Calkin algebra C(H). The description of the norm of an inner derivation on the Calkin algebra can also be derived from Stampfli's paper since he had observed the result for a primitive C*-algebra as well.

Gajendragadkar [17], and independently Hall [19] proved that the equality hold for Von Neumann algebra $\mathfrak{A}$ on a separable Hilbert space. On the other hand, Zsido [43] removed the separability assumption in her result where she established the equality.

The quest for a description in the setting of a general C*-algebra, however, continued. In [33], the case of a quotient of C*-algebra by a closed ideal is treated and Halpern [33] obtained the result for $\mathfrak{A}W^*$- algebra. For the quotients of $\mathfrak{A}W^*$- algebra, the problem remained open until recently, when it was shown that equality holds here also by Somerset [38]. On the other hand, in order to examine the possible behaviour of the norms of derivations, Archbold [5] introduced two constants $K(\mathfrak{A})$ and $K_s(\mathfrak{A})$ which he defined to be the smallest numbers in $[0, \infty]$ such that;

$$d(A, Z(\mathfrak{A})) \leq K(\mathfrak{A}) \|\Delta_A|\mathfrak{A}\|$$

$\forall A \in \mathfrak{A}$ and

$$d(A, Z(\mathfrak{A})) \leq K_s(\mathfrak{A}) \|\Delta_A|\mathfrak{A}\|$$

$\forall A \in \mathfrak{A}, A = A^*$.

Clearly $K(\mathfrak{A}) = K_s(\mathfrak{A}) = 0$ when $\mathfrak{A}$ is commutative. When $\mathfrak{A}$ is non commutative, it follows that

$$K(\mathfrak{A}) \geq \frac{1}{2}$$

and

$$K_s(\mathfrak{A}) \geq \frac{1}{2}.$$

Starting with his thesis, Somerset [38] took up techniques introduced by Archbold and, developing them much further, on the other hand showed that the equality hold for a unital C*-algebra $\mathfrak{A}$ if $\mathfrak{A}$ is quasi- standard or a quotient of an $\mathfrak{A}W^*$- algebra.

As boundedly centrally closed C*-algebra are quasi standard, Pere Ara [33], however, used a far more direct approach to show that equality holds here also.

Practically, all the above mentioned investigations rest on Stampfli's work [40]. There is therefore need to investigate other algebras where equality holds.

On the other hand, Barraa and Boumazgour [7], established that if $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ is a norm ideal on $B(H)$, then the restriction of an inner derivation implemented by $A \in B(H)$ on a norm ideal is a bounded linear operator on $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ and that $\|\Delta_A|\mathfrak{J}\| \leq 2d(A)$ where $d(A) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$. This called for a need in this study to investigate this inequality on the quotient algebra.

In order to examine the extent to which Stampfli's equality applies, L. Fialkow [15] introduced the notion of S - universal operators. An operator $A \in B(H)$ is S - universal if $\|\Delta_A|\mathfrak{J}\| = 2d(A)$ for each norm ideal $\mathfrak{J}$ in $B(H)$. He then went ahead to study the criteria of S - universality for subnormal operators where he established that a subnormal operator is S - universal if and only if the diameter of the spectrum is equal to twice the radius of the smallest disk containing it. He then left open the case of an arbitrary hyponormal operator and in 2001, Barraa and Boumazgour [7] established that the same conclusion holds true for an arbitrary hyponormal operator. This was preceded by the work of the same authors in [8] where they extended the same result and provided a necessary and sufficient condition of S - universality for any non-zero operator $A \in B(H)$.

Thus, the need for further investigation of the condition of S - universality is undoubted in this study.

## 1.5 Statement of the problem

Let $\mathfrak{A}$ be an algebra. A derivation on an algebra $\mathfrak{A}$ is a linear mapping $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A}$ such that $\Delta(AB) = \Delta(A)B + A\Delta(B), \ \ \forall\, A, B \in \mathfrak{A}$. Fixing $A \in \mathfrak{A}$, a mapping from $\mathfrak{A}$ to $\mathfrak{A}$ defined by $\Delta_A(B) = AB - BA, \ \ \forall B \in \mathfrak{A}$ is called an inner derivation. The norm of an inner derivation is defined as

$$\|\Delta_A|\mathfrak{A}\| = \sup\{\|\Delta_A(B)\| : B \in \mathfrak{A}, \|B\| = 1\}.$$

By a simple application of triangle inequality and submultiplicity of the norm, it follows that

$$\|\Delta_A|\mathfrak{A}\| \leq 2d(A, Z(\mathfrak{A})), \tag{1.5.1}$$

where $d(A, Z(\mathfrak{A}))$ denotes the distance from $A$ to $Z(\mathfrak{A})$, the centre of $\mathfrak{A}$. Stampfli established that equality hold in (1.5.1) when $\mathfrak{A}$ is primitive C* - algebra and wondered whether the same result would hold in other algebras. We therefore continue to investigate this equality to investigate this equality for the algebra of bounded linear operators on a Hilbert space $H$.

Let $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ be a norm ideal in $B(H)$, the algebra of all bounded linear operators on a Hilbert space. It is clear that the restriction of an inner derivation on a norm ideal $\mathfrak{J}$ is a bounded linear operator on the ideal $\mathfrak{J}$ and that

$$\|\Delta_A|\mathfrak{J}\| \leq 2d(A), \tag{1.5.2}$$

for any $A \in B(H)$. We thus investigate inequality (1.5.2) on the quotient algebra $B(H)/\mathfrak{J}$, and explore the relationships between $\|\Delta_A|B(H)\|$,

$\|\Delta_A|\mathfrak{J}\|$ and $\|\Delta_A|B(H)/\mathfrak{J}\|$. Further, we include hyponormal operators where we partially answer the question by Barraa and Boumazgour as to whether equality $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$ holds true, where $N, A$ are arbitrary normal and hyponormal operators respectively, and $C_2$ being Hilbert Schmidt class operators.

Finally, we investigate the relationship between the diameter of the numerical range and the norm of inner derivation implemented by an operator $A \in B(H)$.

## 1.6 Objectives of the study

The main purpose of this study is to:

1. Establish Stampfli's equality for the algebra of bounded linear operators on a Hilbert space.

2. Investigate the relationship between norms of derivations of algebras, ideals and quotient algebras.

3. Investigate norms of inner derivations implemented by hyponormal and S - universal operators.

## 1.7 Significance of the study

We hope that the results obtained in this study, which to the best of our knowledge have never been investigated, are a contribution to the field of derivations and will provide motivation for further research to pure mathematicians in this area of study. In the operator - algebraic formulation of quantum theory, we hope that these results shall be useful to the theoretical physicist and applied mathematicians alike.

## 1.8 Research methodology

The major approaches used in this study are majorly borrowed from the previous works of mathematicians Stampfli, Fialkow, Kyle, Agure, Barraa and Boumazgour in the same area. We've also revisited existing theories in literature especially from operator algebra, topology and analysis in general. Various research journals by mathematicians in this area of study provided us with insights wherever it was necessary.

# Chapter 2

# NORMS OF DERIVATIONS

## 2.1 Introduction

In this chapter we study derivations, then concentrate on norms of inner derivations. We give exhaustively elementary algebraic properties of derivations and explore basic results on inner derivations. We mention here that most literature on properties of derivations can be found in [12]. Finally, we embark on Stampfli's equality problem where we establish the equality for $B(H)$, the algebra of bounded linear operators on a Hilbert space $H$, which actually forms the major result in this chapter.

## 2.2 Basic results on inner derivations

In this section, we present simple results on derivations.

**Definition 2.2.1.** A derivation on an algebra $\mathfrak{A}$ is a linear mapping $\Delta$ of $\mathfrak{A}$ into $\mathfrak{A}$ such that; $\Delta(AB) = \Delta(A)B + A\Delta(B), \quad \forall A, B \in \mathfrak{A}$.

**Definition 2.2.2.** Given $A \in \mathfrak{A}$, a mapping $\Delta_A$ of $\mathfrak{A}$ into $\mathfrak{A}$ defined by $\Delta_A(B) = AB - BA$, for all $B \in \mathfrak{A}$ is called an inner derivation.

The following proposition then follows immediately from the definitions above,

**Proposition 2.2.3.** *Inner derivation is a derivation.*

*Proof.* We first prove that $\Delta_A : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a linear mapping. Let $A, B, C \in \mathfrak{A}$, $\alpha, \beta \in \mathbb{K}$. Since $\alpha B + \beta C \in \mathfrak{A}$, then it follows that,

$$\begin{aligned} \Delta_A(\alpha B + \beta C) &= A(\alpha B + \beta C) - (\alpha B + \beta C)A \\ &= A\alpha B + A\beta C - \alpha BA - \beta CA \\ &= \alpha AB + \beta AC - \alpha BA - \beta CA \\ &= \alpha AB - \alpha BA + \beta AC - \beta CA \\ &= \alpha(AB - BA) + \beta(AC - CA) \\ &= \alpha\Delta_A(B) + \beta\Delta_A(C). \end{aligned}$$

Thus, $\Delta_A : \mathfrak{A} \longrightarrow \mathfrak{A}$ is linear.

To complete this proof, it suffices to show that $\Delta_A$ further satisfies the condition

$$\Delta_A(BC) = \Delta_A(B)C + B\Delta_A(C).$$

Let $A, B, C \in \mathfrak{A}$, then,

$$\begin{aligned}
\Delta_A(BC) &= A(BC) - (BC)A \\
&= ABC - BCA \\
&= ABC - BCA + BAC - BAC \\
&= ABC - BAC + BAC - BCA \\
&= (AB - BA)C + B(AC - CA) \\
&= (\Delta_A(B))C + B(\Delta_A(C)).
\end{aligned}$$

Thus

$$\Delta_A(BC) = (\Delta_A(B))C + B(\Delta_A(C)).$$

Hence, $\Delta_A : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a derivation. □

**Lemma 2.2.4.** *If $\mathfrak{A}$ is a normed algebra, then each inner derivation $\Delta_A$ is bounded and $\|\Delta_A\| \leq 2\|A\|$.*

*Proof.* For a fixed $A \in \mathfrak{A}$,

$$\Delta_A(C) = AC - CA \quad \text{for all}\, C \in \mathfrak{A}.$$

Implying that,

$$\begin{aligned}
\|\Delta_A(C)\| &= \|AC - CA\| \\
&\leq \|AC\| + \|CA\| \\
&\leq \|A\|\|C\| + \|C\|\|A\| \\
&= 2\|A\|\|C\|.
\end{aligned}$$

Hence by the definition of $\|\Delta_A\|$, it immediately follows that

$$\|\Delta_A\| \leq 2\|A\|.$$

This completes our proof. $\square$

**Lemma 2.2.5.** *$\mathfrak{A}$ is commutative if and only if 0 is the only inner derivation.*

*Proof.* First, assume that $\mathfrak{A}$ is commutative, then $\forall A, B \in \mathfrak{A},\ AB = BA$.
$\forall\, \Delta_A$ inner derivation, we have

$$\begin{aligned}\Delta_A(B) &= AB - BA \\ &= AB - AB \\ &= 0.\end{aligned}$$

Conversely, assume that 0 is the only inner derivation, then for all $B \in \mathfrak{A}$,

$$\begin{aligned}\Delta_A(B) = 0 &\Longrightarrow AB - BA = 0 \\ &\Longrightarrow AB = BA \\ &\Longrightarrow \mathfrak{A} \text{ is commutative.}\end{aligned}$$

$\square$

**Example 5.** *Let $\mathfrak{A}$ be an algebra with unit and let $a$ be an element of $\mathfrak{A}$ that is not algebraic, that is, such that the elements $1, a, a^2, \ldots$ are linearly independent. Let $B$ be the subalgebra of $\mathfrak{A}$ generated by $1, a$ and define a* *mapping $\Delta$ of $B$ into $B$ by*

$$\Delta(\alpha_0 + \alpha_1 a + \alpha_2 a^2 + ... + \alpha_n a^n) \;=\; \alpha_1 + 2\alpha_2 a + ... + n\alpha_n a^{n-1}.$$

*Then $\Delta$ is a derivation on $B$ which is not inner since $B$ is commutative and $\Delta \neq 0$.*

We next show that a simple application of triangle inequality in $\|\Delta_A|\mathfrak{A}\|$ where $\mathfrak{A}$ is non - commutative and submultiplicity of the norm gives $\|\Delta_A|\mathfrak{A}\| \leq 2d(A, Z(\mathfrak{A}))$.

**Lemma 2.2.6.** *If $\mathfrak{A}$ is non - commutative, then $\|\Delta_A|\mathfrak{A}\| \leq 2d(A, Z(\mathfrak{A}))$.*

*Proof.* By definition,

$$\begin{aligned}
\|\Delta_A|\mathfrak{A}\| &= \sup\{\|\Delta_A(B)\| : B \in \mathfrak{A},\ \|B\| = 1\} \\
&= \sup\{\|AB - BA\| : B \in \mathfrak{A},\ \|B\| = 1\} \\
&= \sup\{\|AB - SB + SB - BA\| : B \in \mathfrak{A},\ \|B\| = 1,\ S \in Z(\mathfrak{A})\} \\
&= \sup\{\|AB - SB + BS - BA\| : B \in \mathfrak{A},\ \|B\| = 1,\ S \in Z(\mathfrak{A})\} \\
&= \sup\{\|(A - S)B + B(S - A)\| : B \in \mathfrak{A},\ \|B\| = 1,\ S \in Z(\mathfrak{A})\} \\
&\leq \sup\{\|A - S\|\|B\| + \|B\|\|S - A\| : B \in \mathfrak{A},\ \|B\| = 1,\ S \in Z(\mathfrak{A})\} \\
&\leq 2\inf\{\|A - S\| :\ S \in Z(\mathfrak{A})\} \\
&= 2d(A, Z(\mathfrak{A})).
\end{aligned}$$

That is

$$\|\Delta_A|\mathfrak{A}\| \;\leq\; 2d(A, Z(\mathfrak{A})).$$

$\square$

**Note 2.2.7.** *We note that when $\mathfrak{A}$ is $B(H)$, then $d(A, Z(\mathfrak{A}))$ simply becomes $d(A)$ since on a Hilbert space, the centre consists only of complex scalars.*

## 2.3 Algebraic properties of derivations

In this section we present the elementary algebraic properties of derivations in propositions 2.3.1 and 2.3.2. This enables us to give Lemma 2.3.3 for arbitrary inner derivations. We realize from Propositions 2.3.1 and 2.3.2 that the concept of derivation is just a generalization of the concept of differentiation of polynomials while Lemma 2.3.3 generalizes the formula for calculating inner derivations.

**Proposition 2.3.1.** *Let $\Delta$ be a derivation on an algebra $\mathfrak{A}$. Then the following statements hold:*

1. *Leibnitz rule*
   $\Delta^n(AB) = \sum_{r=0}^{n} \binom{n}{r} (\Delta^{n-r}(A))(\Delta^r(B)) \quad (n \in \mathbb{N}; A, B \in \mathfrak{A}).$
2. $\Delta(A^n) = nA^{n-1}\Delta(A)$ $(n - 1 \in \mathbb{N})$ *if and only if* $A\Delta(A) = \Delta(A)A$.
3. *If* $\Delta^2(A) = 0$, *then* $\Delta^n(A^n) = n!(\Delta(A))^n, \quad n \in \mathbb{N}$.

*Proof.* We shall use mathematical induction[1] to prove this Proposition as organized in three parts below,

[1]We note that $\Delta^0(A) = A$, which implies that the operator $\Delta$ does not act on $A$ at all.

**Part 1.** We prove that

$$\Delta^n(AB) = \sum_{r=0}^{n} \tbinom{n}{r} (\Delta^{n-r}(A))(\Delta^r(B)) \quad (n \in \mathbb{N}; A, B \in \mathfrak{A}).$$

When $n = 1$,

$$\begin{aligned}
\Delta(AB) &= \tbinom{1}{0}(\Delta(A))(\Delta^0(B)) + \tbinom{1}{1}(\Delta^0(A))(\Delta(B)) \\
&= \frac{1!}{(1-0)!0!}\Delta(A)B + \frac{1!}{(1-1)!1!}A\Delta(B) \\
&= \Delta(A)B + A\Delta(B).
\end{aligned}$$

Hence it is true for $n = 1$.

Now, assume that it is true for $n = k$, that is,

$$\Delta^k(AB) = \sum_{r=0}^{k} \tbinom{k}{r} (\Delta^{k-r}(A))(\Delta^r(B)).$$

We prove that it is true for $n = k + 1$,

$$\begin{aligned}
\Delta^{k+1}(AB) &= \Delta(\Delta^k(AB)) \\
&= \Delta\Big(\sum_{r=0}^{k} \tbinom{k}{r} \Delta^{k-r}(A)\Delta^r(B)\Big) \\
&= \sum_{r=0}^{k} \tbinom{k}{r} \Delta\Big(\Delta^{k-r}(A)\Delta^r(B)\Big) \\
&= \sum_{r=0}^{k} \tbinom{k}{r} \Big(\Delta^{k-r+1}(A)\Delta^r(B) + \Delta^{k-r}(A)\Delta^{r+1}(B)\Big) \\
&= \sum_{r=0}^{k+1} \frac{(k+1)!}{(k+1-r)!r!} \Big(\Delta^{k-r+1}(A)\Delta^r(B)\Big) \\
&= \sum_{r=0}^{k+1} \tbinom{k+1}{r} \Big(\Delta^{k-r+1}(A)\Delta^r(B)\Big).
\end{aligned}$$

Hence it is true for all $n$.

**Part 2.** We now show that

$$\Delta(A^n) = nA^{n-1}\Delta(A) \ (n-1 \in \mathbb{N}) \text{ if and only if } A\Delta(A) = \Delta(A)A.$$

We first assume that $\Delta(A^n) = nA^{n-1}\Delta(A)$ and prove that $A\Delta(A) = \Delta(A)A$.
For $n = 2$,

$$\Delta(A^2) = 2A\Delta(A). \tag{2.3.1}$$

But

$$\begin{aligned} \Delta(A^2) &= \Delta(AA) \\ &= \Delta(A)A + A\Delta(A). \end{aligned}$$

That is

$$\Delta(A^2) = \Delta(A)A + A\Delta(A). \tag{2.3.2}$$

Equating (2.3.1) and (2.3.2), we obtain

$$2A\Delta(A) = \Delta(A)A + A\Delta(A),$$

$\Longrightarrow \quad A\Delta(A) = \Delta(A)A$. Hence true for $n = 2$.
Assume it is true for $n = k$, that is
$\Delta(A^k) = kA^{k-1}\Delta(A) \Longrightarrow A\Delta(A) = \Delta(A)A$. Then for $n = k+1$, we have

$$\Delta(A^{k+1}) = (k+1)A^k\Delta(A).$$

But

$$\begin{aligned}\Delta(A^{k+1}) &= \Delta(A^k A)\\ &= \Delta(A^k)A + A^k\Delta(A)\\ &= kA^{k-1}\Delta(A)A + A^k\Delta(A).\end{aligned}$$

This implies that;

$$kA^{k-1}\Delta(A)A + A^k\Delta(A) = (k+1)A^k\Delta(A).$$

$$kA^{k-1}\Delta(A)A + A^k\Delta(A) = kA^k\Delta(A) + A^k\Delta(A).$$

$$\therefore\ kA^{k-1}\Delta(A)A = kA^{k-1}A\Delta(A).$$

$$\Longrightarrow \Delta(A)A = A\Delta(A).$$

Thus, it is true for all $n$.

Conversely, we assume that $A\Delta(A) = \Delta(A)A$ and prove that $\Delta(A^n) = nA^{n-1}\Delta(A)$.

For $n = 2$,

$$\begin{aligned}\Delta(A^2) &= \Delta(AA)\\ &= \Delta(A)A + A\Delta(A)\\ &= A\Delta(A) + A\Delta(A)\\ &= 2A\Delta(A).\ \text{Hence true.}\end{aligned}$$

Assume that it is true for $n = k$, that is $A\Delta(A) = \Delta(A)A \implies \Delta(A^k) = kA^{k-1}\Delta(A)$.

We prove that it is true for $n = k + 1$,

$$\begin{aligned}
\Delta(A^{k+1}) &= \Delta(A^k A) \\
&= \Delta(A^k)A + A^k\Delta(A) \\
&= kA^{k-1}\Delta(A)A + A^k\Delta(A) \\
&= kA^{k-1}A\Delta(A) + A^k\Delta(A) \\
&= kA^k\Delta(A) + A^k\Delta(A) \\
&= (k+1)A^k\Delta(A).
\end{aligned}$$

Hence it is true for all $n$.

**Part 3.** Finally, we prove that

If $\Delta^2(A) = 0$, then $\Delta^n(A^n) = n!(\Delta(A))^n, \quad n \in \mathbb{N}$.

Let $\Delta^2(A) = 0$. Then by induction

$$\Delta(\Delta(A))^k = 0 \qquad k \in \mathbb{N}. \tag{2.3.3}$$

Assume that;

$$\Delta^{n-1}(A^{n-1}) = (n-1)!(\Delta(A))^{n-1}.$$

Then by (2.3.3),

$$\Delta^n(A^{n-1}) = 0.$$

Therefore, by Leibnitz rule,

$$\begin{aligned}
\Delta^n(A^n) &= \Delta^n(AA^{n-1}) \\
&= \sum_{r=0}^{n} \tbinom{n}{r}(\Delta^r(A))(\Delta^{n-r}(A^{n-1})) \\
&= \tbinom{n}{0}A\Delta^n(A^{n-1})) + \tbinom{n}{1}\Delta(A)\Delta^{n-1}(A^{n-1}) + \ldots + \tbinom{n}{n}\Delta^n(A)A^{n-1} \\
&= \tbinom{n}{1}\Delta(A)\Delta^{n-1}(A^{n-1}), \quad \text{other terms vanish.} \\
&= \frac{n!}{(n-1)!1!}(\Delta(A))(\Delta^{n-1}(A^{n-1})) \\
&= \frac{n(n-1)!}{(n-1)!1!}(\Delta(A)\Delta^{n-1}(A^{n-1})) \\
&= n\Delta(A)\Delta^{n-1}(A^{n-1}) \\
&= n(\Delta(A))(n-1)!(\Delta(A))^{n-1} \\
&= n(n-1)!(\Delta(A))(\Delta(A))^{n-1} \\
&= n!(\Delta(A))^n.
\end{aligned}$$

□

**Proposition 2.3.2.** *Let $\Delta$ be a derivation on an algebra $\mathfrak{A}$, and let $\ell$ be an idempotent in $\mathfrak{A}$. Then the following statements hold;*

**(i)** $\ell\Delta(\ell)\ell = 0$.

**(ii)** *If $\ell\Delta(\ell) = \Delta(\ell)\ell$, then $\Delta(\ell) = 0$.*

**(iii)** *If $\mathfrak{A}$ has a unit element $e$, then $\Delta(e) = 0$.*

*Proof.* We give a detailed proof in three cases.

**Case 1.** We prove that $\ell\Delta(\ell)\ell = 0$.

We have

$$\begin{aligned}\Delta(\ell) &= \Delta(\ell^2)\\ &= \Delta(\ell\ell)\\ &= \Delta(\ell)\ell + \ell\Delta(\ell).\end{aligned}$$

That is

$$\Delta(\ell) = \Delta(\ell)\ell + \ell\Delta(\ell),$$

$$\begin{aligned}\Longrightarrow \ell\Delta(\ell) &= \ell\Delta(\ell)\ell + \ell^2\Delta(\ell)\\ &= \ell\Delta(\ell)\ell + \ell\Delta(\ell),\end{aligned}$$

$$\begin{aligned}\Longrightarrow \ell\Delta(\ell)\ell &= \ell\Delta(\ell) - \ell\Delta(\ell)\\ &= 0.\end{aligned}$$

Thus

$$\ell\Delta(\ell)\ell = 0.$$

**Case 2.** We prove that if $\ell\Delta(\ell) = \Delta(\ell)\ell$, then $\Delta(\ell) = 0$.

We have

$$\begin{aligned}
\Delta(\ell) &= \Delta(\ell^2) \\
&= \Delta(\ell\ell) \\
&= \Delta(\ell)\ell + \ell\Delta(\ell) \\
&= \Delta(\ell)\ell^2 + \ell^2\Delta(\ell).
\end{aligned}$$

Since $\ell\Delta(\ell) = \Delta(\ell)\ell$

$$\begin{aligned}
\therefore\ \ell^2\Delta(\ell) &= \ell\Delta(\ell)\ell \\
&= 0 \quad by(i)
\end{aligned}$$

and

$$\begin{aligned}
\ell\Delta(\ell)\ell &= \Delta(\ell)\ell^2 \\
&= 0 \quad by(i).
\end{aligned}$$

Thus;

$$\begin{aligned}
\Delta(\ell) &= \ell^2\Delta(\ell) + \Delta(\ell)\ell^2 \\
&= \ell\Delta(\ell)\ell + \ell\Delta(\ell)\ell \\
&= 2\ell\Delta(\ell)\ell \\
&= 0.
\end{aligned}$$

Hence,

$$\Delta(\ell) = 0.$$

**Case 3.** We prove that if $\mathfrak{A}$ has a unit element $e$, then $\Delta(e) = 0$.

If $e$ is a unit element of $\mathfrak{A}$, then;

$$\begin{aligned}
\Delta(e) &= \Delta(e^2) \\
&= \Delta(ee) \\
&= \Delta(e)e + e\Delta(e) \\
&= \Delta(e)e^2 + e^2\Delta(e) \\
&= 2e\Delta(e)e \\
&= 0 \quad (by \quad (i).)
\end{aligned}$$

$$\Longrightarrow \quad \Delta(e) = 0.$$

$\square$

We end this section by giving the following Lemma which generalizes the formula for calculating inner derivations.

**Lemma 2.3.3.** *Let $\mathfrak{A}$ be an algebra with inner derivation $\Delta_A$, then*

$$(\Delta_A)^n(X) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XA^r.$$

*Proof.* We shall prove this lemma by induction.

For $n = 1$

$$\begin{aligned}
\Delta_A(X) &= \sum_{r=0}^{1}(-1)^r\tbinom{1}{r}A^{1-r}XA^r \\
&= \tbinom{1}{0}AX - \tbinom{1}{1}XA \\
&= AX - XA.
\end{aligned}$$

$$\Longrightarrow \quad \Delta_A(X) = AX - XA. \quad \text{Hence true.}$$

Let it be true for $n = k$, that is

$$(\Delta_A)^k(X) = \sum_{r=0}^{k}(-1)^r\tbinom{k}{r}A^{k-r}XA^r.$$

We prove that it is true for $n = k + 1$,

$$\begin{aligned}
(\Delta_A)^{k+1}(X) &= \Delta_A(\Delta_A^k(X)) \\
&= \Delta_A\Big(\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}A^{k-r}XA^r\Big) \\
&= \sum_{r=0}^{k}(-1)^r\tbinom{k}{r}\Big(\Delta_A(A^{k-r}XA^r)\Big) \\
&= \sum_{r=0}^{k}(-1)^r\tbinom{k}{r}\Big(A(A^{k-r}XA^r) - (A^{k-r}XA^r)A\Big) \\
&= \sum_{r=0}^{k}(-1)^r\tbinom{k}{r}\Big(A^{k-r+1}XA^r - A^{k-r}XA^{r+1}\Big) \\
&= \sum_{r=0}^{k+1}(-1)^r\tbinom{k+1}{r}A^{k-r+1}XA^r.
\end{aligned}$$

Thus it is true for all $n$. $\square$

## 2.4 Norms of inner derivation

We shall be interested in the algebra of bounded linear operators on a Hilbert space, $B(H)$. The norm of an inner derivation implemented by an element $A \in B(H)$ is defined as,

$$\|\Delta_A|B(H)\| = \sup\{\|\Delta_A(B)\| : B \in B(H),\ \|B\| = 1\}.$$

The aim of this section is to present Stampfli's equality of Theorem 2.4.4 which states that $\|\Delta_A|B(H)\| = 2\inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|$. The first inequality $\|\Delta_A|B(H)\| \leq 2\inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|$ follows easily from the following remark,

**Remark 1.** For any $A \in B(H)$, $\|\Delta_A|B(H)\| = \|\Delta_{A-\lambda I}|B(H)\| \leq 2\inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|$ for all $\lambda \in \mathbb{C}$.

*Proof.*

$$\begin{aligned}
\Delta_A|B(H) &= \Delta_A(X) \quad \text{for } X \in B(H) \\
&= AX - XA \\
&= (A - \lambda I)X - X(A - \lambda I) \\
&= \Delta_{A-\lambda I}(X) \\
&= \Delta_{A-\lambda I}|B(H).
\end{aligned}$$

Hence it follows immediately that;

$$\begin{aligned}
\|\Delta_A|B(H)\| &= \|\Delta_{A-\lambda I}|B(H)\| \\
&\leq 2\|A - \lambda I\| \quad \text{since } B(H) \text{ is a normed algebra.}
\end{aligned}$$

This implies that

$$\|\Delta_A|B(H)\| \leq 2 \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|. \tag{2.4.1}$$

□

**Definition 2.4.1.** A Banach space $\mathfrak{X}$ is said to be **uniformly convex** if, for any $x_n$ and $y_n$ in the unit ball of $\mathfrak{X}$, the statement that

$$\|x_n + y_n\| \longrightarrow 2 \ as \ n \longrightarrow \infty$$

implies that

$$\|x_n - y_n\| \longrightarrow 0 \ as \ n \longrightarrow \infty$$

**Example 6.** *All inner product spaces are uniformly convex, [31].*

J. G. Stampfli [40] established when equality holds in equation (2.4.1) above. He then wondered whether the same result would hold in other algebras. It has been shown satisfactorily that equality sometimes hold and that strict inequality is also possible as we can see in the following examples by B. E. Johnson [21].

**Example 7.** *Let $1 < p < \infty$ and $p \neq 2$. Then there is a rank - one operator $A \in B(\ell^p)$ such that*

$$\|\Delta_A\| < 2 \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\|.$$

Johnson [21] also provided examples of spaces where equality does hold as is the case in the following example,

**Example 8.** *Let $\ell_n^1 = (\mathbb{R}^n, \|.\|_1)$ which is a real Banach space with norm $\|x\| = \sum |x_i|$. Then*

$$\|\Delta_A\| = 2 \inf_{\lambda \in \mathbb{R}} \|A - \lambda I\|,$$

*for any $A \in B(\ell_n^1(\mathbb{R}))$.*

Subsequently, Kyle [24] established the equality for the uniformly convex Banach space. In particular, he propagated the following theorem;

**Theorem 2.4.2.** *([24]) Let $\mathfrak{X}$ be a uniformly convex Banach space. Then the following conditions are equivalent:*

**(i)** *$\mathfrak{X}$ is isometric to a Hilbert space*

**(ii)** $\|\Delta_A\| = 2 \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ *holds for any $A \in B(H)$*

**(iii)** $\|\Delta_A\| = 2 \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ *holds for any rank - one operator $A \in B(H)$.*

In this study, we shall use the concept of tensor products to create rank two operators that will enable us establish Stampfli's [40] equality for the algebra of bounded linear operators on a Hilbert space, $B(H)$, which forms our major result in this chapter. We first give the following definition;

**Definition 2.4.3.** The maximal numerical range of $A \in B(H)$ is

$$W_o(A) = \{\lambda \in \mathbb{C} : \lambda = \lim_n \langle Ax_n, x_n \rangle, \text{ where } \{x_n\} \in H \text{ and } \|Ax_n\| \longrightarrow \|A\|\}$$

The set $W_o(A)$ is known to be non - empty, closed, and convex, [40].

**Theorem 2.4.4.** *Let $H$ be a Hilbert space and $B(H)$ the algebra of bounded linear operators on $H$. Then for $A \in B(H)$,*

$$\|\Delta_A|B(H)\| = 2d(A).$$

*Proof.* Let $\mu \in W_o(A)$. Then it follows that there is a sequence $\{x_n\} \in H$ of unit vectors such that

$$\mu = \lim_n \langle Ax_n, x_n \rangle,$$

and

$$\|A\| = \lim_n \|Ax_n\|.$$

Set $Ax_n = \alpha_n x_n + \beta_n y_n$ for $n \in \mathbb{N}$, where $y_n \perp x_n$ and $\|y_n\| = 1$ so that

$$\alpha_n = \langle Ax_n, x_n \rangle \longrightarrow \mu$$

and

$$|\alpha_n|^2 + |\beta_n|^2 = \|Ax_n\|^2 \longrightarrow \|A\|^2 \quad as \ n \longrightarrow \infty.$$

Define the rank - 2 operators $V_n \in B(H)$ by

$$V_n = (x_n \otimes x_n - y_n \otimes y_n) o P_n$$

where $P_n$ is the orthogonal projection onto $[x_n, y_n]$.

Here

$$(u \otimes v)x = \langle x, u \rangle v$$

for $u. v. x \in H$. Thus:

$$\begin{aligned}
V_n x_n &= (x_n \otimes x_n - y_n \otimes y_n) o P_n x_n \\
&= (x_n \otimes x_n - y_n \otimes y_n) x_n \\
&= (x_n \otimes x_n) x_n - (y_n \otimes y_n) x_n \\
&= \langle x_n, x_n \rangle x_n - \langle x_n, y_n \rangle y_n \\
&= \|x_n\|^2 x_n - 0.y_n \\
&= x_n.
\end{aligned}$$

We also find that,

$$\begin{aligned}
V_n y_n &= (x_n \otimes x_n - y_n \otimes y_n) o P_n y_n \\
&= (x_n \otimes x_n - y_n \otimes y_n) y_n \\
&= (x_n \otimes x_n) y_n - (y_n \otimes y_n) y_n \\
&= \langle y_n, x_n \rangle x_n - \langle y_n, y_n \rangle y_n \\
&= 0.x_n - \|y_n\|^2 . y_n \\
&= -y_n.
\end{aligned}$$

Moreover $\|V_n\| = 1 \ \forall \ n$ as we can see here below,

$$\begin{aligned}
\|V_n\| &= \sup\{\|V_n x_n\| : x_n \in H, \|x_n\| = 1\} \\
&= \sup\{\|x_n\| : x_n \in H, \|x_n\| = 1\} \\
&= 1.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
\lim_n \|AV_nx_n - V_nAx_n\| &= \lim_n \|Ax_n - V_nAx_n\| \\
&= \lim_n \|(\alpha_n x_n + \beta_n y_n) - V_n(\alpha_n x_n + \beta_n y_n)\| \\
&= \lim_n \|\alpha_n x_n + \beta_n y_n - \alpha_n V_n x_n - \beta_n V_n y_n\| \\
&= \lim_n \|\alpha_n x_n + \beta_n y_n - \alpha_n x_n + \beta_n y_n\| \\
&= \lim_n \|2\beta_n y_n\| \\
&= \lim_n 2|\beta_n|\|y_n\| \\
&= \lim_n 2|\beta_n|.
\end{aligned}$$

But since $|\alpha_n|^2 + |\beta_n|^2 = \|Ax_n\|^2$, it follows that;

$$\begin{aligned}
|\beta_n|^2 &= \|Ax_n\|^2 - |\alpha_n|^2 \\
|\beta_n| &= (\|Ax_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}} \\
\lim_n |\beta_n| &= \lim_n (\|Ax_n\|^2 - |\alpha_n|^2)^{\frac{1}{2}} \\
&= (\|A\|^2 - |\mu|^2)^{\frac{1}{2}}.
\end{aligned}$$

So that we have

$$\begin{aligned}
\lim_n \|AV_nx_n - V_nAx_n\| &= \lim_n 2|\beta_n| \\
&= 2(\|A\|^2 - |\mu|^2)^{\frac{1}{2}},
\end{aligned}$$

$$\lim_n \|AV_nx_n - V_nAx_n\| = 2(\|A\|^2 - |\mu|^2)^{\frac{1}{2}}. \tag{2.4.2}$$

But by definition,

$$\begin{aligned}
\|\Delta_A|B(H)\| &= \sup\{\|\Delta_A(V_n)\| : V_n \in B(H),\ \|V_n\| = 1\} \\
&= \sup\{\|AV_n - V_nA\| : V_n \in B(H),\ \|V_n\| = 1\} \\
&\geq \|AV_n - V_nA\|,\ V_n \in B(H),\ \|V_n\| = 1.
\end{aligned}$$

Since

$$\|AV_n - V_nA\| = \sup\{\|(AV_n - V_nA)x_n\| : x_n \in H,\ \|x_n\| = 1\},$$

it follows that

$$\|\Delta_A|B(H)\| \geq \limsup_n \|AV_nx_n - V_nAx_n\|. \tag{2.4.3}$$

Considering equations (2.4.2) and (2.4.3), it follows that

$$\|\Delta_A|B(H)\| \geq 2(\|A\|^2 - |\mu|^2)^{\frac{1}{2}}. \tag{2.4.4}$$

Next, we note that

$$\|\Delta_A|B(H)\| = \|\Delta_{A-\lambda_o I}|B(H)\|$$

for some scalar $\lambda_o \in \mathbb{C}$.

We can then observe that if $0 \in W_o(A - \lambda_o I)$, then equation (2.4.4) would read,

$$\begin{aligned}
\|\Delta_A|B(H)\| &\geq 2(\|A - \lambda_o I\|^2 - 0)^{\frac{1}{2}} \\
&= 2\|A - \lambda_o I\|.
\end{aligned}$$

Thus

$$\|\Delta_A|B(H)\| \geq 2 \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|. \tag{2.4.5}$$

From equations (2.4.1) and (2.4.5), we obtain our result. □

**Note 2.4.5.** *The above Theorem 2.4.4 was first established by Stampfli [40] who included two approaches in proving this result. Also, the non - trivial part of the argument developed in our proof above, which is to find $\lambda_o \in \mathbb{C}$ so that $0 \in W_o(A - \lambda_o I)$, is well documented in [40].*

# Chapter 3

# NORM IDEALS AND S - UNIVERSALITY

## 3.1 Introduction

In this chapter, we concentrate on norms of inner derivations on norm ideals and give several results with respect to it. We define a symmetric norm ideal of the algebra of bounded linear operators on a Hilbert space, $B(H)$. Further, we explore the relationships between the norm of inner derivations restricted to the algebra $B(H)$, norm ideal $\mathfrak{J}$ and the quotient of $B(H)$ by a closed norm ideal, $\mathfrak{J}$. Major tools and the approach used in this chapter are majorly borrowed from the previous works of Agure [1], Barraa and Boumazgour [7, 8], Fialkow [15], Halmos [20], other references were also useful.

In section 3.2 of this chapter, we extend the inequality $\|\Delta_A|\mathfrak{J}\| \leq 2d(A)$ from a norm ideal, $\mathfrak{J}$ to the quotient algebra, $B(H)/\mathfrak{J}$. We then explore its applications to S - universal operators where we establish the conditions under which the equality $\|\Delta_{[A]}|B(H)/\mathfrak{J}\| = \|\Delta_A|\mathfrak{J}\|$ is satisfied.

Section 3.3 satisfactorily considers the case of a hyponormal operator while section 3.4 deals with inner derivations and the numerical range, where we establish the interesting relationship between the norm of inner derivation on norm ideals and the diameter of the numerical range. Finally, we explore its applications to S - universality and make attempt to answer related questions.

## 3.2 Inner derivation on norm ideals

Recall from definition 1.2.40 that for $A \in B(H)$, the inner derivation induced by $A$ is the operator $\Delta_A$ defined on $B(H)$ by $\Delta_A(X) = AX - XA$, for all $X \in B(H)$. The norm of an inner derivation $\Delta_A$ on $H$ has been computed by J. Stampfli, see Theorem 2.4.4 as

$$\|\Delta_A|B(H)\| = 2d(A) \tag{3.2.1}$$

where $d(A) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$.

**Definition 3.2.1.** Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. A norm ideal $(\mathfrak{J}, \|.\|)$ in $B(H)$ consists of a proper two-sided ideal $\mathfrak{J}$ together with a norm $\|.\|_{\mathfrak{J}}$ satisfying the following conditions;

- $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$ is a Banach space.
- $\|AXB\|_{\mathfrak{J}} \leq \|A\|\|X\|_{\mathfrak{J}}\|B\|$ for all $X \in \mathfrak{J}$ and all operators A and B in $B(H)$.

- $\|X\|_{\mathfrak{J}} = \|X\|$ for X a rank one operator.

**Note 3.2.2.** *The norm ideal defined above is sometimes referred to as symmetric norm ideal*[1]*.*

The following remark is an important characterization of the quotient of a normed algebra $\mathfrak{A}$ by a closed norm ideal $\mathfrak{J}$,

**Remark 2.** ([26]) If $\mathfrak{J}$ is a closed ideal in a normed algebra $\mathfrak{A}$, then the quotient $\mathfrak{A}/\mathfrak{J}$ is a normed algebra when multiplication is defined as $(X+\mathfrak{J})(Y+\mathfrak{J}) = (XY+\mathfrak{J})$ for $X$, $Y \in \mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}$ is endowed with the quotient norm; $\|X+\mathfrak{J}\| = \inf\{\|X+K\| : K \in \mathfrak{J}\}$.

In this section, we shall be interested in norms of inner derivations on norm ideals. Let $\mathfrak{J}$ be a norm ideal in $B(H)$ and let $A \in B(H)$. If $X \in \mathfrak{J}$, then $\Delta_A(X) \in \mathfrak{J}$ and

$$\|\Delta_A|\mathfrak{J}\| = \sup\{\|\Delta_A(X)\| : X \in \mathfrak{J}, \|X\| = 1\}.$$

We give the following simple proposition which indicates that the restriction of an inner derivation on a norm ideal is a bounded linear operator on the ideal.

**Proposition 3.2.3.** *Let $\mathfrak{J}$ be a norm ideal in $B(H)$ and let $A \in B(H)$. Then $\|\Delta_A|\mathfrak{J}\| \leq 2d(A)$.*

[1] *We shall write $\mathfrak{J}$ to denote a symmetric norm ideal $(\mathfrak{J}, \|.\|_{\mathfrak{J}})$.*

*Proof.* For any $X \in \mathfrak{J}$, we have

$$\begin{aligned}
\|\Delta_A(X)\|_{\mathfrak{J}} &= \|AX - XA\|_{\mathfrak{J}} \\
&= \|(A-\lambda I)X - X(A-\lambda I)\|_{\mathfrak{J}}, \quad \text{for } \lambda \in \mathbb{C} \\
&\leq \|(A-\lambda I)X\|_{\mathfrak{J}} + \|X(A-\lambda I)\|_{\mathfrak{J}} \\
&\leq \|A-\lambda I\|\|X\|_{\mathfrak{J}} + \|X\|_{\mathfrak{J}}\|A-\lambda I\| \\
&= 2\,\|A-\lambda I\|\|X\|_{\mathfrak{J}} \\
&\leq 2d(A)\|X\|_{\mathfrak{J}}.
\end{aligned}$$

This implies that

$$\|\Delta_A(X)\|_{\mathfrak{J}} \;\leq\; 2d(A)\,\|X\|_{\mathfrak{J}}.$$

By definition of $\|\Delta_A|\mathfrak{J}\|$, it follows that

$$\|\Delta_A|\mathfrak{J}\| \leq 2d(A). \tag{3.2.2}$$

□

The following corollary therefore follows,

**Corollary 3.2.4.** *Let $\mathfrak{J}$ be a norm ideal in $B(H)$ and $A \in B(H)$. Then*

$$\|\Delta_A|\mathfrak{J}\| \;\leq\; \|\Delta_A|B(H)\|.$$

*Proof.* By Theorem 2.4.4, for any $A \in B(H)$, we have

$$\|\Delta_A|B(H)\| \;=\; 2d(A).$$

It therefore follows from Proposition 3.2.3 that

$$\|\Delta_A|\mathfrak{J}\| \leq \|\Delta_A|B(H)\|.$$

$\square$

Next, we shall show that the inequality (3.2.2) is also possible for the quotient algebra $B(H)/\mathfrak{J}$. See Theorem 3.2.5.

Let us first note that addition in $B(H)/\mathfrak{J}$ is defined as; $(X+\mathfrak{J})+(Y+\mathfrak{J}) = (X+Y)+\mathfrak{J}$, for $X, Y \in B(H)$.

**Theorem 3.2.5.** *Let $\mathfrak{J}$ be a norm ideal in $B(H)$ and $A \in B(H)$. Then*

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq 2d(A).$$

*Proof.* Let us first consider the following definitions;
$[X] = X + \mathfrak{J}$, $\|[X]\| = \inf\{\|X+K\| : K \in \mathfrak{J}\}$ and
$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| = \sup\{\|\Delta_{[A]}([X])\| : [X] \in B(H)/\mathfrak{J}, \|[X]\| = 1\}$, where $[X]$ is the canonical image of X in $B(H)/\mathfrak{J}$.
Now,

$$\begin{aligned}
\|\Delta_{[A]}([X])\|_{B(H)/\mathfrak{J}} &= \|[A][X] - [X][A]\|_{B(H)/\mathfrak{J}} \\
&= \|([A]-\lambda I)[X] - [X]([A]-\lambda I)\|_{B(H/\mathfrak{J}}, \quad \lambda \in \mathbb{C} \\
&\leq \|([A]-\lambda I)[X]\|_{B(H)/\mathfrak{J}} + \|[X]([A]-\lambda I)\|_{B(H)/\mathfrak{J}} \\
&\leq \|[A]-\lambda I\|\|[X]\|_{B(H)/\mathfrak{J}} + \|[X]\|_{B(H)/\mathfrak{J}}\|[A]-\lambda I\| \\
&= 2\|[A]-\lambda I\|\|[X]\|_{B(H)/\mathfrak{J}}.
\end{aligned}$$

This implies that,

$$\left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} \;\leq\; 2\,\|[A]-\lambda I\|\|[X]\|_{B(H)/\mathfrak{J}}\,.$$

Thus, by definition of $\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\|$, it follows that

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \leq 2d([A]), \tag{3.2.3}$$

where $d([A]) = \inf_{\lambda\in\mathbb{C}} \|[A]-\lambda I\|$.

We next establish the relationship between $d([A])$ and $d(A)$.

By definition;

$$\begin{aligned} d([A]) &= \inf\left\{\|[A]-\lambda I\| : \lambda\in\mathbb{C}\right\} \\ &= \inf\left\{\|A+\mathfrak{J}-\lambda I\| : \lambda\in\mathbb{C}\right\} \\ &= \inf\left\{\|A-\lambda I+\mathfrak{J}\| : \lambda\in\mathbb{C}\right\}. \end{aligned}$$

But since the map $B(H) \longrightarrow B(H)/\mathfrak{J}$ is continuous and

$$\|A+\mathfrak{J}\| \;\leq\; \|A\|,$$

it follows from above that

$$\begin{aligned} d([A]) &= \inf\left\{\|A-\lambda I+\mathfrak{J}\| : \lambda\in\mathbb{C}\right\} \\ &\leq \inf\left\{\|A-\lambda I\| : \lambda\in\mathbb{C}\right\} \\ &= d(A). \end{aligned}$$

Thus

$$d([A]) \leq d(A).$$

Hence from equation (3.2.3), it follows that,

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq 2d(A). \tag{3.2.4}$$

□

The following corollary is immediate,

**Corollary 3.2.6.** *For $A \in B(H)$ and $\mathfrak{J}$ a norm ideal in $B(H)$, then*

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \;\leq\; \|\Delta_A|B(H)\|.$$

*Proof.* By Theorem 2.4.4, for any $A \in B(H)$, we have

$$\|\Delta_A|B(H)\| \;=\; 2d(A).$$

It thus follows immediately from Theorem 3.2.5 that

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \;\leq\; \|\Delta_A|B(H)\|.$$

□

**Remark 3.** It is clear from above that Corollary 3.2.4 relates the norm of inner derivation on the algebra and that on the norm ideal, while Corollary 3.2.6 relates the norm on the norm ideal and that on the quotient algebra. But one would naturally ask about the relationship between the norm on the norm ideal and that on the quotient algebra.

We mention here that relating the norm of a derivation on the quotient algebra and on the ideal has remained a difficult problem in the past. Partly, this is due to the fact that ideals being subspaces of the algebra, ranges from the trivial ideal to the whole algebra itself. In the next section, we shall realize that this relationship is possible.

### 3.2.1 Applications to S - universality

In order to examine the extent to which the identity (3.2.1) applies, L. Fialkow [15] introduced the notion of S - universal operators. Further, Fialkow studied the criteria for S - universality for subnormal operatos and posed several questions in this context.

**Definition 3.2.7.** An operator $A \in B(H)$ is S - universal if

$$\|\Delta_A|\mathfrak{J}\| = 2d(A),$$

for each norm ideal $\mathfrak{J}$ in $B(H)$.

In this subsection, we give certain results with respect to this concept of S - universality. The following Lemma, which gave a big breakthrough into this study, provides a clear relationship between the norm of inner derivation on the quotient algebra and that on the norm ideal.

**Lemma 3.2.8.** *If $A \in B(H)$ is S - universal, and $\mathfrak{J}$ a norm ideal in $B(H)$, then,*

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq \|\Delta_A|\mathfrak{J}\|.$$

*Proof.* From equation (3.2.2) and definition 3.2.7 of S - universality, we have respectively

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \;\leq\; 2d(A),$$

and

$$\|\Delta_A|\mathfrak{J}\| \;=\; 2d(A).$$

It therefore follows that

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \;\leq\; 2d(A) = \|\Delta_A|\mathfrak{J}\|\,.$$

Thus,

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \leq \|\Delta_A|\mathfrak{J}\|\,. \tag{3.2.5}$$

□

**Remark 4.** From the above Lemma 3.2.8, the following question seems natural; When does equality hold in equation (3.2.5)? We satisfactorily provide an answer to this question in Theorem 3.2.10. We shall first state the following proposition by Agure [1].

**Proposition 3.2.9.** *([1]) Let $\mathfrak{A}$ be a $C^*$ - algebra. Then there is a primitive ideal $\mathfrak{J}$ such that $\|[AX - XA]\|_{B(H)/\mathfrak{J}} > \|AX - XA\| - \epsilon,\ \ \epsilon > 0$.*

**Theorem 3.2.10.** *Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space $H$, $\mathfrak{J}$ be a primitive norm ideal in $B(H)$. Then for an $S$ - universal operator $A \in B(H)$,*

$$\left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| \;=\; \|\Delta_A|\mathfrak{J}\|\,,$$

*where* $[A]$ *is the canonical image of* $A$ *in* $B(H)/\mathfrak{J}$.

*Proof.* It is clear by Lemma 3.2.8 that

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \leq \|\Delta_A|\mathfrak{J}\|.$$

We therefore need to establish the reverse inequality. It suffices to show that

$$\|\Delta_{[A]}|B(H)/\mathfrak{J}\| \geq \|\Delta_A|\mathfrak{J}\|.$$

By definition

$$\|\Delta_A|\mathfrak{J}\| = \sup\{\|\Delta_A(X)\| : X \in \mathfrak{J}, \quad \|X\| = 1\}, \tag{3.2.6}$$

which implies that

$$\|\Delta_A|\mathfrak{J}\| \geq \|\Delta_A(X)\|, \; X \in \mathfrak{J}, \; \|X\| = 1.$$

For any $\epsilon > 0$, we can find $X \in \mathfrak{J}$ with $\|X\| = 1$ such that,

$$\|\Delta_A|\mathfrak{J}\| < \|\Delta_A(X)\| + \epsilon,$$

so that

$$\|\Delta_A(X)\| > \|\Delta_A|\mathfrak{J}\| - \epsilon.$$

Also, it follows from Proposition 3.2.9 that for all primitive ideals $\mathfrak{J}$ in a

C*-algebra, $B(H)$, there exists $\epsilon > 0$ such that

$$\|[AX - XA]\|_{B(H)/\mathfrak{J}} \;>\; \|AX - XA\| - \epsilon.$$

This implies that

$$\|\Delta_A(X)\| \;<\; \|[AX - XA]\|_{B(H)/\mathfrak{J}} + \epsilon.$$

Therefore

$$\begin{aligned}
\|\Delta_A|\mathfrak{J}\| - \epsilon &< \|\Delta_A(X)\| \\
&< \|[AX - XA]\|_{B(H)/\mathfrak{J}} + \epsilon \\
&= \|[A][X] - [X][A]\|_{B(H)/\mathfrak{J}} + \epsilon \\
&= \left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} + \epsilon.
\end{aligned}$$

Thus

$$\|\Delta_A|\mathfrak{J}\| \;\leq\; \left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} + 2\epsilon.$$

But by definition,

$$\left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} \;\leq\; \left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\|.$$

So that,

$$\begin{aligned}
\|\Delta_A|\mathfrak{J}\| &\leq \left\|\Delta_{[A]}([X])\right\|_{B(H)/\mathfrak{J}} + 2\epsilon \\
&\leq \left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\| + 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\|\Delta_A|\mathfrak{J}\| \leq \left\|\Delta_{[A]}|B(H)/\mathfrak{J}\right\|. \tag{3.2.7}$$

Considering equations (3.2.5) and (3.2.7), we obtain our result. □

## 3.3 Hyponormal operators

In the previous section 3.2, we studied inner derivations on norm ideals and realized that $\|\Delta_A|B(H)/\mathfrak{J}\| \leq 2d(A)$ for any $A \in B(H)$, see Theorem 3.2.5.

In this section, we are going to be interested in inner derivations implemented by normal and hyponormal operators on norm ideals. For definitions of normal operators, hyponormal operators, compact operators, S - universal operators and sequence of singular values, refer definitions 1.2.16, 1.2.17, 1.2.18 and 3.2.7. We shall then define the schatten $p$ - ideal $C_p(H)$, $1 \leq p \leq \infty$, introduced by R. Schatten [35], which, see Proposition 3.3.2, form a class of norm ideals.

**Definition 3.3.1.** The space $C_p(H)$ is a class of compact operators $X$ such that $\sum_j S_j^p(X) < \infty$, where $\{S_j(X)\}_j$ denotes the sequence of singular values of $X$. For $X \in C_p(H)$ $(1 \leq p \leq \infty)$, we set $\|X\|_p = (\sum_j S_j^p(X))^{\frac{1}{p}}$, where, by convention, $\|X\|_\infty = S_1(X)$ is the usual operator norm of $X$.

The following propositions give significant characterization of this class of operators, $C_p(H)$, [35].

**Proposition 3.3.2.** *Let $C_p(H)$ denote the schatten $p$ - ideal. Then $(C_p(H), \|.\|_p)$ is a norm ideal.*

**Proposition 3.3.3.** *Let $C_p(H)$ denote the schatten $p$ - ideal. Then $(C_2(H), \|.\|_2)$ is a Hilbert space with inner product defined by $\langle X, Y\rangle = tr(XY^*)$, $(X, Y \in C_2(H))$ where tr stands for the usual trace functional and $Y^*$ denotes the adjoint of $Y$.*

We shall write $\Delta_A|C_2$ instead of $\Delta_A|C_2(H)$ to denote inner derivation on $C_2(H)$. In fact in this section we will be particularly interested in the norm ideal $(C_2, \|.\|_2)$.
Before giving the main result of this section (see Theorem 3.3.7), we shall first state the following results from literature on hyponormal and normal operators.

**Theorem 3.3.4.** *([7]) An hyponormal operator $A \in B(H)$ is $S$ - universal if and only if $diam(\sigma(A)) = 2R_A$ where $R_A$ is the radius of the smallest disc containing the spectrum.*

In establishing the above result in [7], the authors used the following theorem due to B.Sz. Nagy and C. Foias [30];

**Theorem 3.3.5.** *([30]) For every hyponormal operator $A$ on a Hilbert space $H$ there exists a normal operator $N$ and a unitary operator $U$ on some Hilbert space $K$, and a contraction $R$ of $H$ into $K$, such that;*

**(a)** $A = R^*NR$

**(b)** $\|N\| = \|A\|$

**(c)** $NU = UN = N^*$

**(d)** *$\|R^*Ug\| \leq \|R^*g\|$ for all $g \in K$*

**(e)** *The manifolds $L_n = U^n RH$ (n= 0, 1, 2,...) form a non- decreasing sequence and span $K$*

**(f)** *For any complex scalars $\alpha, \beta$,*

$$\sigma(\alpha N + \beta N^*) \subseteq \sigma_l(\alpha A + \beta A^*)$$

*($\sigma_l$; left spectrum).*

As a consequence to Theorem 3.3.5 above, Barraa and Boumazgour in [7] established the following Corollary,

**Corollary 3.3.6.** *([7]) Let $A$ be a hyponormal operator and let $N$ be a normal operator given by Theorem 3.3.5. Then $d(A) = \|A\|$ if and only if $d(N) = \|N\|$.*

In concluding their paper [7] with a remark, an open question was posed as to whether the equality

$$\|\Delta_N|C_2\| \quad = \quad \|\Delta_A|C_2\|$$

holds true, where $A$, $N$ are arbitrary hyponormal and normal operators defined as in Theorem 3.3.5 above.
Our result in this section is the following,

**Theorem 3.3.7.** *Let $A$ and $N$ be defined as in Theorem 3.3.5 above. If $A$ is $S$ - universal, then*

$$\|\Delta_N|C_2\| \quad = \quad \|\Delta_A|C_2\| \, .$$

*Proof.* Since $A$ is S - universal, then by definition 3.2.7 of S -universal operators, assertion (b) of Theorem 3.3.5 and Corollary 3.3.6, it follows that

$$\|\Delta_A|C_2\| = \|\Delta_A|B(H)\| = 2d(A) = 2\|A\| = 2\|N\| = 2d(N) = \|\Delta_N|B(K)\|.$$

Then we have that

$$\begin{aligned} \|\Delta_A|C_2\| = 2\|A\| &\iff \|\Delta_A|C_2\| = \sup\{\|\Delta_A(X)\| : X \in C_2,\ \|X\| = 1\} = 2\|A\| \\ &\iff \exists\{X_n\} \in C_2(H) \text{ with } \|X_n\| = 1 \text{ such that} \\ &\qquad \|AX_n - X_nA\|_2 \longrightarrow 2\|A\| \text{ as } n \longrightarrow \infty. \end{aligned}$$

Since

$$\begin{aligned} \|AX_n - X_nA\|_2 &\leq \|AX_n\|_2 + \|X_nA\|_2 \\ &\leq \|A\| + \|X_n\|_2\|A\| \\ &= \|A\| + \|A\| \\ &= 2\,\|A\|, \end{aligned}$$

we deduce that

$$\|AX_n\|_2 \longrightarrow \|A\|,$$

and

$$\|X_nA\|_2 \longrightarrow \|A\|, \text{ as } n \longrightarrow \infty.$$

Now, from the identity;

$$\|AX_n - X_nA\|_2^2 \;=\; \|AX_n\|_2^2 + \|X_nA\|_2^2 - 2\Re(\langle AX_n, X_nA\rangle),$$

we conclude that $-\Re(\langle AX_n, X_nA\rangle) \longrightarrow \|A\|^2$ as $n \longrightarrow \infty$, where $\Re$ denotes the real part.

But for every sequence $\{X_n\} \in C_2(H)$ and $\|X_n\| = 1$, there exists a corresponding sequence $\{RX_nR^*\} \in C_2(K)$ such that $\|RX_nR^*\| \leq 1$. Moreover,

$$\begin{aligned}\langle NRX_nR^*, NRX_nR^*\rangle &= tr((NRX_nR^*)(RX_nR^*N)^*)\\ &= \langle AX_n, X_nA\rangle.\end{aligned}$$

Hence, $\Re(\langle NRX_nR^*, RX_nR^*N\rangle) = \Re(\langle AX_n, X_nA\rangle) \longrightarrow -\|A\|^2 = -\|N\|^2$, as $n \longrightarrow \infty$.

Also

$$\begin{aligned}|\Re(\langle NRX_nR^*, RX_nR^*N\rangle)| &\leq \|NRX_nR^*\|_2\|RX_nR^*N\|_2\\ &\leq \|N\|^2.\end{aligned}$$

So it follows that $\|NRX_nR^*\|_2 \longrightarrow \|N\|$ and $\|RX_nR^*N\|_2 \longrightarrow \|N\|$, as $n \longrightarrow \infty$.

Therefore

$$\begin{aligned}\|N(RX_nR^*) - (RX_nR^*)N\|_2^2 &= \|NRX_nR^*\|_2^2 + \|RX_nR^*N\|_2^2 \\ &\quad -2\Re(\langle NRX_nR^*, NRX_nR^*\rangle) \\ &\longrightarrow \|N\|^2 + \|N\|^2 - (-2\|N\|^2) \\ &= 4\|N\|^2 \text{ as } n \longrightarrow \infty.\end{aligned}$$

In other words,

$$\|\Delta_N(RX_nR^*)\|_2^2 \longrightarrow 4\|N\|^2 \ as\ n \longrightarrow \infty.$$

Implying that

$$\|\Delta_N(RX_nR^*)\|_2 \longrightarrow 2\|N\| \ as\ n \longrightarrow \infty. \tag{3.3.1}$$

Since by definition, $\|\Delta_N|C_2\| \geq \|\Delta_N(RX_nR^*)\|$, it follows that $\|\Delta_N|C_2\| \geq 2\|N\|$. But it is clear that $\|\Delta_N|C_2\| \leq 2\|N\|$, so that $\|\Delta_N|C_2\| = 2\|N\|$. Thus by taking into consideration our assumptions at the start of this proof, it immediately follows that $\|\Delta_N|C_2\| = \|\Delta_A|C_2\|$ which completes this proof. □

**Remark 5.** Theorem 3.3.7 answers partly the question[2] posed in [7].

[2]The question posed in [7] still remains open for investigation

## 3.4 Inner derivations and the numerical range

This section deals with inner derivation and the numerical range. See definition 1.2.23 for the numerical range and the spectrum.

In order to state our results in this section in detail, we shall first recall some notations and results from literature. For $A \in B(H)$, we denote the spectrum by $\sigma(A)$, the approximate point spectrum by $\sigma_{ap}(A)$, the spectral radius by $r(A)$, the numerical range by $W(A)$, and the numerical radius by $\omega(A)$, see definition 1.2.23.

The relationships between the spectra and the numerical range was widely studied by P. R. Halmos [20] in his Hilbert space problem book. For instance, we give the following results;

**Theorem 3.4.1.** *([20], problem 78) The boundary of the spectrum of an operator is included in the approximate point spectrum.*

The following is an immediate corollary of the above Theorem 3.4.1,

**Corollary 3.4.2.** *For any operator $A$, $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \sigma_{ap}(A)$.*

Further, Halmos mentioned the following theorem which is generally referred to as the spectral inclusion theorem,

**Theorem 3.4.3.** *([18], Spectral inclusion theorem) $\overline{W(A)}$ is a compact convex subset of the plane and $\sigma(A) \subseteq \overline{W(A)}$ , where the bar denotes the closure.*

As an immediate consequence, we have the following,

**Corollary 3.4.4.** *For any operator $A$, $\|A\|$ lies in $\overline{W(A)}$ if and only if $\|A\|$ lies in $\sigma_{ap}(A)$.*

The numerical range and the spectrum of inner derivations on norm ideals has been studied by several authors, see for instance [15] or [37], and several results obtained. We mention some of the results that shall be vital in giving the results of this study.

In [37], S. Shaw considered inner derivations acting on subspaces which satisfy axioms like those of norm ideals. In particular, he proved the following proposition,

**Proposition 3.4.5.** *([37]) Let $A \in B(H)$, then $\overline{W(\Delta_A|C_2)} = \overline{W(A)} - \overline{W(A)}$.*

We note that $\overline{W(A)} - \overline{W(A)} = \{\alpha - \beta : \alpha,\, \beta \in \overline{W(A)}\}$.

Proposition 3.4.5 formed the numerical range analogue of Fialkow's [15] formula for spectra which we state in the following proposition,

**Proposition 3.4.6.** *([15]) For an operator $A \in B(H)$, $\sigma(\Delta_A|C_2) = \sigma(A) - \sigma(A)$.*

We also note that $\sigma(A) - \sigma(A) = \{\alpha - \beta : \alpha,\, \beta \in \sigma(A)\}$.

Fialkow's work [15] followed from the work of A. Brown and C. Pearcy [6] who studied the multiplication operators, $L_A$ and $R_A$ (see definition 1.2.24) in detail. Specifically they established the following Theorem,

**Theorem 3.4.7.** *([6]) For $A \in B(H)$, $\sigma(L_A) = \sigma(R_A) = \sigma(A)$ and that*

$$\|L_A|C_2\| = \|A\| = \|R_A|C_2\|. \tag{3.4.1}$$

The following is a corollary to Theorem 3.4.7 above,

**Corollary 3.4.8.**

$$\sigma(-L_{A^*}|C_2 R_A|C_2) = -\sigma(A^*)\sigma(A) \textit{ for any } A \in B(H). \tag{3.4.2}$$

In this section, we are going to be interested in the relationship between the diameter of the numerical range and the norm of inner derivation on norm ideals.
We begin by establishing the inequality between the diameter of the numerical range and the norm of inner derivation implemented by $A \in B(H)$ on norm ideal $\mathfrak{J}$ of $B(H)$, see Theorem 3.4.10. The same result was first established by L. Fialkow [15]. We shall first state the following Lemma,

**Lemma 3.4.9.** *Let $A \in B(H)$ be non - zero and $\mathfrak{J}$ be an ideal in $B(H)$. If $B \in \mathfrak{J}$ with $Bx_n = y_n$ and $x_n, y_n \in H$ such that $\|y_n\| = \|x_n\| = 1\ \forall n$, then $B$ is unitary.*

*Proof.*

$$\begin{aligned}
\langle B^*Bx_n, x_n\rangle &= \langle Bx_n, Bx_n\rangle \\
&= \|Bx_n\|^2 \\
&= \|y_n\|^2 \\
&= \|x_n\|^2 \\
&= \langle x_n, x_n\rangle \\
&= \langle Ix_n, x_n\rangle.
\end{aligned}$$

Thus

$$B^*B = I.$$

Similarly, it is easy to show that $BB^* = I$ which completes the proof. □

**Theorem 3.4.10.** *For any operator $A \in B(H)$ and each norm ideal $\mathfrak{J}$ in $B(H)$, diam$(W(A)) \leq \|\Delta_A|\mathfrak{J}\|$.*

*Proof.* By definition,

$$\begin{aligned}\|\Delta_A|\mathfrak{J}\| &= \sup\{\|\Delta_A(B)\| : B \in \mathfrak{J},\ \|B\| = 1\} \\ &= \sup\{\|AB - BA\| : B \in \mathfrak{J},\ \|B\| = 1\}.\end{aligned}$$

It therefore follows that

$$\|\Delta_A|\mathfrak{J}\| \geq \|AB - BA\|, B \in \mathfrak{J},\ \|B\| = 1.$$

Let us consider $\|AB - BA\|$,
So $\exists\ \{x_n\} \in H$ with $\|x_n\| = 1\ \forall\ n$ such that
$\|AB - BA\| \geq \|ABx_n - BAx_n\| \geq \|ABx_n\| - \|BAx_n\|$.
Thus

$$\|\Delta_A|\mathfrak{J}\| \geq \|ABx_n\| - \|BAx_n\|. \tag{3.4.3}$$

But

$$\begin{aligned}|\langle ABx_n, Bx_n\rangle| &\leq \|ABx_n\|\|Bx_n\| \\ &\leq \|ABx_n\|\|B\|\|x_n\| \\ &= \|ABx_n\|.\end{aligned}$$

Similarly,

$$|\langle BAx_n, Bx_n\rangle| \leq \|BAx_n\|.$$

So that equation (3.4.3) becomes;

$$\|\Delta_A|\mathfrak{J}\| \geq \|ABX_n\| - \|BAX_n\| \geq |\langle ABx_n, Bx_n\rangle| - |\langle BAx_n, Bx_n\rangle|.$$

By letting $Bx_n = y_n$ with $\|y_n\| = \|x_n\| = 1$, we have by Lemma 3.4.9 that

$$\langle ABx_n, Bx_n\rangle = \langle Ay_n, y_n\rangle,$$

and

$$\begin{aligned}\langle BAx_n, Bx_n\rangle &= \langle Ax_n, B^*Bx_n\rangle \\ &= \langle Ax_n, x_n\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Delta_A|\mathfrak{J}\| &\geq |\langle Ay_n, y_n\rangle - \langle Ax_n, x_n\rangle| \\ &= |\alpha - \beta|; \alpha, \beta \in W(A).\end{aligned}$$

This implies that

$$\|\Delta_A|\mathfrak{J}\| \geq \sup\{|\alpha - \beta| : \alpha, \beta \in W(A)\}.$$

Hence

$$\|\Delta_A|\mathfrak{J}\| \geq \text{diam}(W(A)). \tag{3.4.4}$$

□

**Remark 6.** Theorem 3.4.10 above lead us to the following question; When does equality hold in equation (3.4.4) above?

### 3.4.1 Application to S - universality

We begin this subsection by stating the following Theorem by Barraa and Boumazgour [8],

**Theorem 3.4.11.** *([8]) Let $A$, $B \in B(H)$ be non - zero. Then the equation $\|A+B\| = \|A\| + \|B\|$ holds if and only if $\|A\|\|B\| \in \overline{W(A^*B)}$*

We shall satisfactorily provide an answer to the question in Remark 6 above in Theorem 3.4.13. This will form the major result in this section. To do this, we need to first establish the condition when the diam$(W(A))$ attains its optimal value, $2\|A\|$, which we immediately establish in the following Theorem,

**Theorem 3.4.12.** *Let $A \in B(H)$ be S - universal. Then*

$$diam(W(A)) = 2\|A\|.$$

*Proof.* Since $A$ is S - universal (see definition 3.2.7), we have

$$\|\Delta_A|C_2\| = \|\Delta_A|B(H)\|.$$

By Stampfli [40], for any $A \in B(H)$,

$$\|\Delta_A|B(H)\| = 2\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\|.$$

By a compactness argument, considering that $\Delta_A|C_2$ is compact, $\exists\ \mu \in \mathbb{C}$ such that

$$\inf_{\lambda\in\mathbb{C}}\|A-\lambda I\| = \|A-\mu I\|.$$

Hence

$$\|\Delta_A|C_2\| = 2\|A-\mu I\|.$$

Since

$$\Delta_A|C_2 = \Delta_{A-\mu I}|C_2 = L_{A-\mu I}|C_2 - R_{A-\mu I}|C_2,$$

it follows that,

$$\|L_{A-\mu I}|C_2 - R_{A-\mu I}|C_2\| = 2\|A-\mu I\|.$$

On the other hand, by Theorem 3.4.7 we have;

$$\|L_{A-\mu I}|C_2\| = \|A-\mu I\|$$

and

$$\|R_{A-\mu I}|C_2\| = \|A-\mu I\|.$$

Hence,

$$\|L_{A-\mu I}|C_2 - R_{A-\mu I}|C_2\| \quad = \quad \|L_{A-\mu I}|C_2\| + \|R_{A-\mu I}|C_2\|.$$

Without loss of generality, we may assume that $\mu = 0$ and then

$$\|L_A|C_2 - R_A|C_2\| \quad = \quad \|L_A|C_2\| + \|R_A|C_2\|.$$

Then by Theorem 3.4.11 due to Barraa and Boumazgour, this is equivalent to

$$\|L_A|C_2\|\|R_A|C_2\| \in \overline{W(-L_{A^*}|C_2R_A|C_2)}.$$

As remarked in the introduction this implies that

$$\|L_A|C_2\|\|R_A|C_2\| \in \sigma(-L_{A^*}|C_2R_A|C_2).$$

But from equations (3.4.1) and (3.4.2),

$$\sigma(-L_{A^*}|C_2R_A|C_2) \quad = \quad -\sigma(A^*)\sigma(A),$$

and

$$\|A\|^2 = \|L_{A^*}|C_2\|\|R_A|C_2\|.$$

So $\exists\ \alpha, \beta \in \sigma(A)$ such that $\|A\|^2 = -\overline{\alpha}\beta$.

Since $|\alpha| \leq \|A\|$ and $|\beta| \leq \|A\|$, one can find $\theta \in \mathbb{R}$ such that

$\alpha = \|A\|e^{i\theta}$ and

$\beta = -\|A\|e^{i\theta}$. Since

$$\sigma(\Delta_A|C_2) \quad = \quad \sigma(A) - \sigma(A),$$

it follows that

$$r(\Delta_A|C_2) \quad = \quad \mathrm{diam}(\sigma(A)).$$

So

$$r(\Delta_A|C_2) = \mathrm{diam}(\sigma(A)) \geq |\alpha - \beta| = 2\|A\|.$$

By the spectral inclusion theorem; $\sigma(A) \subseteq \overline{W(A)}$,

$$\therefore \quad \mathrm{diam}(\sigma(A)) \quad \leq \quad \mathrm{diam}(W(A)).$$

Hence,

$$\mathrm{diam}(W(A)) \geq \mathrm{diam}(\sigma(A)) \geq 2\|A\|.$$

That is

$$\mathrm{diam}(W(A)) \geq 2\|A\|. \tag{3.4.5}$$

Conversely, we need to establish that

$$\mathrm{diam}(W(A)) \quad \leq \quad 2\|A\|.$$

So by definition

$$\mathrm{diam}(W(A)) \quad = \quad \sup\left\{|\alpha - \beta| : \alpha, \beta \in W(A)\right\}.$$

This implies that; $\exists\ x, y \in H$ with $\|x\| = \|y\| = 1$ such that;
$\alpha = \langle Ax, x\rangle$ and $\beta = \langle Ay, y\rangle$.
$\therefore\ \ \alpha - \beta = \langle Ax, x\rangle - \langle Ay, y\rangle$.

So that

$$|\alpha - \beta| = |\langle Ax, x\rangle - \langle Ay, y\rangle|.$$

But

$$\begin{aligned}
|\langle Ax, x\rangle - \langle Ay, y\rangle| &\leq |\langle Ax, x\rangle| + |\langle Ay, y\rangle| \\
&\leq \|Ax\|\|x\| + \|Ay\|\|y\| \\
&\leq \|A\|\|x\|\|x\| + \|A\|\|y\|\|y\| \\
&= \|A\|\|x\|^2 + \|A\|\|y\|^2 \\
&= \|A\| + \|A\| \\
&= 2\|A\|.
\end{aligned}$$

Hence,

$$\sup\{|\langle Ax, x\rangle - \langle Ay, y\rangle| : x, y \in H,\ \|x\| = \|y\| = 1\} \leq 2\|A\|.$$

Thus

$$\mathrm{diam}(W(A)) \leq 2\|A\|. \tag{3.4.6}$$

Now, from equations (3.4.6) and (3.4.5), we obtain our result. □

We now proceed to answer the question in Remark 6 in the following result,

**Theorem 3.4.13.** *Let $A \in B(H)$ be $S$ - universal and $\mathfrak{J}$ a norm ideal in $B(H)$. Then*

$$\mathit{diam}(W(A)) = \|\Delta_A|\mathfrak{J}\|.$$

*Proof.* From Theorem 3.4.10, for any $A \in B(H)$

$$\operatorname{diam}(W(A)) \leq \|\Delta_A|\mathfrak{J}\|\,. \tag{3.4.7}$$

We therefore need to establish the reverse inequality, that is,

$$\operatorname{diam}(W(A)) \quad \geq \quad \|\Delta_A|\mathfrak{J}\|\,.$$

Since $A$ is S - universal, then by Theorem 3.4.12, we have

$$\operatorname{diam}(W(A)) = 2\|A\|,$$

and by definition of S - universal operator,

$$\|\Delta_A|B(H)\| \quad = \quad \|\Delta_A|\mathfrak{J}\|.$$

But by Lemma 2.2.4, since $B(H)$ is a normed algebra, then $\|\Delta_A|B(H)\| \leq 2\|A\|$. It therefore follows that,

$$\|\Delta_A|\mathfrak{J}\| = \|\Delta_A|B(H)\| \leq 2\|A\| = \operatorname{diam}(W(A)).$$

Hence

$$\operatorname{diam}(W(A)) \geq \|\Delta_A|\mathfrak{J}\|. \tag{3.4.8}$$

Now, from equations (3.4.7) and (3.4.8), we obtain our result. □

The following Corollaries are immediate,

**Corollary 3.4.14.** *If $A \in B(H)$ is $S$ - universal, then*

$$diam(W(A)) \;=\; \|\Delta_A|B(H)\|.$$

*Proof.* This follows immediately from Theorem 3.4.13 and the definition 3.2.7 of an S - universal operator. □

**Corollary 3.4.15.** *If $A \in B(H)$ is $S$ - universal, then $\|\Delta_A|B(H)\| = \|\Delta_A|\mathfrak{J}\| = 2\|A\|$.*

*Proof.* The proof of this Corollary follows immediately from Theorem 3.4.12, Theorem 3.4.13 and the Corollary 3.4.14 above. □

The next result which is the last in this series and even in this thesis, considers the Hilbert - Schmidt class $C_2(H)$ and establishes the necessary and sufficient condition for an operator $A \in B(H)$ to be S - universal. Before giving the next result, we state the following results by Barraa and Boumazgour [8],

**Theorem 3.4.16.** *([8]) Let $A \in B(H)$ be non - zero. Then $\|\Delta_A|C_2\| = \|\Delta_A|B(H)\|$ if and only if $r(\Delta_A|C_2) = \|\Delta_A|B(H)\|$*

**Corollary 3.4.17.** *([8]) For $A \in B(H)$, the following are equivalent:*

1. *$A$ is $S$ - universal*
2. *$diam(W(A)) = 2\inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$*
3. *$diam(\sigma(A)) = 2\inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$*

Our result states as follows;

**Theorem 3.4.18.** *Let $A \in B(H)$ be non- zero. Then $A$ is $S$ - universal if and only if*

$$r(\Delta_A|C_2) = \omega(\Delta_A|C_2).$$

*Proof.* Assume that $A$ is S - universal. Since,

$$\overline{W(\Delta_A|C_2)} = \overline{W(A)} - \overline{W(A)}$$

then it follows that;

$$\omega(\Delta_A|C_2) = \operatorname{diam}(W(A)).$$

Also, since

$$\sigma(\Delta_A|C_2) = \sigma(A) - \sigma(A),$$

it follows that;

$$r(\Delta_A|C_2) = \operatorname{diam}(\sigma(A)).$$

Since $A$ is S - universal, then by corollary 3.4.14,

$$\operatorname{diam}(W(A)) = \|\Delta_A|B(H)\|$$

Also, by the definition of S - universality, we have

$$\|\Delta_A|B(H)\| = \|\Delta_A|C_2\|.$$

Thus;

$$\omega(\Delta_A|C_2) = \text{diam}(W(A)) = \|\Delta_A|B(H)\| = \|\Delta_A|C_2\|. \tag{3.4.9}$$

But by Theorem 3.4.16 due to Barraa and Boumazgour [8], equation (3.4.9) results to (3.4.10) below;

$$\omega(\Delta_A|C_2) = \|\Delta_A|B(H)\| = r(\Delta_A|C_2). \tag{3.4.10}$$

Hence

$$r(\Delta_A|C_2) \quad = \quad \omega(\Delta_A|C_2).$$

Conversely, assume that

$$r(\Delta_A|C_2) \quad = \quad \omega(\Delta_A|C_2).$$

Then since

$$\omega(\Delta_A|C_2) \quad = \quad \text{diam}(W(A))$$

and

$$r(\Delta_A|C_2) \quad = \quad \text{diam}(\sigma(A));$$

it follows that;

$$\text{diam}(W(A)) \quad = \quad \text{diam}(\sigma(A)).$$

Thus, by corollary 3.4.17 above, it follows immediately that $A$ is S - universal. $\square$

We end this work with the following remark,

**Remark 7.** Can the condition established in Theorem 3.4.18 hold if we restrict the derivations on $C_p$, $(1 \leq p \leq \infty)$, that is the Schatten p - class? What about on any norm ideal $\mathfrak{J}$. We predict that if it can hold for any norm ideal $\mathfrak{J}$? then it will hold for the whole algebra, $B(H)$.

# Chapter 4

# SUMMARY AND RECOMMENDATION

## 4.1 Summary

In this chapter, we summarize our work by highlighting the main results based on our objectives of study.

In chapter one, we gave the background information with respect to the study of derivations which enabled us to state the problems with a lot of ease.

In chapter two, we considered basic results on inner derivations and further exhausted the elementary properties of inner derivations. These together with the theory of tensor products highlighted in chapter one enabled us to establish Stampfli's equality for the algebra of bounded linear operators on a Hilbert space.

Chapter three contains the main results of this study on norms inner derivations on norm ideals and the quotient algebra. First, we have extended the inequality (1.5.2) by Barraa and Boumazgour to the quotient algebra. This enabled us to give the relationships between norms of inner derivations restricted to algebra, ideal and the quotient algebra. Further, we introduced the concept of S - universal operators where we realized that for an S - universal operator, the norm of inner derivation on the quotient algebra equals that on the ideal if the ideal is primitive.

We then focussed on hyponormal operators where we partially answered the question by Barraa and Boumazgour as to whether equality hold between the norm of inner derivations implemented by hyponormal and normal operators respectively on the Hilbert - Schmidt class operators. Specifically, we've established that equality hold if the hyponormal operator is S - universal.

Moreover, we have established the inequality between the norm of inner derivation on norm ideals and the diameter of the numerical range, and further realized that equality holds when the operator is S - universal.

Finally, we have provided a necessary and sufficient condition for any non-zero operator to be S - universal, where we have shown that an operator is S - universal if and only if the numerical and spectral radii of inner derivations on Hilbert - Schmidt class are equal.

## 4.2 Recommendation.

From this study, it has clearly emerged that the study of derivations is still an interesting and active area of research in pure mathematics, and still calls for a special attention.

The notion of S - universality was introduced in 1979 by L. Fialkow, but it did not attract the attention of mathematicians until in 2001 when

Barraa and Boumazgour studied some applications of these operators. Even though this study explores reasonable norm properties and several 'beautiful' applications of S - universal operators, a lot is still not known about the structural properties of these operators, for example, the spectra, the essential spectra, the numerical range, the essential numerical range, among others. We therefore invite researchers into the study of structural properties of S - universal operators. Moreover, the conditions for S - universality for any operator can also be investigated.
On the other hand, the Stampfli's equality can be investigated for other algebras other than the ones where it has been established.

# References


[1] Agure, J. O., *Certain Properties of Norms of Derivations*, Far East J. Math. Sci. **5** (1997), 29-39.

[2] Apostol, C. and Zsido, L., *Ideals in $W^*$-algebras and the function $\eta$ of A. Brown and C. Pearcy,* Journ. Math. Pures Appl. **18** (1973), 1151-1170.

[3] Archbold, R. J. and Somerset, D. W. B., *Quasi standard $C^*$ - algebras,* Math. Proc. Camb. Phil. Soc. **107** (1990), 349-360.

[4] Archbold, R. J. and Somerset, D. W. B., *Inner derivations and primal ideals of $C^*$ - algebras II*, Proc. London Math. Soc. **88** (2004), 162-166.

[5] Archbold, R. J., *On the Norm of inner derivation of $C^*$-algebras,* Proc. Amer. Math. Soc. **17** (1966), 273-291.

[6] Brown, A. and Pearcy, C., *Spectra of tensor products of operators,* Proc. Amer. Math. Soc. **17** (1966), 162-166.

[7] Barraa, M. and Boumazgour, M., *Norm of a derivation and Hyponormal operators,* Extracta Mathematicae **16** (2001), 229-233.

[8] Barraa, M. and Boumazgour, M., *Inner derivations and norm equality,* Proc. Amer. Math. Soc. **130** (2001), 471-476.

[9] Barraa, M. and Boumazgour, M., *A lower bound of the Norm of Operator $X \to AXB + BXA$*, Extracta Mathematicae **16** (2001), 223-227.

[10] Barraa, M. and Boumazgour, M., *Norm equality for a basic elementary operator,* J. Math. Anal. Appl. **286** (2003), 359-362.

[11] Batty, C. J. K., *Derivations of tensor products of $C^*$ - algebras,* J. London Math. Soc. **17** (1978), 129-140.

[12] Bonsall, F. F. and Dancun, J., *Complete normed algebras,* Springer-Verlag, New York, (1973).

[13] Boumazgour, M., *Norm inequalities for sums of two basic elementary operators,* J. Math. Anal. Appl. **342** (2008), 386-393.

[14] Elliot, G. A., *On derivations of $AW^*$ - algebras,* Tohoku Math. Journ. **30** (1978), 263-276.

[15] Fialkow, L., *A note on norm ideals and the operator $X \longrightarrow AX - XB$*, Israel. J. Math. **32** (1979), 331-348.

[16] Franka, M. B., *Tensor products of $C^*$-algebras, operator spaces and Hilbert $C^*$-modules,* Math. Comm. **4** (1999), 257-268.

[17] Gajandragadkar, P., *Norm of Derivations of a Von Neumann Algebra,* Trans. Amer. Math. Soc. **170** (1972), 165-170.

[18] Gustafson, K. E. and Rao, D. K. M., *Numerical range: the field of values of linear operators and matrices,* Springer-Verlag, New York, (1997).

[19] Hall, A. A., *On the Norms of inner Derivations,* Phd Thesis, University of New Castle upon Tyne, (1972).

[20] Halmos, P. R., *A Hilbert space problem book,* Van Nostrand, Princeton, (1970).

[21] Johnson, B. E., *Norms of Derivations on $\mathfrak{L}(\mathfrak{X})$,* Pacific Journ. Math. **38** (1971), 465-469.

[22] Kadison, R. V., Lance, E. C. and Ringrose, J. R., *Derivations and Automorphisms of Operator Algebras III,* Math. Journ. Functional Analysis **1** (1967), 204-227.

[23] Kaniuth, E., *Minimal primal ideal spaces and norms of inner derivations of tensor products of $C^*$ - algebras,* Math. Proc. Camb. Phil. Soc. **119** (1996), 297-308.

[24] Kyle, J., *Norms of Derivations,* Journ. London Math. Soc. **16** (1977), 297-312.

[25] Masamichi, T., *Theory of Operator Algebras I,* Springer-Verlag, New York, (1979).

[26] Murphy, J. G., *$C^*$-algebras and Operator Theory,* Academic Press, Oval Road, London, (1990).

[27] Murray, F. J. and Von-Neumman, J., *On rings of operators,* Ann. Math. **37** (1936), 116-229.

[28] Murray, F. J. and Von-Neumman, J., *On rings of operators II,* Trans. Amer. Math. Soc. **41** (1937), 208-248.

[29] Murray, F. J. and Von-Neumman, J., *On rings of operators IV,* Ann. Math. **44** (1943), 716-808.

[30] Nagy, B. and Foias, C., *An application of dilation theory to hyponormal operators,* Acta Sci. Math., **37** (1975), 155-159.

[31] Narici, L. and Beckenstein, E., *Topological vector spaces,* Marcel Dekker Inc., New York, (1985).

[32] Nyamwala, F. O., *Norms of inner derivations and of symmetrised two-sided multiplication operators,* MSc. Thesis, Maseno University, (2005).

[33] Pere, A. and Martin, M., *Local Multipliers of C*-algebra,* Springer-Verlag, Britain, (2003).

[34] Sakai, S., *Derivations of W*-algebras,* Ann. Math. **83** (1966), 273-279.

[35] Schatten, R., *Norm ideals of completely continuous operators,* Springer-Verlag, Berlin, (1960).

[36] Seddik, A., *Rank one operators and norm of elementary operators,* Linear Algebra and its Applications **424** (2007), 177-183.

[37] Shaw, S. Y., *On numerical ranges of generalized derivations and related properties,* J. Austral. Math. Soc. (Series A) **36** (1984), 134-142.

[38] Somerset, D. W. B., *The inner derivations and the primitive ideal space of a C*-algebra,* Journ. Operator Theory **29** (1993), 307-321.

[39] Somerset, D. W. B., *Inner derivations and primal ideals of C*-algebras,* Journ. London Math. Soc. **50** (1994), 568-580.

[40] Stampfli, J. G., *The Norm of a Derivation,* Pacific Math. Journal **33** (1970), 737-747.

[41] Von-Neumman, J., *Zur Algebra der Funktionaloperation und Theorie der normalen Operatoren,* Ann. Math. **102** (1929), 370-427.

[42] Von-Neumman, J., *On rings of operators III,* Ann. Math. **41** (1940), 94-161.

[43] Zsido, L., *The norm of a derivation in a $W^*$ - algebra,* Proc. Amer. Math. Soc. **38** (1973), 147-150.