RELATIVISTIC DYNAMICS BASED ON FINSLER GEOMETRY

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Abstract

Physicists have been ruled by a great passion to identify electromagnetism with gravitation and to relate all other fundamental forces of nature. In order to achieve this, they have attempted to construct various Unified Field Theories that relate gravitation and electromagnetism and other fundamental forces of nature based on an extended metric tensor. Many of these gravitational theories are developed within the framework of Riemann geometry. However, there have also been recent attempts to formulate theories of gravitation within the framework of Finsler geometry. Unlike Riemann geometry in which the metric tensor is a function of spacetime coordinates only, the metric tensor in Finsler geometry is a function of both spacetime coordinates and the corresponding velocities normally referred to as tangent vectors. If the metric-dependance on the velocity is ignored, then Finsler geometry naturally reduces to Riemann geometry. Thus Finsler geometry is a natural extension of Riemann geometry. Unfortunately, none of the current attempts has been able to establish an exact general conservation law and field equations to describe the dynamics and evolution of our universe. In contrast to these attempts, our work establishes a general and exact conservation law, which through Noether theorem i.e., with every distribution of matter and fields, there is always an associated tensor called energy-momentum tensor, yields the desired relativistic field equations for the description of the dynamics and evolution of our universe. In order to develop a relativistic theory of dynamics in a non-inertial reference frame interpreted as a Finsler space where events are specified by both spacetime coordinates and corresponding velocities, we have closely followed the procedure usually applied in

Chapter 1

Introduction

1.1 Introduction

The formulation of a successful relativistic theory of the dynamics of a physical system is based on the nature of the appropriate reference frame, whether inertial or non-inertial. Einstein distinguished between inertial and non-inertial frames by the following illustration [1], which we quote: 'Let now K be an inertial system. Masses which are sufficiently far from each other and from other bodies are then, with respect to K, free from acceleration. We shall also refer these masses to a system of coordinates K', uniformly accelerated with respect to K. Relatively to K' all the masses have equal and parallel acceleration; with respect to K', they behave just as if a gravitational field were present and K' were unaccelerated.'

1.1.1 Inertial reference frame

According to the clear elaboration by Einstein, an inertial reference frame K, is defined as a reference frame in which there is no accelerating force, and is suitable for describing the dynamics of free particles governed by the special theory of relativity, [1, 2-3]. An inertial frame is therefore specified by rectilinear motion with constant velocity $\mathbf{v}$ according to

$$\frac{d\mathbf{v}}{dt} = 0 \quad ; \qquad \Rightarrow d\mathbf{v} = 0 \tag{1.1}$$

where we have used the bold face to represent velocity as a vector. This representation will be used throughout this thesis for vectors and tensors. The system undergoes displacement in space according to

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad ; \qquad \Rightarrow d\mathbf{r} = \mathbf{v}\, dt \tag{1.2}$$

Hence, in an inertial reference frame, only the position in space changes with time, but the velocity does not change such that after a time duration dt, we have

$$d\mathbf{v} = 0 \quad ; \qquad d\mathbf{r} = \mathbf{v}\, dt \tag{1.3}$$

according to equations (1.1) and (1.2). An event and any physical quantity characterizing dynamics in an inertial reference frame is therefore specified by the four-component spacetime coordinate

$$X = (x_0, x^1, x^2, x^3) = (ct,\ \mathbf{r}) \tag{1.4}$$

with

$$x_0 = x^0 = ct \ ; \qquad \mathbf{r} = (x^1, x^2, x^3) \equiv (x, y, z) \tag{1.5}$$

where c is the speed of light. An event interval is specified through

$$dX = (c\,dt, d\mathbf{r}) = (dx^0, dx^1, dx^2, dx^3) = (dx_0, dx^1, dx^2, dx^3) \tag{1.6}$$

The corresponding velocity four-vector follows immediately from equation (1.6) in the form

$$\mathbf{u} = \frac{dX}{dt} = (c, \mathbf{v}) \tag{1.7}$$

which on introducing the proper time τ according to

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} \ ; \qquad \frac{d\tau}{dt} = \frac{1}{\gamma} \ ; \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.8}$$

gives the standard relativistic form

$$\mathbf{u} = \frac{1}{\gamma}\frac{dX}{d\tau} = \frac{1}{\gamma}\,\mathbf{u}_\tau = (c, \mathbf{v}) \tag{1.9}$$

so that

$$\mathbf{u}_\tau = \frac{dX}{d\tau} = \gamma\,\mathbf{u} = \gamma(c, \mathbf{v}). \tag{1.10}$$

The distance ds between neighbouring points in an inertial reference frame is defined through the square of the event interval in the form

$$ds^2 = dX^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3 \tag{1.11}$$

where $g_{\mu\nu}$ is the spacetime metric. Expanding equation (1.11) and using equation (1.6) to obtain

$$ds^2 = c^2dt^2 - d\mathbf{r}^2 = c^2dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{1.12}$$

gives the metric for dynamics in an inertial frame in the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.13}$$

The set of equations (1.1)-(1.13) capture the main physical and mathematical properties for formulating relativistic dynamics in an inertial reference frame.

1.1.2 Non-inertial reference frame

We now proceed to develop the basic physical and mathematical features of a non-inertial reference frame. According to the above elaboration by Einstein, K' is a non-inertial reference frame satisfying the principle of equivalence. A particle in the non-inertial reference frame K' undergoes acceleration due to gravity according to

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} \tag{1.14}$$

where $\mathbf{v}$ is the instantaneous velocity defined in equation (1.2), while $\mathbf{g}$ is the gravitational field intensity, generally referred to as acceleration due to gravity. From equations (1.2) and (1.14), we observe that in such a non-inertial reference frame, a particle changes both position in space and velocity as time varies in the form

$$d\mathbf{v} = \mathbf{g}dt \ ; \qquad d\mathbf{r} = \mathbf{v}dt \tag{1.15}$$

These are the basic physical properties characterizing dynamics in a non-inertial reference frame in accordance with Einstein's equivalence principle. They generalize the physical properties summarized in equation (1.3), to which they reduce in an inertial reference frame where $\mathbf{g} = 0$.

An important physical consequence of the properties in equation (1.15) is that events and physical quantities characterizing dynamics in a non-inertial reference frame are to be specified by both spacetime coordinates and corresponding velocity four-vectors defined according to equations (1.4) and (1.7)-(1.10). Hence, a dynamical quantity χ, which describes an event in a non-inertial reference frame, depends on both spacetime coordinate X and corresponding velocity four-vector $\mathbf{u}_\tau = \gamma \ \mathbf{u}$ in the form

$$\chi = \chi(X, \mathbf{u}_\tau) = \chi(X, \dot{X}) \ ; \qquad \dot{X} = \frac{dX}{d\tau} \tag{1.16}$$

This specification of events in a non-inertial frame presented here is more general than the specification of events within the framework of the general theory of relativity [4-6], which ignores the varying velocity associated with the gravitational acceleration. The fundamental proposal of the present thesis is that a relativistic theory of dynamics in a non-inertial reference frame must treat both spacetime position and corresponding velocity four-vector (or equivalently, momentum four-vector) as variables specifying events.

A geometrical framework which admits both spacetime coordinates and corresponding velocity four-vectors, generally referred to as tangent vectors, in specifying the metric tensor, is Finsler geometry. In this respect, we interpret the non-inertial reference frame as defined in this thesis to be a Finsler space.

1.2 Finsler space

A Finsler space [4- 7, 10-11] is a real differentiable manifold M endowed with a non-negative scalar function $F(x, y)$ of two sets of arguments, namely, the points $x \equiv x^{\mu}$ and $y \equiv y^{\mu}$ such that $x^{\mu} \in M$ and $y^{\mu} \in T_x M$. x^{μ} denotes spacetime coordinates, y^{μ} velocity four-vectors also referred to as tangent vectors and $T_x M$ is the tangent space to M at the point x. $\mu = 0, 1, 2, 3$ and so shall be the values of any other greek letter used in this thesis. The function $F(x, y)$ is positively homogeneous of degree one with respect to y^{μ} such that

$$F(x, ky) = kF(x, y) \ ; \qquad y^{\mu} = \frac{1}{c}\frac{dx^{\mu}}{d\tau} \tag{1.17}$$

for fixed $k > 0$. c is the velocity of light and τ is the proper time. If we square $F(x, ky)$ in equation (1.17) and perform partial differentiation (twice) w.r.t k, we obtain (for $k = 1$)

$$F^2(x, y) = \frac{1}{2} y^{\mu} y^{\nu} \frac{\partial^2 F^2(x, y)}{\partial y^{\mu} \partial y^{\nu}} \tag{1.18}$$

from which follows the metric tensor of Finsler geometry $g_{\mu\nu}(x, y)$ defined as

$$g_{\mu\nu}(x, y) = \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^\mu \partial y^\nu}. \tag{1.19}$$

It is clear from the above discussion that the metric tensor specifying Finsler geometry depends on both spacetime coordinates x^μ and corresponding tangent vectors y^μ; it is therefore more general than the metric tensor specifying Riemann geometry, which depends on spacetime coordinates x^μ only. As we are aware, Riemann geometry provides the geometrical framework for Einstein's general theory of relativity [12-16].

In this thesis, we develop a relativistic theory of dynamics in a non-inertial reference frame through application of the mathematical properties of Finsler geometry, following closely the procedure usually applied in formulating the general theory of relativity through tensor analysis within Riemann geometry. Work based on Finsler geometry has attracted a great number of researchers [8- 9, 17- 19]. In particular, similar attempts to formulate a theory of gravitation based on Finsler geometry have been made earlier by other researchers with various degrees of success [8- 9]. In contrast to those attempts, our thesis establishes a general conservation law, which by Noether's theorem, yields the desired general field equations within Finsler space. We observe that the generalized relativistic field equations provide the appropriate theoretical framework for accurate cosmological models of the universe.

1.3 Statement of the problem

For a very long time, physicists have been searching for the correct generalized theory of gravity. The earlier attempts were unsuccessful as they were formulated in the context of general relativity based on a limited theoretical framework, i.e., Riemann geometry. Current attempts to address the same problem have expanded the working theoretical framework to Finsler geometry. However, the models and assumptions employed have made it difficult to obtain an exact and generalized theory of gravity.

It is therefore the goal of this thesis to develop an exact generalized theory that would characterize relativistic dynamics in a non-inertial reference frame based on Finsler geometry.

1.4 Significance of the study

The exact and generalized field theory formulated in this thesis provides the appropriate theoretical framework for accurate cosmological models of the universe. It further provides the starting point for developing theoretical models for physics beyond Einsten's general relativity and the Standard Model in quantum field theory. It may account for observed effects in the universe such as anisotropy, acceleration and expansion of the universe, as well as the Lorentz invariance violation effects, which are the major challenges of theoretical physics at the moment. It is observed that our generalized relativistic field equations governing relativistic dynamics in non-inertial reference frames have the necessary physical and mathematical ingredients to address these problems.

1.5 Aims and objectives of the study

The overall objectives of this study are:

- To develop a relativistic theory of dynamics in a non-inertial reference frame through application of mathematical properties of Finsler geometry.
- To relate the developed field theory to the challenging problems of theoretical physics such as anisotropy, acceleration of the universe and Lorentz invariance violation effects.

Chapter 2

Literature review

The search for an elaborate field theory of gravitation has been a long on-going research. Until the 1910s, Sir Isaac Newton's law of universal gravitation was accepted as the correct and complete theory of gravitation. However, this theory is only accurate in its predictions regarding everyday phenomena for systems in which the velocities are small compared to the speed of light and where gravitational potentials are weak enough so as not to cause large velocities.

In order to understand fully the dynamics of systems whose speed approach the speed of light, a relativistic theory of dynamics is required. This led to the advent of Einstein's special theory of relativity in 1905 [20-23]. With this theory uniting the concepts of space and time into that of four dimensional flat space-time (named Minkowski space-time after the mathematician Hermann Minkowski), a problem became discernible with Newtonian theory. However, even with the advent of special relativity, gravitational effects such as redshift could not be fully explained. In order to account for such effects, there was need to search for a theory of gravitation compatible with the principle of relativity. The basis for the development of such a theory was the idea first conceived by the mathematician Ceorg Bernhard Riemann in 1854 [20-22]. According to Riemann, the crucial ingredient of the gravitational theory is the concept of gravitation, not as a force, but as a manifestation of the curvature of space-time.

After a decade of searching for new concepts, Einstein came up with the theory of general relativity based on Riemann's idea in 1915 [22]. This theory has been regarded as the prototype of all modern gravitational theories. In Einstein's hands, gravitation theory was thus transformed from a theory of forces into the first dynamical theory of geometry, the geometry of four dimensional curved space-time.

The underlying principle behind the general theory of relativity is Einstein's strong equivalence principle i.e., physics is the same in the presence and absence of gravitational fields. Less well tested than the weak Einstein's equivalence principle (the motion of a particle is independent of its internal structure or composition), the strong version requires Newton's constant expressed in atomic units to be the same number everywhere in strong or weak gravitational fields.

Observing that there is very little experimental evidence bearing on this assertion, Dicke and his student Carl Brans proposed in 1961 [20] some modification to the general theory of relativity. In the Brans-Dicke theory, the reciprocal of the gravitational constant is itself a one-component field, the scalar field, that is generated by matter in accordance with an additional equation. Then the scalar field as well as matter play a role in determining the metric via a modified version of Einstein's equations. The Brans-Dicke theory reduces to Newtonian theory for systems with small velocities and weak potentials i.e., it has a Newtonian limit. In fact, Brans-Dicke theory is distinguishable from general relativity only by the value of its single dimensionless parameter which determines the effectiveness of matter in producing the scalar field. The larger the parameter, the closer the Brans-Dicke theory predictions are to general relativity.

However, constancy of this parameter is not conceptually required. In the generic scalar-tensor theory studied by Peter Bergmann, Robert Wagoner, and Kenneth Nordtvedt (BWN theory) [20, 24], this parameter is itself a general function of the scalar field. It remains true that in regions of space-time where this parameter is numerically large, the theory's predictions approach those of general relativity. Initially a popular alternative to General Relativity, the Brans-Dicke theory lost favor as it became clear that the dimensionless parameter must be very large, an artificial requirement in some views.

Bekenstein developed a variable mass theory (VMT) [20] as a special case of the BWN theory devised to test the necessity for the strong equivalence principle, hence the theory is not different from the general theory of relativity.

In 1983, Modified Newtonian Dynamics (MOND) theory [20, 24] was developed. This was a non-metric and non-relativistic gravitational theory based on Newton's gravitational theory. It is not therefore suitable for describing relativistic dynamics in the universe.

It is clear that apart from general theory of relativity, there exist a number of alternative theories of gravitation. They all employ the Riemann geometric model of spacetime borrowed from general relativity, and differ only by the field equations which describe the self-consistent dynamics of spacetime and matter. The cosmological models based on such theories differ accordingly. Common to them, however, is the fact that spacetime being Riemann and, consequently, locally isotropic, preserves its local isotropy during the evolution of the universe.

The preservation of local isotropy of space and Lorentz transformations is now questionable. There are some indications [25] that in our epoch, spacetime, on the average, has a weak relic local anisotropy, and that it therefore should be described by Finsler geometry rather than Riemann geometry. A strong local anisotropy of spacetime must have occurred at an early stage in the evolution of the universe as a result of high temperature phase transitions in its geometric structure, caused by a breaking of higher gauge symmetries and by the appearance of massive elementary particles. If this was the case, it is natural to assume that the local anisotropy of space decreased to its present low level ($< 10^{-10}$) [25] due to the expansion of the universe. The existence of a local anisotropy of spacetime is indicated by an anisotropy of the relic background radiation filling the universe; a breaking of the discrete spacetime symmetries in weak interactions; and the absence of the effect of cutoff of the spectrum of primary ultra-high energy cosmic protons i.e., of the so-called GZK cutoff. The idea of a possible violation of the usual Lorentz transformations at Lorentz factors ($\gamma > 5 \times 10^{10}$), and of a corresponding generalization of the relativistic theories was suggested first in [25]. Its motivation rested on a discrepancy, assumed at the time, between the theoretical predictions and the experimental data relating to the behavior of the spectrum of primary ultra-high energy cosmic protons. If the usual Lorentz transformations would correctly link inertial frames at relative velocities very close to the velocity of light, then, in the case of uniformly distributed sources, the energy spectrum of primary cosmic protons should show a cutoff (due to inelastic collisions of the protons with cosmic background radiation photons) at proton energies $\sim 5 \times 10^{19} eV$. However, as it has now been firmly established, such a prediction is at variance with present experimental data [25].

Other researchers who studied Einstein's gravitational theory include Elie Cartan, hence Einstein-Cartan theory. The motivation behind Cartan's modification of Einstein's general theory of relativity was the notion of torsion. Cartan proposed to relate torsion tensor to the density of intrinsic angular momentum well before the introduction of the modern concept of spin [16]. He seemed to have used the idea of asymmetric affine connection mentioned first by Eddington in 1922 [16]. In this way, Cartan expanded the Riemann spacetime with affine symmetric connection (V_4 theory) to Riemann-Cartan spacetime with affine asymmetric connection (U_4 theory) to accommodate the new geometric property of spacetime, spin tensor. In the Einstein-Cartan theory (sometimes called Riemann-Cartan theory), torsion is associated with the antisymmetric part of an asymmetric connection and is responsible for dislocation of spacetime leading to its quantization. The early universe cosmological problem is associated with elementary particle physics [16]. Elementary particles are not characterized by mass alone but also spin which occurs in units of $\frac{\hbar}{2}$. As a mass distribution in spacetime is described by energy-momentum tensor, so a spin distribution is described in a field theory by a spin density tensor. Similarly, as mass is connected with the curvature of spacetime, so spin should be connected with another geometric property of spacetime i.e., torsion. In the Einstein-Cartan theory, the field equations in empty space are the same so that the majority of its experimental verifiable consequences for the solar system cannot be distinguished from the predictions of general relativity.

Einstein's theory was further studied by Kibble (1961) and Sciama (1962) sometimes referred to as Sciama-Kibble theory [16]. Sciama's approach is that of a gauge theory of the homogeneous Lorentz group starting from a Riemann background i.e., Einstein theory with affine connection. In his approach, Einstein's theory is manifestly gauge invariant under the local Lorentz group. Kibble's approach on the other hand starts from Minkowski space and a passive interpretation of the Poincaire transformation. Sciama-Kibble theory is invariant under local rotations, hence leads to the same geometry i.e., Einstein-Cartan geometry and is therefore within the U_4 theory involving torsion.

The geometrical framework of Einstein-Cartan or U_4 theory provides a rudimentary basis for its extension. If spacetime turned out to be locally anisotropic, we would be forced to consider a richer geometrical framework that would take care of matter directions. This would naturally lead to the modification of the general theory of relativity using Finsler geometry.

Another step towards Finsler geometry from Einstein-Cartan geometry was the urge to formulate a unified field theory. This, upto now, has attracted a number of researchers over a period of over 50 years, dating back to Randers' [26] original suggestion of a unified metric in the form

$$ds = (\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} + A_\mu\frac{dx^\mu}{d\tau})d\tau$$

Although Randers' spaces are Finsler spaces, Randers' metrics are not absolutely homogeneous, hence the metric is strictly Riemann. Therefore, Randers' metrics fit in the context of general relativity.

Work in [26] tried to formulate a unified field theory based on 'free' geodesic connections in Finsler space. The equation of structure based on 'free' geodesic connection is found to incorporate, as a special case, equations which are closely comparable to Einstein's equations of gravity. The method of 'free' geodesic equation is based on Randers' metric, hence is unsuitable for description of dynamics within the framework of Finsler geometry as the metric is Riemann. Furthermore, in the geodesic theory, the geodesic function defines the Berwald connection which is known to be nonmetric compatible.

Other work [27] addressing this subject develops the physical motivation for Finsler structure using gauge transformations on the tangent space. The author's method is based on the fibre bundle and is aimed at the unification of gravitation and electromagnetism. The theory suffers setbacks due to the use of Randers' metric and assumption of surfaces of constant curvature.

Further recent research in this area is the work by Yazaki [8] and Ikeda [9]. Yazaki [8] considers a generalization of Einstein's field equations in a Finsler space using tensor analysis and the assumption of surface of constant curvature. Ikeda [9] on the other hand approaches the same problem using vector bundles approach under the same assumption of surface of constant curvature and ultimately arrives at field equations in Finsler space. Therefore the field equations derived by these two authors are not general enough to describe general features of dynamics in the universe.

In [28-30], a gravitational theory is developed based on vector bundles and differential forms. The formulation of this theory is based on Modified Newtonian Dynamics(MOND), which as mentioned earlier is a non-relativistic gravitational theory. Furthermore, the field equations in this theory are derived through Berwald covariant differentiation, which is non-metric compatible.

Other recent research [31-32,33] obtain Einstein field equations based on Friedmann-Robertson-Walker (FRW) model. In this model, field equations are derived from curvature tensor of Riemann metric and the model is similar to that formulated earlier by Bekenstein. Even though the model is introduced within a Finsler geometrical setting, field equations similar to Einstein's equations of general relativity are derived from a Riemann metric to account for Lorentz violation effects in terms of the cosmological acceleration. Therefore, relativistic dynamics in this model strictly fall within the context of the general theory of relativity.

In [43], Cohen and Glashow developed Very Special Relativity model to account for Lorentz non-invariance effects. Its generalization to include features of Finsler geometry by other authors [44]. The model is strictly based on Einstein's theory of special relativity, which has its own limitations as discussed above.

Further recent works [34-38] consider Einstein's equations of gravity in Finsler geometry. By imposing nonholonomic constraints on generalized metrics, connections and adapted frames, the author generates solutions in Einstein's gravity. However, the work has an underlying assumption of dynamics on surfaces of constant curvature.

The above recent works by various authors present various theoretical models of gravitational theory. However, most of them have the dynamics based on vector bundles approach and make use of assumption of surface of constant curvature. Other models are based on Berwald covariant differentiation that are non-metric compatible. It is therefore clear that none of this work has been successful in the development of an exact theory of gravitation. Our thesis addresses the problem of formulation of a relativistic theory of dynamics in a non-inertial reference frame within the framework of Finsler geometry. Using tensor analysis but without using the assumption of surface of constant curvature, it turns out that the field equations obtained by this method are more general and exact, hence are suitable for describing dynamics in a non-inertial reference frame.

In the next chapter, we give the method that we shall adopt to achieve the desirable results in Chapter 4. In particular, we derive Cartan's curvature tensors and the corresponding Bianchi identity in Finsler space that we shall later need for the derivation of our field equations.

Chapter 3

Methodology

In this chapter, we give an overview of Finsler geometry, followed by the derivation of Cartan curvature tensors. We then proceed to derive the associated Bianchi identity necessary to obtain the desired conservation laws and field equations in a Finsler space.

3.1 Review of Finsler geometry

As already mentioned in the Introduction, a Finsler space [4-7, 10-11] is a real differentiable manifold M, endowed with a non-negative scalar function $F(x, y)$ of two sets of arguments, namely, the points x^μ and y^μ such that $x^\mu \in M$ and $y^\mu \in T_xM$. The function $F(x, y)$ is positively homogeneous of degree one with respect to y^μ as defined in equation (1.17). If we square $F(x, y)$ in equation (1.17) and perform partial differentiation (twice) w.r.t k, we obtain (for $k = 1$) equation (1.18) from which follows the metric tensor of Finsler geometry as defined by equation (1.19).

The metric function of Finsler space $F(x, y)$ can also be written in terms of Cartan torsion tensor $\mathbf{C}_{\mu\nu\alpha}$ as [10]

$$\mathbf{C}_{\mu\nu\alpha} = \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^\mu \partial y^\nu \partial y^\alpha} \tag{3.1}$$

The Cartan torsion tensor satisfies the property

$$\mathbf{C}_{\mu\nu\alpha}\dot{x}^\mu = 0.$$

If $x^\mu = (x^0, x^1, x^2, x^3)$ are the spacetime coordinates and $y^\mu = (y^0, y^1, y^2, y^3)$ are the corresponding tangent vectors, then at each point (x^μ, y^μ), the metric tensor $g_{\alpha\beta}$ is such that

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\mu, y^\mu). \tag{3.2}$$

Clearly, we recognize the tangent vector y^μ as defined by equation (1.17) to be the velocity four-vector

$$\mathbf{u}_\tau = cy^\mu \tag{3.3}$$

according to our definition in equation (1.10). This four-velocity is the instantaneous velocity of a particle or system as defined by equation (1.17). The rate of change of the four-velocity gives us the gravitational field intensity or acceleration for dynamics of a particle in a non-inertial reference frame within the framework of Finsler geometry. In such a non-inertial reference frame, a particle changes both position in space and velocity as time varies according to equation (1.15). Therefore, events and physical quantities characterizing dynamics in a non-inertial reference frame are to be specified by both spacetime coordinates and corresponding velocity four-vectors. Since physics is formulated in a geometric way, we shall use the appropriate language of differential geometry to describe relativistic dynamics in a curved spacetime. So specification of position and four-velocity vector for a particle undergoing dynamics in a non-inertial reference frame is associated with a vector field which is a function of the independent variables $\mathbf{X}(x, y)$. Since the spacetime information is encoded in the Finsler metric $g_{\mu\nu}(x, y)$, throughout this thesis, $g_{\mu\nu}$ is understood to mean $g_{\mu\nu}(x, y)$ with the lower case x and y denoting spacetime coordinates and tangent vectors respectively. We use upper case $\mathbf{X}$ to denote vector field $\mathbf{X}(x, y)$.

3.2 Curvature tensors in Finsler Geometry

Covariant differentiation in Finsler geometry has been developed in several different forms [7,10-11]. We consider only Cartan's form of covariant differentiation and associated curvature tensors in Finsler geometry, since these are relevant to our model of relativistic dynamics in a non-inertial reference frame.

We denote Cartan's covariant derivative operator by ∇_β and its action on a vector field $\mathbf{X}^\mu(x, y)$ is given by

$$\nabla_\beta \mathbf{X}^\mu = \frac{\partial \mathbf{X}^\mu}{\partial x^\beta} - \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha}\frac{\partial G^\alpha}{\partial \dot{x}^\beta} + \Gamma^{*\mu}{}_{\alpha\beta}\mathbf{X}^\alpha \tag{3.4}$$

where the Cartan connections $\Gamma^{*\mu}{}_{\alpha\beta}$ are defined by

$$\Gamma^*_{\alpha\mu\nu} = \gamma_{\alpha\mu\nu} - \mathbf{C}_{\nu\mu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\alpha} - \mathbf{C}_{\alpha\mu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\nu} + \mathbf{C}_{\alpha\nu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\mu}; \tag{3.5}$$

$$\Gamma_{\alpha\mu\nu} = \Gamma^*_{\alpha\mu\nu} + \mathbf{C}_{\alpha\mu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\nu} = \gamma_{\alpha\mu\nu} - \mathbf{C}_{\nu\mu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\alpha} + \mathbf{C}_{\alpha\nu\beta}\frac{\partial G^\beta}{\partial \dot{x}^\mu} \tag{3.6}$$

with

$$\dot{x}^\alpha = \frac{dx^\alpha}{d\tau} = y^\alpha = Fl^\alpha \ ; \qquad G^\alpha = \frac{1}{2}\gamma^\alpha{}_{\rho\beta}y^\rho y^\beta; \qquad \gamma_{\alpha\mu\nu}\dot{x}^\nu\dot{x}^\mu = \frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial x^\alpha}\dot{x}^\nu\dot{x}^\mu \tag{3.7}$$

In the above, l^α is a unit vector in the direction of $\dot{x}^\alpha$ [10].

The partial covariant derivative of a vector field $\mathbf{X}^\mu$ w.r.t $y = \dot{x}$ is given by

$$\triangleright_\beta \mathbf{X}^\mu = \frac{F\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} + \mathbf{A}^\mu{}_{\alpha\beta}\mathbf{X}^\alpha \tag{3.8}$$

$$\mathbf{A}^\mu{}_{\alpha\beta} = F\mathbf{C}^\mu{}_{\alpha\beta} \tag{3.9}$$

Using Cartan's covariant differentiation (equation (3.4)), we obtain commutation relation of the form [10]

$$\nabla_\alpha\nabla_\beta\mathbf{X}^\mu - \nabla_\beta\nabla_\alpha\mathbf{X}^\mu = \mathbf{K}^\mu_{\nu\beta\alpha}\mathbf{X}^\nu - F\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\nu}\mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma \tag{3.10}$$

which together with equation (3.8) gives

$$\begin{aligned}\nabla_\alpha\nabla_\beta\mathbf{X}^\mu - \nabla_\beta\nabla_\alpha\mathbf{X}^\mu &= \mathbf{K}^\mu_{\nu\beta\alpha}\mathbf{X}^\nu - (\triangleright_\nu\mathbf{X}^\mu - \mathbf{A}^\mu{}_{\nu\tau}\mathbf{X}^\tau)\mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma \\ &= \mathbf{K}^\mu_{\nu\beta\alpha}\mathbf{X}^\nu + \mathbf{A}^\mu{}_{\nu\tau}\mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma\mathbf{X}^\tau - \mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma\triangleright_\nu\mathbf{X}^\mu.\end{aligned} \tag{3.11}$$

If we interchange τ and ν in the second term on the R.H.S of equation

(3.11) and noting that $A^{\mu}{}_{\nu\tau}$ is symmetric in the lower two indices, we get

$$\begin{aligned}\nabla_\alpha\nabla_\beta\mathbf{X}^\mu - \nabla_\beta\nabla_\alpha\mathbf{X}^\mu &= \mathbf{K}^\mu_{\nu\beta\alpha}\mathbf{X}^\nu + \mathbf{A}^\mu{}_{\nu\tau}\mathbf{K}^\tau{}_{\gamma\beta\alpha}l^\gamma\mathbf{X}^\nu - \mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma \triangleright_\nu \mathbf{X}^\mu \\ &= \left(\mathbf{K}^\mu_{\nu\beta\alpha} + \mathbf{A}^\mu{}_{\nu\tau}\mathbf{K}^\tau{}_{\gamma\beta\alpha}l^\gamma\right)\mathbf{X}^\nu - \mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma \triangleright_\nu \mathbf{X}^\mu.\end{aligned} \tag{3.12}$$

Making use of equation (3.7) and (3.9) gives

$$\mathbf{A}^\mu{}_{\nu\tau}l^\gamma = \mathbf{C}^\mu_{\nu\tau}\dot{x}^\gamma \tag{3.13}$$

so that upon using equation (3.13) in equation (3.12), we write

$$\nabla_\alpha\nabla_\beta\mathbf{X}^\mu - \nabla_\beta\nabla_\alpha\mathbf{X}^\mu = \mathbf{R}^\mu{}_{\nu\beta\alpha}\mathbf{X}^\nu - \mathbf{K}^\nu_{\gamma\beta\alpha}l^\gamma \triangleright_\nu \mathbf{X}^\mu \tag{3.14}$$

where we identify

$$\mathbf{R}^\mu{}_{\nu\beta\alpha} = \mathbf{K}^\mu_{\nu\beta\alpha} + \mathbf{C}^\mu{}_{\nu\tau}\mathbf{K}^\tau{}_{\gamma\beta\alpha}\dot{x}^\gamma \tag{3.15}$$

to be the first Cartan curvature tensor defined explicitly by [10]

$$\begin{aligned}\mathbf{R}^\mu{}_{\nu\beta\alpha} &= \Big(\frac{\partial\Gamma^{*\mu}{}_{\nu\beta}}{\partial x^\alpha} - \frac{\partial\Gamma^{*\mu}{}_{\nu\beta}}{\partial\dot{x}^\delta}\frac{\partial G^\delta}{\partial\dot{x}^\alpha}\Big) - \Big(\frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial x^\beta} - \frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial\dot{x}^\delta}\frac{\partial G^\delta}{\partial\dot{x}^\beta}\Big) + \\ &\quad \mathbf{C}^\mu{}_{\nu\tau}\Big(\frac{\partial^2 G^\tau}{\partial x^\alpha\partial\dot{x}^\beta} - \frac{\partial^2 G^\tau}{\partial\dot{x}^\alpha\partial x^\beta} - G^\tau{}_{\beta\delta}\frac{\partial G^\delta}{\partial\dot{x}^\alpha} + G^\tau{}_{\alpha\delta}\frac{\partial G^\delta}{\partial\dot{x}^\alpha}\Big) + \\ &\quad \Gamma^{*\mu}{}_{\tau\alpha}\Gamma^{*\tau}{}_{\nu\beta} - \Gamma^{*\mu}{}_{\tau\beta}\Gamma^{*\tau}{}_{\nu\alpha}\end{aligned} \tag{3.16}$$

while

$$\mathbf{K}^{\mu}{}_{\nu\beta\alpha} = (\frac{\partial\Gamma^{*\mu}{}_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{*\mu}{}_{\nu\beta}}{\partial\dot{x}^{\delta}}\frac{\partial\mathbf{G}^{\delta}}{\partial\dot{x}^{\alpha}}) - (\frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial x^{\beta}} - \frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial\dot{x}^{\delta}}\frac{\partial G^{\delta}}{\partial\dot{x}^{\beta}}) + \Gamma^{*\mu}{}_{\tau\alpha}\Gamma^{*\tau}{}_{\nu\beta} - \Gamma^{*\mu}{}_{\tau\beta}\Gamma^{*\tau}{}_{\nu\alpha} \qquad (3.17)$$

Application of the property of the Cartan torsion tensor $\mathbf{C}_{\mu\nu\alpha}\dot{x}^{\mu} = 0$ to equation (3.16) gives the result

$$\mathbf{R}^{\mu}{}_{\nu\beta\alpha}\dot{x}^{\nu} = \mathbf{K}^{\mu}{}_{\nu\beta\alpha}\dot{x}^{\nu}. \qquad (3.18)$$

Let us now derive the second curvature tensor in Finsler geometry. The commutation relation involving equation (3.8) can be written as

$$\begin{aligned} \triangleright_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu} - \triangleright_{\beta}\triangleright_{\alpha}\mathbf{X}^{\mu} &= \frac{F\partial}{\partial\dot{x}^{\alpha}}(\triangleright_{\beta}\mathbf{X}^{\mu}) + \mathbf{A}^{\mu}{}_{\gamma\alpha}(\triangleright_{\beta}\mathbf{X}^{\gamma}) - \frac{F\partial}{\partial\dot{x}^{\beta}}(\triangleright_{\alpha}\mathbf{X}^{\mu}) + \mathbf{A}^{\mu}{}_{\tau\beta}(\triangleright_{\alpha}\mathbf{X}^{\tau}) \\ &= \{F\frac{\partial}{\partial\dot{x}^{\alpha}}(\frac{F\partial\mathbf{X}^{\mu}}{\partial\dot{x}^{\beta}} + \mathbf{A}^{\mu}{}_{\gamma\beta}\mathbf{X}^{\gamma}) + \mathbf{A}^{\mu}{}_{\gamma\alpha}(\frac{F\partial\mathbf{X}^{\gamma}}{\partial\dot{x}^{\beta}} + \mathbf{A}^{\gamma}{}_{\tau\beta}\mathbf{X}^{\tau})\} \\ &\quad -\{F\frac{\partial}{\partial\dot{x}^{\beta}}(F\frac{\partial\mathbf{X}^{\mu}}{\partial\dot{x}^{\alpha}} + \mathbf{A}^{\mu}{}_{\gamma\alpha}\mathbf{X}^{\gamma}) + \mathbf{A}^{\mu}{}_{\tau\beta}(\frac{F\partial\mathbf{X}^{\tau}}{\partial\dot{x}^{\alpha}} + \mathbf{A}^{\tau}{}_{\gamma\alpha}\mathbf{X}^{\gamma})\}. \end{aligned} \qquad (3.19)$$

Equation (3.19) can be expanded to read

$$\begin{aligned} \triangleright_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu} - \triangleright_{\beta}\triangleright_{\alpha}\mathbf{X}^{\mu} &= \{F(\frac{\partial F}{\partial\dot{x}^{\alpha}}\frac{\partial\mathbf{X}^{\mu}}{\partial\dot{x}^{\beta}} + \frac{F\partial^{2}\mathbf{X}^{\mu}}{\partial\dot{x}^{\alpha}\partial\dot{x}^{\beta}} + \frac{\mathbf{X}^{\gamma}\partial\mathbf{A}^{\mu}{}_{\gamma\beta}}{\partial\dot{x}^{\alpha}} + \frac{\mathbf{A}^{\mu}{}_{\gamma\beta}\partial\mathbf{X}^{\gamma}}{\partial\dot{x}^{\alpha}} + \\ &\quad \mathbf{A}^{\mu}{}_{\gamma\alpha}\frac{\partial\mathbf{X}^{\gamma}}{\partial\dot{x}^{\beta}}) + \mathbf{A}^{\mu}{}_{\gamma\alpha}\mathbf{A}^{\gamma}{}_{\tau\beta}\mathbf{X}^{\tau}\} - \\ &\quad \{F(\frac{\partial F}{\partial\dot{x}^{\beta}}\frac{\partial\mathbf{X}^{\mu}}{\partial\dot{x}^{\alpha}} + \frac{F\partial^{2}\mathbf{X}^{\mu}}{\partial\dot{x}^{\beta}\dot{x}^{\alpha}} + \frac{\mathbf{X}^{\gamma}\partial\mathbf{A}^{\mu}{}_{\gamma\alpha}}{\partial\dot{x}^{\beta}} + \frac{\mathbf{A}^{\mu}{}_{\gamma\alpha}\partial\mathbf{X}^{\gamma}}{\partial\dot{x}^{\beta}} + \\ &\quad \mathbf{A}^{\mu}{}_{\tau\beta}\frac{\partial\mathbf{X}^{\tau}}{\partial\dot{x}^{\alpha}}) + \mathbf{A}^{\mu}{}_{\tau\beta}\mathbf{A}^{\tau}{}_{\gamma\alpha}\mathbf{X}^{\gamma}\} \end{aligned} \qquad (3.20)$$

We observe that the second and eighth terms on the R.H.S of equation

(3.20) cancel. If we denote $\frac{\partial F}{\partial \dot{x}^\nu}$ by $F_{\dot{x}^\nu}$, then equation (3.20) can be written as

$$\begin{aligned} \triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu &= F(F_{\dot{x}^\alpha}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} - F_{\dot{x}^\beta}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha}) + F(\frac{\partial \mathbf{A}^\mu{}_{\gamma\beta}}{\partial \dot{x}^\alpha} - \frac{\partial \mathbf{A}^\mu{}_{\gamma\alpha}}{\partial \dot{x}^\beta})\mathbf{X}^\gamma + \\ & F(\mathbf{A}^\mu{}_{\gamma\beta}\frac{\partial \mathbf{X}^\gamma}{\partial \dot{x}^\alpha} - \mathbf{A}^\mu{}_{\gamma\alpha}\frac{\partial \mathbf{X}^\gamma}{\partial \dot{x}^\beta}) + F(\mathbf{A}^\mu{}_{\gamma\alpha}\frac{\partial \mathbf{X}^\gamma}{\partial \dot{x}^\beta} - \mathbf{A}^\mu{}_{\tau\beta}\frac{\partial \mathbf{X}^\tau}{\partial \dot{x}^\alpha}) + \\ & (\mathbf{A}^\mu{}_{\gamma\alpha}\mathbf{A}^\gamma{}_{\tau\beta}\mathbf{X}^\tau - \mathbf{A}^\mu{}_{\tau\beta}\mathbf{A}^\tau{}_{\gamma\alpha}\mathbf{X}^\gamma). \end{aligned} \tag{3.21}$$

We note that the sixth and seventh terms on the R.H.S of equation (3.21) cancel. Furthermore, since γ is a dummy index, we can let $\gamma \to \tau$ in the fifth term so that it cancels with the eighth term on the R.H.S of equation (3.21), reducing this equation to

$$\begin{aligned} \triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu &= F(F_{\dot{x}^\alpha}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} - F_{\dot{x}^\beta}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha}) + F(\frac{\partial \mathbf{A}^\mu{}_{\gamma\beta}}{\partial \dot{x}^\alpha} - \frac{\partial \mathbf{A}^\mu{}_{\gamma\alpha}}{\partial \dot{x}^\beta})\mathbf{X}^\gamma + \\ & (\mathbf{A}^\mu{}_{\gamma\alpha}\mathbf{A}^\gamma{}_{\tau\beta}\mathbf{X}^\tau - \mathbf{A}^\mu{}_{\tau\beta}\mathbf{A}^\tau{}_{\gamma\alpha}\mathbf{X}^\gamma). \end{aligned} \tag{3.22}$$

Let us now consider the fifth and sixth terms on the R.H.S of this equation as follows: let $\tau \to \nu$ in the fifth term, while $\tau \to \gamma$ and $\gamma \to \nu$ in the sixth term. If we now factorize the vector field $\mathbf{X}^\nu$ outside the bracket term, then equation (3.22) becomes

$$\begin{aligned} \triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu &= F(F_{\dot{x}^\alpha}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} - F_{\dot{x}^\beta}\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha}) + F(\frac{\partial \mathbf{A}^\mu{}_{\gamma\beta}}{\partial \dot{x}^\alpha} - \frac{\partial \mathbf{A}^\mu{}_{\gamma\alpha}}{\partial \dot{x}^\beta})\mathbf{X}^\gamma + \\ & (\mathbf{A}^\mu{}_{\gamma\alpha}\mathbf{A}^\gamma{}_{\nu\beta} - \mathbf{A}^\mu{}_{\gamma\beta}\mathbf{A}^\gamma{}_{\nu\alpha})\mathbf{X}^\nu. \end{aligned} \tag{3.23}$$

If we let

$$\mathbf{S}^\mu{}_{\nu\alpha\beta} = (\mathbf{A}^\mu{}_{\gamma\alpha}\mathbf{A}^\gamma{}_{\nu\beta} - \mathbf{A}^\mu{}_{\gamma\beta}\mathbf{A}^\gamma{}_{\nu\alpha}) \tag{3.24}$$

then we can write equation (3.23) as

$$\triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu = F(F_{\dot{x}^\alpha} \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} - F_{\dot{x}^\beta} \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha}) + F(\frac{\partial \mathbf{A}^\mu{}_{\gamma\beta}}{\partial \dot{x}^\alpha} - \frac{\partial \mathbf{A}^\mu{}_{\gamma\alpha}}{\partial \dot{x}^\beta})\mathbf{X}^\gamma + \mathbf{S}^\mu{}_{\nu\alpha\beta}\mathbf{X}^\nu. \tag{3.25}$$

But the third and the fourth terms on the R.H.S of equation (3.25) forms an identity of the form

$$F(\frac{\partial \mathbf{A}^\mu{}_{\gamma\beta}}{\partial \dot{x}^\alpha} - \frac{\partial \mathbf{A}^\mu{}_{\gamma\alpha}}{\partial \dot{x}^\beta}) = F_{\dot{x}^\alpha}\mathbf{A}^\mu{}_{\gamma\beta} - F_{\dot{x}^\beta}\mathbf{A}^\mu{}_{\gamma\alpha} \tag{3.26}$$

so that equation (3.25) can be written as

$$\triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu = F_{\dot{x}^\alpha}(F \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} + \mathbf{A}^\mu{}_{\gamma\beta}\mathbf{X}^\gamma) - F_{\dot{x}^\beta}(F \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\alpha} + \mathbf{A}^\mu{}_{\gamma\alpha}\mathbf{X}^\gamma) + \mathbf{S}^\mu{}_{\nu\alpha\beta}\mathbf{X}^\nu. \tag{3.27}$$

Using equation (3.8) in the above bracketed terms reduces equation (3.27) to

$$\triangleright_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \triangleright_\alpha \mathbf{X}^\mu = F_{\dot{x}^\alpha} \triangleright_\beta \mathbf{X}^\mu - F_{\dot{x}^\beta} \triangleright_\alpha \mathbf{X}^\mu + \mathbf{S}^\mu{}_{\nu\alpha\beta}\mathbf{X}^\nu. \tag{3.28}$$

From this, it follows that $\mathbf{S}^\mu{}_{\nu\alpha\beta}$ defined in equation (3.24) is the second curvature tensor in Finsler geometry.

We obtain the third curvature tensor through mixed commutation relation involving equations (3.4) and (3.8). By definition, we have

$$\begin{aligned} \nabla_\alpha \triangleright_\beta \mathbf{X}^\mu &= \nabla_\alpha(\triangleright_\beta \mathbf{X}^\mu) \\ &= \nabla_\alpha(F \frac{\partial X^\mu}{\partial \dot{x}^\beta} + A^\mu{}_{\beta\gamma}X^\gamma) \\ &= \nabla_\alpha(F \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta}) + \mathbf{X}^\gamma \nabla_\alpha \mathbf{A}^\mu{}_{\beta\gamma} + \mathbf{A}^\mu{}_{\beta\gamma}\nabla_\alpha \mathbf{X}^\gamma \end{aligned} \tag{3.29}$$

and

$$\triangleright_\beta \nabla_\alpha \mathbf{X}^\mu = F\frac{\partial}{\partial \dot{x}^\beta}(\nabla_\alpha \mathbf{X}^\mu) + \mathbf{A}^\mu{}_{\beta\gamma}\nabla_\alpha \mathbf{X}^\gamma - \mathbf{A}^\gamma{}_{\beta\alpha}\nabla_\gamma \mathbf{X}^\mu \tag{3.30}$$

so that using equation (3.29) and equation (3.30), we have

$$\begin{aligned}\nabla_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \nabla_\alpha \mathbf{X}^\mu &= F\big(\nabla_\alpha(\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta}) - \frac{\partial}{\partial \dot{x}^\beta}(\nabla_\alpha \mathbf{X}^\mu)\big) + \\ &\quad X^\gamma \nabla_\alpha \mathbf{A}^\mu{}_{\beta\gamma} - \mathbf{A}^\mu{}_{\beta\gamma}\nabla_\alpha \mathbf{X}^\gamma + \mathbf{A}^\mu{}_{\beta\gamma}\nabla_\alpha \mathbf{X}^\gamma + \mathbf{A}^\gamma{}_{\beta\alpha}\nabla_\gamma \mathbf{X}^\mu.\end{aligned} \tag{3.31}$$

We note that the fourth and fifth terms on the R.H.S of equation (3.31) cancel, whereas the term in bracket is an identity such that

$$\nabla_\alpha \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\beta} - \frac{\partial}{\partial \dot{x}^\beta}\nabla_\alpha \mathbf{X}^\mu = \frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\delta}\nabla_\gamma(\mathbf{A}^\delta{}_{\beta\alpha})l^\gamma - \frac{\partial \Gamma^{*\mu}{}_{\gamma\alpha}}{\partial \dot{x}^\beta}\mathbf{X}^\gamma. \tag{3.32}$$

Hence, we can rewrite equation (3.31) using equation (3.32) as

$$\nabla_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \nabla_\alpha \mathbf{X}^\mu = F\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\delta}\nabla_\gamma \mathbf{A}^\delta{}_{\beta\alpha}l^\gamma - F\frac{\partial \Gamma^{*\mu}{}_{\gamma\alpha}}{\partial \dot{x}^\beta}\mathbf{X}^\gamma + \mathbf{X}^\gamma \nabla_\alpha \mathbf{A}^\mu{}_{\beta\gamma} + \mathbf{A}^\gamma{}_{\beta\alpha}\nabla_\gamma \mathbf{X}^\mu \tag{3.33}$$

or

$$\nabla_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \nabla_\alpha \mathbf{X}^\mu = F\frac{\partial \mathbf{X}^\mu}{\partial \dot{x}^\delta}\nabla_\gamma \mathbf{A}^\delta{}_{\beta\alpha}l^\gamma - (F\frac{\partial \Gamma^{*\mu}{}_{\gamma\alpha}}{\partial \dot{x}^\beta} - \nabla_\alpha \mathbf{A}^\mu{}_{\beta\gamma})\mathbf{X}^\gamma + \mathbf{A}^\gamma{}_{\beta\alpha}\nabla_\gamma \mathbf{X}^\mu. \tag{3.34}$$

We can rewrite the first term on the R.H.S of equation (3.34) using equation (3.8) so that equation (3.34) becomes

$$\nabla_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu}-\triangleright_{\beta}\nabla_{\alpha}\mathbf{X}^{\mu} = (\triangleright_{\delta}\mathbf{X}^{\mu}-\mathbf{A}^{\mu}{}_{\delta\tau}\mathbf{X}^{\tau})\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma}-(F\frac{\partial\Gamma^{*\mu}{}_{\gamma\alpha}}{\partial\dot{x}^{\beta}}-\nabla_{\alpha}\mathbf{A}^{\mu}{}_{\beta\gamma})\mathbf{X}^{\gamma}+\mathbf{A}^{\gamma}{}_{\beta\alpha}\nabla_{\gamma}\mathbf{X}^{\mu}. \tag{3.35}$$

If we let $\gamma \rightarrow \tau$ in the third and fourth terms on the R.H.S of equation (3.35), we get

$$\begin{aligned}\nabla_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu} - \triangleright_{\beta}\nabla_{\alpha}\mathbf{X}^{\mu} \;&=\; \triangleright_{\delta}\mathbf{X}^{\mu}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma} - \mathbf{A}^{\mu}{}_{\delta\tau}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma}\mathbf{X}^{\tau} - \\ &\quad (F\frac{\partial\Gamma^{*\mu}{}_{\tau\alpha}}{\partial\dot{x}^{\beta}} - \nabla_{\alpha}\mathbf{A}^{\mu}{}_{\beta\tau})\mathbf{X}^{\tau} + \mathbf{A}^{\gamma}{}_{\beta\alpha}\nabla_{\gamma}\mathbf{X}^{\mu}.\end{aligned} \tag{3.36}$$

Equation (3.36) can be rewritten as

$$\begin{aligned}\nabla_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu} - \triangleright_{\beta}\nabla_{\alpha}\mathbf{X}^{\mu} \;&=\; \triangleright_{\delta}\mathbf{X}^{\mu}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma} - (F\frac{\partial\Gamma^{*\mu}{}_{\tau\alpha}}{\partial\dot{x}^{\beta}} + \mathbf{A}^{\mu}{}_{\delta\tau}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma} - \nabla_{\alpha}\mathbf{A}^{\mu}{}_{\beta\tau})\mathbf{X}^{\tau} + \\ &\quad \mathbf{A}^{\gamma}{}_{\beta\alpha}\nabla_{\gamma}\mathbf{X}^{\mu}.\end{aligned} \tag{3.37}$$

Consider the R.H.S of equation (3.37). Let $\delta \rightarrow \nu$ in the first term and $\tau \rightarrow \nu$ in the second, third and fourth terms while $\gamma \rightarrow \nu$ in the last term so that equation (3.37) becomes

$$\nabla_{\alpha}\triangleright_{\beta}\mathbf{X}^{\mu}-\triangleright_{\beta}\nabla_{\alpha}\mathbf{X}^{\mu} = \triangleright_{\nu}\mathbf{X}^{\mu}\nabla_{\gamma}\mathbf{A}^{\nu}{}_{\beta\alpha}l^{\gamma}-(F\frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial\dot{x}^{\beta}}+\mathbf{A}^{\mu}{}_{\delta\nu}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha}l^{\gamma}-\nabla_{\alpha}\mathbf{A}^{\mu}{}_{\beta\nu})\mathbf{X}^{\nu}+\mathbf{A}^{\nu}{}_{\beta\alpha}\nabla_{\nu}\mathbf{X}^{\mu} \tag{3.38}$$

If we now define

$$\mathbf{P}^{\mu}{}_{\nu\alpha\beta} = F\frac{\partial\Gamma^{*\mu}{}_{\nu\alpha}}{\partial\dot{x}^{\beta}} + \mathbf{A}^{\mu}{}_{\delta\nu}\nabla_{\gamma}\mathbf{A}^{\delta}{}_{\beta\alpha} - \nabla_{\alpha}\mathbf{A}^{\mu}{}_{\beta\nu}, \tag{3.39}$$

then equation (3.38) reduces to

$$\nabla_\alpha \triangleright_\beta \mathbf{X}^\mu - \triangleright_\beta \nabla_\alpha \mathbf{X}^\mu = \triangleright_\nu \mathbf{X}^\mu \nabla_\gamma \mathbf{A}^\nu{}_{\beta\alpha} l^\gamma - \mathbf{P}^\mu{}_{\nu\alpha\beta} \mathbf{X}^\nu + \nabla_\nu \mathbf{X}^\mu \mathbf{A}^\nu{}_{\beta\alpha} \quad (3.40)$$

$\mathbf{P}^\mu{}_{\nu\alpha\beta}$ in equation (3.39) then defines the third curvature tensor.

We have thus obtained all the three curvature tensors $\mathbf{R}^\mu{}_{\nu\beta\alpha}$, $\mathbf{S}^\mu{}_{\nu\alpha\beta}$ and $\mathbf{P}^\mu{}_{\nu\alpha\beta}$ in Finsler space through Cartan covariant differentiation as defined in equations (3.15),(3.24) and (3.39) respectively.

3.3 Bianchi Identities in Finsler Geometry

Bianchi identities are usually derived from curvature tensors. In this thesis, we are interested in Bianchi identities derived from Cartan's curvature tensors applied within the framework of Finsler geometry. In particular, we find the second Bianchi identity to be of great relevance to our work.

For the sake of completeness and application below, we state the first Bianchi identity [10] in Finsler geometry as

$$\nabla_\beta \mathbf{K}^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{K}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{K}^\gamma{}_{\mu\beta\nu} + \left(\frac{\partial \Gamma^{*\gamma}{}_{\mu\nu}}{\partial \dot{x}^\delta} \mathbf{K}^\delta{}_{\sigma\alpha\beta} + \frac{\partial \Gamma^{*\gamma}{}_{\mu\beta}}{\partial \dot{x}^\delta} \mathbf{K}^\delta{}_{\sigma\nu\alpha} + \frac{\partial \Gamma^{*\gamma}{}_{\mu\alpha}}{\partial \dot{x}^\delta} \mathbf{K}^\delta{}_{\sigma\beta\nu}\right) \dot{x}^\sigma = 0. \quad (3.41)$$

We proceed to derive the second Bianchi identity that we shall use in the derivation of our generalized field equations within the framework of Finsler geometry. To do this, we use equation (3.13) in equation (3.15) to write

$$\mathbf{R}^{\mu}{}_{\nu\beta\alpha} = \mathbf{K}^{\mu}_{\nu\beta\alpha} + \mathbf{A}^{\mu}{}_{\nu\tau}\mathbf{K}^{\tau}{}_{\gamma\beta\alpha}l^{\gamma}$$

or

$$\mathbf{K}^{\mu}_{\nu\beta\alpha} = \mathbf{R}^{\mu}{}_{\nu\beta\alpha} - \mathbf{A}^{\mu}{}_{\nu\tau}\mathbf{K}^{\tau}{}_{\gamma\beta\alpha}l^{\gamma}.$$

But we can write this equation in the form

$$\mathbf{K}^{\gamma}_{\mu\nu\alpha} = \mathbf{R}^{\gamma}{}_{\mu\nu\alpha} - \mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma}\ . \tag{3.42}$$

Covariant differentiation of equation (3.42) w.r.t x^{β} gives

$$\nabla_{\beta}\mathbf{K}^{\gamma}{}_{\mu\nu\alpha} = \nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha} - \nabla_{\beta}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma} - \mathbf{A}^{\gamma}{}_{\mu\tau}\nabla_{\beta}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma}. \tag{3.43}$$

Cyclic interchange of ν, α, β in equation (3.43) gives other equations as

$$\nabla_{\nu}\mathbf{K}^{\gamma}{}_{\mu\alpha\beta} = \nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta} - \nabla_{\nu}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta}l^{\sigma} - \mathbf{A}^{\gamma}{}_{\mu\tau}\nabla_{\nu}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta}l^{\sigma} \tag{3.44}$$

and

$$\nabla_{\alpha}\mathbf{K}^{\gamma}{}_{\mu\beta\nu} = \nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu} - \nabla_{\alpha}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}l^{\sigma} - \mathbf{A}^{\gamma}{}_{\mu\tau}\nabla_{\alpha}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}l^{\sigma} \tag{3.45}$$

Adding equations (3.43), (3.44) and (3.45) gives

$$\begin{aligned}\nabla_{\beta}\mathbf{K}^{\gamma}{}_{\mu\nu\alpha} + \nabla_{\nu}\mathbf{K}^{\gamma}{}_{\mu\alpha\beta} + \nabla_{\alpha}\mathbf{K}^{\gamma}{}_{\mu\beta\nu} &= \\ \nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha} + \nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta} + \nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu}& \\ -\nabla_{\beta}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma} - \nabla_{\nu}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta}l^{\sigma} - \nabla_{\alpha}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}l^{\sigma}& \\ -\mathbf{A}^{\gamma}{}_{\mu\tau}\big(\nabla_{\beta}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha} + \nabla_{\nu}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta} + \nabla_{\alpha}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}\big)l^{\sigma}.&\end{aligned} \tag{3.46}$$

But applying the first Bianchi identity equation (3.41) to the left hand term and to the bracketed right hand term, we get

$$-(\frac{\partial\Gamma^{*\gamma}{}_{\mu\nu}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\alpha\beta}+\frac{\partial\Gamma^{*\gamma}{}_{\mu\beta}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\nu\alpha}+\frac{\partial\Gamma^{*\gamma}{}_{\mu\alpha}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\beta\nu})\dot{x}^{\sigma}=$$
$$\nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha}+\nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta}+\nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu}-$$
$$\nabla_{\beta}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma}-\nabla_{\nu}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta}l^{\sigma}-\nabla_{\alpha}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}\,l^{\sigma}+$$
$$\mathbf{A}^{\gamma}{}_{\mu\tau}(\frac{\partial\Gamma^{*\tau}{}_{\sigma\nu}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\alpha\beta}+\frac{\partial\Gamma^{*\tau}{}_{\sigma\beta}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\nu\alpha}+\frac{\partial\Gamma^{*\tau}{}_{\sigma\alpha}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\beta\nu})\dot{x}^{\rho}l^{\sigma}. \tag{3.47}$$

Writing $\dot{x}^{\sigma}=F\,l^{\sigma}$ and $\dot{x}^{\rho}=F\,l^{\rho}$ and bringing the L.H.S of equation (3.47) to the right, we have

$$F(\frac{\partial\Gamma^{*\gamma}{}_{\mu\nu}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\alpha\beta}+\frac{\partial\Gamma^{*\gamma}{}_{\mu\beta}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\nu\alpha}+\frac{\partial\Gamma^{*\gamma}{}_{\mu\alpha}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\sigma\beta\nu})l^{\sigma}+$$
$$\mathbf{A}^{\gamma}{}_{\mu\tau}(F\frac{\partial\Gamma^{*\tau}{}_{\sigma\nu}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\alpha\beta}+F\frac{\partial\Gamma^{*\tau}{}_{\sigma\beta}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\nu\alpha}+F\frac{\partial\Gamma^{*\tau}{}_{\sigma\alpha}}{\partial\dot{x}^{\delta}}\mathbf{K}^{\delta}{}_{\rho\beta\nu})l^{\rho}\,l^{\sigma}+\nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha}+$$
$$\nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta}+\nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu}-\nabla_{\beta}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\nu\alpha}l^{\sigma}-\nabla_{\nu}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\alpha\beta}l^{\sigma}-\nabla_{\alpha}\mathbf{A}^{\gamma}{}_{\mu\tau}\mathbf{K}^{\tau}{}_{\sigma\beta\nu}\,l^{\sigma}=0. \tag{3.48}$$

Interchanging ρ and σ in the second bracketed term and letting $\tau\longrightarrow\delta$ in the last three terms on the L.H.S of equation (3.48), we get

$$0=\nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha}+\nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta}+\nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu}+l^{\sigma}\mathbf{K}^{\delta}{}_{\sigma\alpha\beta}(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\nu}}{\partial\dot{x}^{\delta}}-$$
$$\nabla_{\nu}\mathbf{A}^{\gamma}{}_{\mu\delta}+\mathbf{A}^{\gamma}{}_{\mu\tau}\frac{F\partial\Gamma^{*\tau}{}_{\rho\nu}}{\partial\dot{x}^{\delta}}\,l^{\rho})+l^{\sigma}K^{\delta}{}_{\sigma\nu\alpha}(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\beta}}{\partial\dot{x}^{\delta}}-\nabla_{\beta}\mathbf{A}^{\gamma}{}_{\mu\delta}+$$
$$\mathbf{A}^{\gamma}{}_{\mu\tau}\frac{F\partial\Gamma^{*\tau}{}_{\rho\beta}}{\partial\dot{x}^{\delta}}\,l^{\rho})+l^{\sigma}\mathbf{K}^{\delta}{}_{\sigma\beta\nu}(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\alpha}}{\partial\dot{x}^{\delta}}-\nabla_{\alpha}\mathbf{A}^{\gamma}{}_{\mu\delta}+\mathbf{A}^{\gamma}{}_{\mu\tau}\frac{F\partial\Gamma^{*\tau}{}_{\rho\alpha}}{\partial\dot{x}^{\delta}}\,l^{\rho}). \tag{3.49}$$

But we can write

$$F\frac{\partial\Gamma^{*\tau}{}_{\gamma\nu}}{\partial\dot{x}^{\delta}}\,l^{\gamma}=\nabla_{\gamma}\mathbf{A}^{\tau}{}_{\nu\delta}l^{\gamma} \tag{3.50}$$

so that the above equation reads

$$\begin{aligned}\nabla_\beta R^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu} +\\ l^\sigma \mathbf{K}^\delta{}_{\sigma\alpha\beta}\Big(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\nu}}{\partial\dot{x}^\delta} - \nabla_\nu \mathbf{A}^\gamma{}_{\mu\delta} + \mathbf{A}^\gamma{}_{\mu\tau}\nabla_\rho \mathbf{A}^\tau{}_{\nu\delta}\, l^\rho\Big)\\ + l^\sigma \mathbf{K}^\delta{}_{\sigma\nu\alpha}\Big(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\beta}}{\partial\dot{x}^\delta} - \nabla_\beta \mathbf{A}^\gamma{}_{\mu\delta} + \mathbf{A}^\gamma{}_{\mu\tau}\nabla_\rho \mathbf{A}^\tau{}_{\beta\delta}\, l^\rho\Big) +\\ l^\sigma \mathbf{K}^\delta{}_{\sigma\beta\nu}\Big(\frac{F\partial\Gamma^{*\gamma}{}_{\mu\alpha}}{\partial\dot{x}^\delta} - \nabla_\alpha \mathbf{A}^\gamma{}_{\mu\delta} + \mathbf{A}^\gamma{}_{\mu\tau}\nabla_\rho \mathbf{A}^\tau{}_{\alpha\delta}\, l^\rho\Big) = 0. \end{aligned} \qquad (3.51)$$

Using Cartan's third curvature tensor (equation (3.39)), we can replace the terms in bracket and write the above equation as:

$$0 = \nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu} + l^\sigma \mathbf{K}^\delta{}_{\sigma\alpha\beta}\mathbf{P}^\gamma{}_{\mu\nu\delta} + l^\sigma \mathbf{K}^\delta{}_{\sigma\nu\alpha}\mathbf{P}^\gamma{}_{\mu\beta\delta} + l^\sigma \mathbf{K}^\delta{}_{\sigma\beta\nu}\mathbf{P}^\gamma{}_{\mu\alpha\delta}. \qquad (3.52)$$

Using equation (3.18), we write the above equation as

$$0 = \nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu} + l^\sigma\big(\mathbf{R}^\delta{}_{\sigma\alpha\beta}\mathbf{P}^\gamma{}_{\mu\nu\delta} + \mathbf{R}^\delta{}_{\sigma\nu\alpha}P^\gamma{}_{\mu\beta\delta} + \mathbf{R}^\delta{}_{\sigma\beta\nu}\mathbf{P}^\gamma{}_{\mu\alpha\delta}\big). \qquad (3.53)$$

Replacing σ with τ, we get

$$\nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu} + l^\tau\big(\mathbf{R}^\delta{}_{\tau\alpha\beta}\mathbf{P}^\gamma{}_{\mu\nu\delta} + \mathbf{R}^\delta{}_{\tau\nu\alpha}\mathbf{P}^\gamma{}_{\mu\beta\delta} + \mathbf{R}^\delta{}_{\tau\beta\nu}\mathbf{P}^\gamma{}_{\mu\alpha\delta}\big) = 0. \qquad (3.54)$$

or

$$\nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha} + \nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta} + \nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu} + \big(\mathbf{R}^\delta{}_{\tau\alpha\beta}\mathbf{P}^\gamma{}_{\mu\nu\delta} + \mathbf{R}^\delta{}_{\tau\nu\alpha}\mathbf{P}^\gamma{}_{\mu\beta\delta} + \mathbf{R}^\delta{}_{\tau\beta\nu}\mathbf{P}^\gamma{}_{\mu\alpha\delta}\big) l^\tau = 0. \qquad (3.55)$$

This is the desired second Bianchi identity. If we let $\mathbf{R}^\mu{}_{\nu\alpha\beta}\, l^\nu = \mathbf{R}^{*\mu}{}_{\alpha\beta}$, then this equation is rewritten as

$$\nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha}+\nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta}+\nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu}+\left(\mathbf{P}^\gamma{}_{\mu\nu\delta}\mathbf{R}^{*\delta}{}_{\alpha\beta}+\mathbf{P}^\gamma{}_{\mu\beta\delta}\mathbf{R}^{*\delta}{}_{\nu\alpha}+\mathbf{P}^\gamma{}_{\nu\alpha\delta}\mathbf{R}^{*\delta}{}_{\beta\nu}\right)=0. \tag{3.56}$$

Further letting

$$\mathbf{P}^\beta{}_{\tau\sigma\mu}\mathbf{R}^{*\mu}{}_{\nu\alpha}=\mathbf{Q}^\beta{}_{\tau\sigma\nu\alpha}, \tag{3.57}$$

equation (3.56) takes the form

$$\nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha}+\nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta}+\nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu}+\left(\mathbf{Q}^\gamma{}_{\mu\nu\alpha\beta}+\mathbf{Q}^\gamma{}_{\mu\alpha\beta\nu}+\mathbf{Q}^\gamma{}_{\mu\beta\nu\alpha}\right)=0. \tag{3.58}$$

If we now define

$$\mathbf{D}^\gamma{}_{\mu\nu\alpha\beta}=\mathbf{Q}^\gamma{}_{\mu\nu\alpha\beta}+\mathbf{Q}^\gamma{}_{\mu\alpha\beta\nu}+\mathbf{Q}^\gamma{}_{\mu\beta\nu\alpha}, \tag{3.59}$$

then equation (3.58) becomes

$$\nabla_\beta \mathbf{R}^\gamma{}_{\mu\nu\alpha}+\nabla_\nu \mathbf{R}^\gamma{}_{\mu\alpha\beta}+\nabla_\alpha \mathbf{R}^\gamma{}_{\mu\beta\nu}+\mathbf{D}^\gamma{}_{\mu\nu\alpha\beta}=0. \tag{3.60}$$

This is the general form of the second Bianchi identity with respect to Cartan covariant differentiation in Finsler space [10]. It agrees exactly with the Bianchi identity obtained and used before in [8] as a starting point for deriving field equations within Finsler space.

Chapter 4

Results and Discussion

In this chapter, we start by contraction of the general form of the second Bianchi identity equation (3.60) obtained in the previous chapter. The contraction of the second Bianchi identity is based on two alternative forms, giving rise to the desired conservation law in two forms. On application of Noether theorem, general relativistic field equations suitable for describing dynamics in a non-inertial reference frame are obtained in two forms.

4.1 Contracted Bianchi identities

We now proceed to obtain the contracted Bianchi identities based on equation (3.60). Applying the antisymmetry property $\mathbf{R}^{\rho}{}_{\mu\beta\nu} = -\mathbf{R}^{\rho}{}_{\mu\nu\beta}$, we write equation (3.60) in the form

$$\nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha} + \nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta} - \nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\nu\beta} + \mathbf{D}^{\gamma}{}_{\alpha\beta\mu\nu} = 0 \qquad (4.1)$$

Contracting this in γ, ν and multiplying through by the metric tensor yields

$$g^{\mu\alpha}\nabla_\beta \mathbf{R}_{\mu\alpha} + g^{\mu\alpha}\nabla_\nu \mathbf{R}^\nu{}_{\mu\alpha\beta} - g^{\mu\alpha}\nabla_\alpha \mathbf{R}_{\mu\beta} + g^{\mu\alpha}\mathbf{D}_{\alpha\beta\mu} = 0. \tag{4.2}$$

But the second term in equation (4.2) can be written as

$$\begin{aligned}
\nabla_\nu g^{\mu\alpha}\mathbf{R}^\nu{}_{\mu\alpha\beta} &= \nabla_\nu g^{\nu\sigma} g^{\mu\alpha}\mathbf{R}_{\sigma\mu\alpha\beta} \\
&= \nabla_\nu g^{\nu\sigma} g^{\mu\alpha}\ \mathbf{R}_{\mu\sigma\beta\alpha} \\
&= \nabla_\nu g^{\nu\sigma}\mathbf{R}^\alpha{}_{\sigma\beta\alpha} \\
&= -\nabla_\nu g^{\nu\sigma}\mathbf{R}_{\sigma\beta} \\
&= -\nabla_\nu \mathbf{R}^\nu{}_\beta
\end{aligned}$$

which we substitute in equation (4.2) to obtain

$$\nabla_\beta \mathbf{R} - \nabla_\nu \mathbf{R}^\nu{}_\beta - \nabla_\alpha \mathbf{R}^\alpha{}_\beta + \mathbf{D}^\mu{}_{\beta\mu} = 0 \tag{4.3}$$

or

$$\nabla_\beta \mathbf{R} - \nabla_\nu \mathbf{R}^\nu{}_\beta - \nabla_\alpha \mathbf{R}^\alpha{}_\beta + \mathbf{D}_\beta = 0. \tag{4.4}$$

If we let $\nu \longrightarrow \alpha$, then equation (4.4) becomes

$$\nabla_\beta \mathbf{R} - 2\nabla_\alpha \mathbf{R}^\alpha{}_\beta + \mathbf{D}_\beta = 0$$

or

$$\nabla_\alpha \mathbf{R}^\alpha{}_\beta - \frac{1}{2}\nabla_\beta\ \mathbf{R} = \frac{1}{2}\mathbf{D}_\beta. \tag{4.5}$$

Using the metric tensor, we write the second term on the L.H.S of equation (4.5) as

$$\nabla_\beta \mathbf{R} = \nabla_\alpha g^\alpha{}_\beta \mathbf{R} \tag{4.6}$$

so that equation (4.5) becomes

$$\nabla_\alpha \mathbf{R}^\alpha{}_\beta - \frac{1}{2}\nabla_\alpha g^\alpha{}_\beta \mathbf{R} = \frac{1}{2}\mathbf{D}_\beta$$

or

$$\nabla_\alpha \left(\mathbf{R}^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta}\mathbf{R}\right) = \frac{1}{2}\mathbf{D}^\beta. \tag{4.7}$$

Using tensor calculus, we can express

$$\mathbf{D}^\beta = \mathbf{D}^{\beta\mu\nu}{}_{\mu\nu} \tag{4.8}$$

which we substitute in equation (4.7) to obtain an alternative form

$$\nabla_\alpha \left(\mathbf{R}^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta}\mathbf{R}\right) = \frac{1}{2}\mathbf{D}^{\beta\mu\nu}{}_{\mu\nu}. \tag{4.9}$$

This form of the contracted Bianchi identities coincides with that obtained in [8]. We observe that in the absence of the term on the R.H.S., equation (4.9) takes the form giving Einstein's equations of general relativity (in the limit of constant velocity). However, this term does not just vanish in Finsler geometry, since it is associated with the varying velocity in the tangent space. In order to rewrite equation (4.9) as a conservation equation, the term on the R.H.S must be written in the form of a covariant derivative. In an attempt to rewrite $\frac{1}{2}\mathbf{D}^{\beta\mu\nu}{}_{\mu\nu}$ as a covariant derivative, Yazaki [8] used an approximation (assumption) that the tangent space is a surface of constant curvature and therefore applied the tensor identity [8-10]

$$\mathbf{R}^{\alpha}{}_{\mu\nu} = (y_\mu \delta^\alpha{}_\nu - y_\nu \delta^\alpha{}_\mu)K \tag{4.10}$$

with constant K to derive approximate conservation laws and associated field equations. We observe that this form is not appropriate for deriving exact conservation laws and field equations within Finsler space. The field equations obtained by Yazaki [8] based on direct contraction of the Bianchi identities in equation (3.60) are therefore not exact and are less general.

In this thesis, we follow an alternative approach to derive exact conservation law and field equations based on these Bianchi identities. To rewrite $\frac{1}{2}\mathbf{D}^{\beta\mu\nu}{}_{\mu\nu}$ in the form of a covariant derivative, we make some generalized assumptions of two forms with the first form given as $\mathbf{Q}^{\gamma}{}_{\rho\sigma\alpha\beta} = y^\tau y_\beta \nabla_\tau \mathbf{S}^{\gamma}{}_{\rho\sigma\alpha}$ and the second form as $\mathbf{Q}^{\gamma}{}_{\rho\sigma\alpha\beta} = g^\tau{}_\beta \nabla_\tau \mathbf{S}^{\gamma}{}_{\rho\sigma\alpha}$. The resulting conservation law and field equations shall thus be given in two forms.

4.2 Exact Conservation law

Bianchi identity in equation (3.60) provide the starting point for deriving a general conservation law and the associated field equations within Finsler space. The resulting conservation law and field equations shall be given in two forms.

4.2.1 The first form of conservation law

To obtain the first form, we start by rewriting each term on the R.H.S of equation (3.59) in the form

$$\mathbf{Q}^{\gamma}{}_{\rho\sigma\alpha\beta} = y^{\tau} y_{\beta} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\rho\sigma\alpha}. \tag{4.11}$$

If we use equation (4.11) in equation (3.59), we get

$$\mathbf{D}^{\gamma}{}_{\mu\nu\alpha\beta} = (y^{\tau} y_{\beta} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\nu\alpha} + y^{\tau} y_{\nu} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\alpha\beta} + y^{\tau} y_{\alpha} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\beta\nu}). \tag{4.12}$$

Using equation (4.12) in equation (3.60), we get

$$\nabla_{\beta} \mathbf{R}^{\gamma}{}_{\mu\nu\alpha} + \nabla_{\nu} \mathbf{R}^{\gamma}{}_{\mu\alpha\beta} + \nabla_{\alpha} \mathbf{R}^{\gamma}{}_{\mu\beta\nu} + (y^{\tau} y_{\beta} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\nu\alpha} + y^{\tau} y_{\nu} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\alpha\beta} + y^{\tau} y_{\alpha} \nabla_{\tau} \mathbf{S}^{\gamma}{}_{\mu\beta\nu}) = 0. \tag{4.13}$$

Contracting γ, ν, and applying the antisymmetry property, equation (4.13) becomes

$$\nabla_{\beta} \mathbf{R}_{\mu\alpha} + \nabla_{\nu} \mathbf{R}^{\nu}{}_{\mu\alpha\beta} - \nabla_{\alpha} \mathbf{R}_{\mu\beta} + (y^{\tau} y_{\beta} \nabla_{\tau}\ \mathbf{S}_{\mu\alpha} + y^{\tau} y_{\nu} \nabla_{\tau} \mathbf{S}^{\nu}{}_{\mu\alpha\beta} - y^{\tau} y_{\alpha} \nabla_{\tau} \mathbf{S}_{\mu\beta}) = 0. \tag{4.14}$$

Using the metric tensor, we write equation (4.14) as

$$g^{\mu\alpha} \nabla_{\beta} \mathbf{R}_{\mu\alpha} + g^{\mu\alpha} \nabla_{\nu} \mathbf{R}^{\nu}{}_{\mu\alpha\beta} - g^{\mu\alpha} \nabla_{\alpha} \mathbf{R}_{\mu\beta} + g^{\mu\alpha} (y^{\tau} y_{\beta} \nabla_{\tau} \mathbf{S}_{\mu\alpha} + y^{\tau} y_{\nu} \nabla_{\tau} \mathbf{S}^{\nu}{}_{\mu\alpha\beta} - y^{\tau} y_{\alpha} \nabla_{\tau} \mathbf{S}_{\mu\beta}) = 0 \tag{4.15}$$

Application of the properties of the metric tensor as in equation (4.2), puts equation (4.15) in the form

$$\nabla_{\beta} \mathbf{R} - \nabla_{\nu} \mathbf{R}^{\nu}{}_{\beta} - \nabla_{\alpha} \mathbf{R}^{\alpha}{}_{\beta} + \nabla_{\tau} (y^{\tau} y_{\beta} \mathbf{S} - y^{\tau} y_{\nu} \mathbf{S}^{\nu}{}_{\beta} - y^{\tau} y_{\alpha} \mathbf{S}^{\alpha}{}_{\beta}) = 0. \tag{4.16}$$

Since ν is a dummy variable, we replace it with α so that equation (4.16) gives

$$\nabla_\beta \mathbf{R} - \nabla_\alpha \mathbf{R}^\alpha{}_\beta - \nabla_\alpha \mathbf{R}^\alpha{}_\beta + \nabla_\tau (y^\tau y_\beta \mathbf{S} - y^\tau y_\alpha \mathbf{S}^\alpha{}_\beta - y^\tau y_\alpha \mathbf{S}^\alpha{}_\beta) = 0$$

or

$$\nabla_\beta \mathbf{R} - 2\nabla_\alpha \mathbf{R}^\alpha{}_\beta + \nabla_\tau y^\tau y_\beta \mathbf{S} - 2\nabla_\tau y^\tau y_\alpha \mathbf{S}^\alpha{}_\beta = 0. \tag{4.17}$$

Using equation (4.6) in equation (4.17), we get

$$\nabla_\alpha g^\alpha{}_\beta \mathbf{R} - 2\nabla_\alpha \mathbf{R}^\alpha{}_\beta + \nabla_\tau y^\tau y_\beta \mathbf{S} - 2\nabla_\alpha \mathbf{S}^\alpha{}_\beta = 0 \tag{4.18}$$

or

$$\nabla_\alpha \mathbf{R}^{\alpha\beta} - \frac{1}{2}\nabla_\alpha \mathbf{R} g^{\alpha\beta} - \frac{1}{2}\nabla_\tau y^\tau y^\beta \mathbf{S} + \nabla_\alpha \mathbf{S}^{\alpha\beta} = 0. \tag{4.19}$$

If we further let $\tau \longrightarrow \alpha$, then this equation yields a conservation law of the form

$$\nabla_\alpha[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R} g^{\alpha\beta}) + (\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S} y^\alpha y^\beta)] = 0 \tag{4.20}$$

or

$$\nabla_\alpha[(\mathbf{R}^{\alpha\beta} + \mathbf{S}^{\alpha\beta}) - \frac{1}{2}(\mathbf{R} g^{\alpha\beta} + \mathbf{S} y^\alpha y^\beta)] = 0. \tag{4.21}$$

Equation (4.21) gives the first form of an exact conservation law in Finsler space.

4.2.2 The second form of conservation law

Let us now proceed to obtain the second form of the conservation law by rewriting the terms on the R.H.S of equation (3.59) using an alternative form

$$\mathbf{Q}^{\gamma}{}_{\rho\sigma\alpha\beta} = g^{\tau}{}_{\beta}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\rho\sigma\alpha} \tag{4.22}$$

to write equation (3.59) as

$$\mathbf{D}^{\gamma}{}_{\mu\nu\alpha\beta} = (g^{\tau}{}_{\beta}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\nu\alpha} + g^{\tau}{}_{\nu}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\alpha\beta} + g^{\tau}{}_{\alpha}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\beta\nu}). \tag{4.23}$$

Using equation (4.23) in equation (3.60), we get

$$\nabla_{\beta}\mathbf{R}^{\gamma}{}_{\mu\nu\alpha}+\nabla_{\nu}\mathbf{R}^{\gamma}{}_{\mu\alpha\beta}+\nabla_{\alpha}\mathbf{R}^{\gamma}{}_{\mu\beta\nu}+(g^{\tau}{}_{\beta}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\nu\alpha}+g^{\tau}{}_{\nu}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\alpha\beta}+g^{\tau}{}_{\alpha}\nabla_{\tau}\mathbf{S}^{\gamma}{}_{\mu\beta\nu}) = 0. \tag{4.24}$$

Contracting γ , ν, we express equation (4.24) as

$$\nabla_{\beta}\ \mathbf{R}_{\mu\alpha}+\nabla_{\nu}\mathbf{R}^{\nu}{}_{\mu\alpha\beta}-\nabla_{\alpha}\mathbf{R}_{\mu\beta}+(g^{\tau}{}_{\beta}\nabla_{\tau}\mathbf{S}_{\mu\alpha}+g^{\tau}{}_{\nu}\nabla_{\tau}\mathbf{S}^{\nu}{}_{\mu\alpha\beta}-g^{\tau}{}_{\alpha}\nabla_{\tau}\ \mathbf{S}_{\mu\beta}) = 0. \tag{4.25}$$

Using the metric tensor, we can write equation (4.25) as

$$g^{\mu\alpha}\nabla_{\beta}\mathbf{R}_{\mu\alpha}+g^{\mu\alpha}\nabla_{\nu}\mathbf{R}^{\nu}{}_{\mu\alpha\beta}-g^{\mu\alpha}\nabla_{\alpha}\mathbf{R}_{\mu\beta}+g^{\mu\alpha}(g^{\tau}{}_{\beta}\nabla_{\tau}\mathbf{S}_{\mu\alpha}+g^{\tau}{}_{\nu}\nabla_{\tau}\mathbf{S}^{\nu}{}_{\mu\alpha\beta}-g^{\tau}{}_{\alpha}\nabla_{\tau}\mathbf{S}_{\mu\beta}) = 0. \tag{4.26}$$

Once again, we apply the properties of the metric tensor as in equation (4.2), to put equation (4.26) in the form

$$\nabla_{\beta}\mathbf{R} - \nabla_{\nu}\mathbf{R}^{\nu}{}_{\beta} - \nabla_{\alpha}\mathbf{R}^{\alpha}{}_{\beta} + \nabla_{\tau}(g^{\tau}{}_{\beta}\mathbf{S} - g^{\tau}{}_{\nu}\mathbf{S}^{\nu}{}_{\beta} - g^{\tau}{}_{\alpha}\mathbf{S}^{\alpha}{}_{\beta}) = 0. \tag{4.27}$$

Since ν is a dummy variable, we replace it with α so that equation (4.27) gives

$$\nabla_{\beta}\mathbf{R} - \nabla_{\alpha}\mathbf{R}^{\alpha}{}_{\beta} - \nabla_{\alpha}\mathbf{R}^{\alpha}{}_{\beta} + \nabla_{\tau}(g^{\tau}{}_{\beta}\mathbf{S} - g^{\tau}{}_{\alpha}\mathbf{S}^{\alpha}{}_{\beta} - g^{\tau}{}_{\alpha}\mathbf{S}^{\alpha}{}_{\beta}) = 0 \tag{4.28}$$

or

$$\nabla_\beta \mathbf{R} - 2\nabla_\alpha \mathbf{R}^\alpha{}_\beta + \nabla_\tau g^\tau{}_\beta \mathbf{S} - 2\nabla_\tau g^\tau{}_\alpha \mathbf{S}^\alpha{}_\beta. \tag{4.29}$$

Using equation (4.6) in equation (4.29), we get

$$\nabla_\alpha g^\alpha{}_\beta \mathbf{R} - 2\nabla_\alpha \mathbf{R}^\alpha{}_\beta + \nabla_\tau g^\tau{}_\beta \mathbf{S} - 2\nabla_\alpha \mathbf{S}^\alpha{}_\beta = 0 \tag{4.30}$$

or

$$\nabla_\alpha \mathbf{R}^{\alpha\beta} - \frac{1}{2}\nabla_\alpha \mathbf{R} g^{\alpha\beta} - \frac{1}{2}\nabla_\tau g^{\tau\beta}\mathbf{S} + \nabla_\alpha \mathbf{S}^{\alpha\beta} = 0. \tag{4.31}$$

If we further let $\tau \to \alpha$, then this equation yields a conservation law of the form

$$\nabla_\alpha[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R} g^{\alpha\beta}) + (\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S} g^{\alpha\beta})] = 0 \tag{4.32}$$

or

$$\nabla_\alpha[(\mathbf{R}^{\alpha\beta} + \mathbf{S}^{\alpha\beta}) - \frac{1}{2}(\mathbf{R} + \mathbf{S}) g^{\alpha\beta}] = 0. \tag{4.33}$$

This is the second form of the exact conservation law based on Cartan covariant differentiation in Finsler space. Equations (4.21) and (4.33) are the alternative forms of an exact conservation law in Finsler space. This is the first time they are being derived within the framework of Finsler geometry.

4.3 Generalized relativistic field equations

We proceed to derive the general field equations associated with exact conservation law in the alternative forms of equations (4.20) and (4.32). To do this, we apply Noether's theorem which states that with every distribution of matter and fields, there is always an associated tensor (energy-momentum tensor). We obtain the general relativistic field equations in two equivalent forms associated with the first and second forms of the exact conservation law.

4.3.1 First form of field equations

Application of Noether theorem to our conservation equations (4.20) and (4.21) yields

$$[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) + (\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S}y^{\alpha}y^{\beta})] = k\mathcal{T}^{\alpha\beta} \tag{4.34}$$

and

$$[(\mathbf{R}^{\alpha\beta} + \mathbf{S}^{\alpha\beta}) - \frac{1}{2}(\mathbf{R}g^{\alpha\beta} + \mathbf{S}y^{\alpha}y^{\beta})] = k\mathcal{T}^{\alpha\beta} \tag{4.35}$$

respectively, where k is a constant and $\mathcal{T}^{\alpha\beta}$ is a two-component energy-momentum tensor in Finsler space. These equations form the first form of the exact field equations in a non-inertial reference frame derived within the framework of Finsler geometry. In particular, equation (4.34) gives the field equations in two different parts with the first part $\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}$ corresponding to coordinate space and the second part $\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S}y^{\alpha}y^{\beta}$ to momentum space (tangent space). Our field equations are different from those obtained in earlier work by Yazaki [8]. Yazaki [8] derived approximate forms of the conservation law in equations (4.20) as

$$\nabla_{\alpha}[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) - \frac{1}{2}\mathbf{S}y^{\alpha}y^{\beta}] = 0, \tag{4.36}$$

which he used through Noether's theorem to write down the corresponding approximate forms of generalized field equations in Finsler space as

$$[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) - \frac{1}{2}\mathbf{S}y^{\alpha}y^{\beta}] = k\mathbf{T}^{\alpha\beta} \tag{4.37}$$

His derivation was essentially based on an approximation (assumption) that the tangent space is a surface of constant curvature. Based on this assumption, he applied the tensor identity in equation (4.10) which is less general than the tensor form in equation (4.11) used in this thesis. In particular, in the limit of constant velocity and absence of the Ricci tensor $\mathbf{S}^{\alpha\beta}$ in equation (4.34), our field equations reduces to that of Yazaki [8] in equation (4.37). Therefore, Yazaki's [8] conservation law and field equations in equations (4.36) and (4.37) are approximate forms of our exact conservation law and field equations (4.20) and (4.34) respectively. Furthermore, the term $k\mathcal{T}^{\alpha\beta}$ on the R.H.S of equation (4.34) is a conserved tensor corresponding to a two-component energy-momentum tensor in Finsler space, which may be written as

$$\mathcal{T}^{\alpha\beta} = \mathbf{T}^{\alpha\beta} + \mathbf{t}^{\alpha\beta} \tag{4.38}$$

$\mathbf{T}^{\alpha\beta}$ is the generalized energy-momentum tensor corresponding to the coordinate space and $\mathbf{t}^{\alpha\beta}$ is the energy-momentum tensor in the momentum space (tangent space). Equations (4.20) and (4.34) form the first part of our main results and they are appearing for the first time in literature on Finsler spaces.

4.3.2 Second form of field equations

Application of Noether theorem to our conservation equations (4.32) and (4.33) yields

$$[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) + (\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S}g^{\alpha\beta})] = k\mathcal{T}^{\alpha\beta} \tag{4.39}$$

and

$$[(\mathbf{R}^{\alpha\beta} + \mathbf{S}^{\alpha\beta}) - \frac{1}{2}(\mathbf{R} + \mathbf{S})g^{\alpha\beta}] = k\mathcal{T}^{\alpha\beta} \tag{4.40}$$

respectively. Equations (4.39) and (4.40) are the second form of our field equations in Finsler space. The terms $\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}$ and $\mathbf{S}^{\alpha\beta} - \frac{1}{2}\mathbf{S}g^{\alpha\beta}$ on the L.H.S of equation (4.39) correspond to the coordinate space and momentum space respectively. Ikeda [9], following an approach based on vector bundles, assuming surface of constant curvature based on tensor identity (4.11) on horizontal or vertical Finsler space arrived at approximate conservation law and field equations in the form

$$\nabla_\alpha[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) - \frac{1}{2}\mathbf{S}g^{\alpha\beta}] = 0 \tag{4.41}$$

and

$$[(\mathbf{R}^{\alpha\beta} - \frac{1}{2}\mathbf{R}g^{\alpha\beta}) - \frac{1}{2}\mathbf{S}g^{\alpha\beta}] = k\mathcal{T}^{\alpha\beta} \tag{4.42}$$

respectively. In particular, in the limit of constant velocity and absence of the Ricci tensor $\mathbf{S}^{\alpha\beta}$ in equation (4.39), our field equations reduces to that of Ikeda [9] in equation (4.42). Its also clear that Ikeda's [9] conservation law and field equations in equations (4.41) and (4.42) are approximate forms of our exact conservation law and field equations (4.32) and (4.39) respectively. If we redefine

$$\mathcal{R} = \mathbf{R} + \mathbf{S} \quad ; \quad \mathbf{R}^{\alpha\beta} + \mathbf{S}^{\alpha\beta} = (\mathbf{R} + \mathbf{S})^{\alpha\beta} = \mathcal{R}^{\alpha\beta}; \tag{4.43}$$

and put equation (4.39) in form of equation (4.40), we get

$$\mathcal{R}^{\alpha\beta} - \frac{1}{2}\mathcal{R}g^{\alpha\beta} = k\mathcal{T}^{\alpha\beta} \tag{4.44}$$

where $\mathcal{T}^{\alpha\beta}$ is as defined in equation (4.38). Equation (4.44) puts our field equations in a familiar form as in general relativity. Equation (4.44) is the second form of the exact field equations in a non-inertial reference frame derived within the framework of Finsler geometry.

Equations (4.32) and (4.44) form the second part of our main results in this thesis and they are being derived for the first time within Finsler space.

We consider equations (4.34) and (4.44) to provide two equivalent alternative forms of generalized relativistic field equations governing dynamics in non-inertial reference frames. The inclusion of velocity as a variable expands the mathematical framework from Riemann geometry to Finsler geometry, with the physical consequence that the velocity-dependent curvature terms, in particular $\mathbf{S}^{\alpha\beta}$ and $\mathbf{S}$ (together with any velocity-dependent contribution from $\mathbf{R}^{\alpha\beta}$ and $\mathbf{R}$) in the field equations can account for the anisotropy of the gravitational field, which is associated with dark matter and dark energy responsible for the acceleration and expansion of the universe as observed in cosmic microwave background radiation and astrophysics experiments studying the origin and evolution of large scale structures in the universe [39-40]. We note that $\mathbf{S}^{\alpha\beta}$ and $\mathbf{S}$, which naturally arise in these equations, effectively play the role of the cosmological constant usually included in Einstein's field equations by hand [1-3]. Our generalized relativistic field equations (4.34) and (4.44) thus provide the appropriate theoretical framework for accurate cosmological models of the universe.

We observe that the Lorentz invariance violation effects established in experiments on the Standard Model of particle interactions in quantum field theory can be understood as effects of dynamics in non-inertial reference frames, which can therefore be accounted for in theoretical models based on Finsler geometry as elaborated in our thesis. In other words, the general theoretical framework for dynamics in non-inertial reference frames developed here provides the physical ingredients for understanding Lorentz invariance violation effects. This is justified by the fact that besides its importance as a physical principle governing dynamics in relativistic quantum mechanics or quantum field theory, the Lorentz invariance also forms the cornerstone of Einstein's general theory of relativity [41-42].

In general, equations (4.34) and (4.44) governing relativistic dynamics in non-inertial reference frames where both spacetime coordinates and corresponding velocities are variables, constitute the starting point for developing theoretical models for physics beyond Einstein's general relativity and the Standard Model in quantum field theory. Indeed, recent work by various authors [28-38,41-44] present models generally based on Finsler geometry (nonholonomic manifold) to account for the anisotropy of the gravitational field and Lorentz invariance violation effects. In particular, the models in [28-30,34-38] are to be understood as special or approximate versions of our field equations (4.34) and (4.44) in the sense that the field equations in those works are not based on the exact conservation laws, while at the same time some of them [28-30] are based on the Berwald covariant differentiation which is known to be nonmetric compatible [34-36, 38]. On the other hand, models in [14-15, 25-28] attempt to account for the Lorentz invariance violation either within the Finsler geometric framework [41-42,43-44] or starting with the Cohen-Glashow model of Very Special Relativity [43] with its generalizations incorporating basic elements of Finsler geometry by other authors in [32,44].

Chapter 5

Summary, Conclusion and Recommendations

In this chapter, we give a summary and concluding remarks of our work, particularly of the main results. We shall end by making recommendations for this work.

5.1 Summary and Conclusion

We observe that our field equations, presented in equivalent alternative forms in equations (4.34) and (4.44), are more general and exact, since they are based on more generalized forms of conservation law derived in equations (4.20) and (4.32). The derivation of the conservation law, appearing in the literature on Finsler geometry for the first time in this thesis, is not restricted to surfaces of constant curvature.

In earlier work, Ikeda [9] and Yazaki [8] derived approximate forms of the conservation law in equations (4.32) and (4.20), which they used through Noether's theorem to write down the corresponding approximate forms of generalized field equations in Finsler space. Their derivations were essentially based on an approximation (assumption) that the tangent space is a surface of constant curvature and therefore applied the tensor identity in equation (4.10) which is less general than the tensor forms in equations (4.11) and (4.22) used in this thesis. In particular, this approximate identity yielded conservation law and corresponding approximate field equations as given by equations (4.36) and (4.37) respectively [8].

On the other hand, Ikeda [9], following an approach based on vector bundles, assuming surface of constant curvature on horizontal or vertical Finsler space separately arrived at approximate conservation law and field equations in the form given by equations (4.41) and (4.42) respectively. We observe that Yazaki's [8] equations in (4.36)-(4.37) constitute approximate forms of our field equations in (4.20) and (4.34), while Ikeda's [9] equations in (4.41)-(4.42) constitute approximate forms of our alternative forms of field equations in (4.32) and (4.44).

Equivalent equations have been obtained in various forms through the method of vector bundles within nonholonomic frames in Finsler geometry in [35-38]. The assumptions of surface of constant curvature employed in [8-9] makes the Yazaki [8] and Ikeda [9] equations (4.37) and (4.42) above, and similar equations based on the same assumption in [35-38], less general than the exact field equations (4.34) and (4.44) derived from the exact conservation laws in equations (4.20) and (4.32) in the present thesis.

We therefore observe that our generalized equations (4.34) and (4.44) derived for the first time in literature in this thesis based on Finsler geometry constitute a new field theory that is more general and exact in contrast to the current gravitational field theories. The current gravitational theories should be therefore understood as approximate forms of our field equations derived in this thesis.

We also note that equations (4.34) and (4.44) give the interaction between the horizontal (x-field) and the vertical (y-field or field in the tangent space) fields. The effective field is therefore the sum of the horizontal and vertical fields. The energy-momentum tensor, $\mathcal{T}^{\alpha\beta}$, is divided into two parts, i.e., $\mathbf{T}^{\alpha\beta}$ which is the usual energy-momentum tensor in general relativity, and $\mathbf{t}^{\alpha\beta}$ which is the energy-momentum tensor that is linked to the tensors $\mathbf{S}^{\alpha\beta}$ and $\mathbf{R}^{\alpha\beta}$. As observed earlier, the additional velocity-dependent terms can account for important features such as anisotropy of the gravitational field, acceleration and expansion of the universe and Lorentz invariance violation effects.

Therefore, our generalized relativistic field equations (4.34) and (4.44) governing relativistic dynamics in non-inertial reference frames have the necessary physical and mathematical ingredients to address the problems of anisotropy, acceleration and expansion of the universe, as well as the Lorentz invariance violation effects, which are the major challenges of theoretical physics at the moment.

5.2 Recommendation

In order to obtain exact and generalized solution for describing relativistic dynamics in a non-inertial reference frame, detailed calculations with our field equations (4.34) and (4.44) is recommended.

It is further recommended that the resulting solution from our field equation be analyzed to reveal the full features of relativistic dynamics in a non-inertial reference frame. This will address the current problems of anisotropy, acceleration and expansion of the universe as well as Lorentz violation effects.

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