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Deficiency indices and spectrum of fourth order difference equations with unbounded coefficients

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Using subspace theory together with appropriate smoothness and decay conditions, we calculated the deficiency indices and absolutely continuous spectrum of fourth order difference equations with unbounded coefficients. In particular, we found the absolutely continuous spectrum to be $\mathbb{R}$ with a spectral multiplicity one.

1 Introduction

In this paper, we have considered a symmetric fourth order difference equation of the form

$$\begin{aligned}\tau\hat{y}(t) &= \triangle^2\big[p_2(t)\triangle^2\hat{y}(t-2)\big] - \triangle\big[p_1(t)\triangle\hat{y}(t-1)\big] + p_0(t)\hat{y}(t)\\ &\quad - i\big\{\triangle\big(q_2(t)\triangle^2\hat{y}(t-2)\big) + \triangle^2\big(q_2(t)\triangle\hat{y}(t-1)\big)\big\}\\ &\quad + i\big\{\triangle\big(q_1(t)\hat{y}(t)\big) + \big(q_1(t)\triangle\hat{y}(t-1)\big)\big\}\\ &= zw(t)\hat{y}(t)\end{aligned} \tag{1.1}$$

defined on a weighted Hilbert space $\ell^2_w(\mathbb{N})$ with $w(t) > 0$ as the weight function, $t \in \mathbb{N}$, $p_2(t) \neq 0,\ p_k,\ q_j$, $k = 0, 1, 2$, $j = 1, 2$ are real valued functions and their second difference, that is, $\triangle^2 p_k$ and $\triangle^2 q_j$ exists and tend to zero as $t \to \infty$. Here, of course, z is the spectral parameter. In (1.1), the notation $\triangle$ refers to a forward difference operator, that is, for any mapping f, $\triangle f(t) = f(t+1) - f(t)$. The notations that will be used in this study are largely standard and follow closely those of [28]. In most cases, the underlying interval will be taken as $I = [a, \infty)$ for a large regular end-point $a,\ a > 0$, but Remlings's results [23] will be used to extend the spectral results to the integral interval $[0, \infty)$.

There has been a number of papers on difference equations and difference operators but in most cases the studies have been focussed either on the Hamiltonian systems of the difference equations or deficiency indices with little or no emphasis on spectral theory, see for example [11]–[14], [17], [28]. In the situations where the studies have gone beyond deficiency indices, the authors have only considered difference equations or operators of second order with bounded coefficients [13]. Behncke and one of the authors [7], [8] went further to investigate the absolutely continuous spectrum of higher order difference operators with almost constant coefficients but in the case of unbounded coefficients, they only managed to study a fourth order difference operator with even order coefficients only. This is due to the difficulty involved in computing the roots of a fourth order polynomial and more so if the coefficients are unbounded.

Another difficulty that is always encountered in the difference systems is that the minimal operator generated by (1.1) may be neither densely defined nor single valued, its maximal operator may not be well defined and

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thus the selfadjoint extension operator for minimal operator cannot be discussed by application of von Neumann theory for densely defined Hermitian operators. This, therefore, requires the theory of Hermitian subspaces where the von Neumann theory has been extended in order to discuss the selfadjoint extension of the minimal Hermitian subspaces and for more details see [25]–[27], [29] and the references cited therein.

In order to avoid the first difficulty mentioned above, we have employed the techniques in [15, Sect. 3.3] which have also been used in [10] to approximate the roots of the associated characteristic polynomial which are pertinent ingredients in the analysis of the deficiency indices of the minimal subspace generated by (1.1) and also the absolutely continuous spectrum of the selfadjoint extension subspace of this minimal subspace. We have managed to circumvent the second problem by doing our analysis through the subspaces generated by (1.1).

It is a well known fact, at least by now, that the study of absolutely continuous spectrum of difference equations parallel that of their continuous counterparts, the differential equations and in that light, this study can be considered as a continuation of [7, Sect. 5] as well as the discrete counterpart of the paper by Behncke [4]. Though it should be noted that there is a big difference in terms of the approach used. Whereas the minimal operator is densely defined and the maximal operator is well defined in the continuous case, the same is not true in the discrete case and hence the only sure way to do the analysis is via the subspace theory. The case of discrete Kummer-Liouville transformation of the coefficients is being pursued by Behncke and one of the authors and therefore will appear in a different paper. Singular continuous spectrum would be assumed to be absent in our case since singular continuous spectrum is not bounded under finite rank perturbations and therefore asymptotic summation as outlined in the paper by Benzaid and Lutz [11] cannot handle this component of the spectrum.

Our main results state that if the coefficients p_k, q_j, $k = 0, 1, 2$, $j = 1, 2$, are allowed to be unbounded, satisfy appropriate smoothness and decay conditions, and the Hamiltonian satisfies the definiteness and regular conditions, the deficiency indices of the minimal subspace is (n, n), where $2 \leq n \leq 4$ and the absolutely continuous spectrum of the selfadjoint extension subspace is the whole of $\mathbb{R}$ and has spectral multiplicity one. These results are, however, invariant under higher order smoothness and decay conditions. If the minimal difference operator generated by (1.1) is densely defined and the corresponding maximal difference operator is well defined, then similar results are also true for the difference operators.

The spectral multiplicity has been studied using the theory of M-matrix as developed by Hinton and Schneider [18] which is a generalisation of the Weyl-Titchmarsh m-function and relates the asymptotics of the eigenfunctions of higher order difference equations to the spectrum of their selfadjoint realisations [28]. The M-matrix is the Borel transform of the spectral measure and the latter can be recovered from the M-matrix [23].

The paper is divided into five sections, namely; 1. Introduction, 2. Hamiltonian systems and Subspaces, 3. Bounded coefficients, 4. Unbounded coefficients and 5. $q_1 \to \infty$ as $t \to \infty$.

2 Hamiltonian systems and subspaces

Discrete Hamiltonian systems originated from the discretisation of continuous Hamiltonian systems and from the discrete processes acting in accordance with the Hamiltonian principle such as discrete physical problems and discrete control problems. Thus like in the differential case, the coefficients will be assumed to be real-valued functions and will be allowed to be unbounded and satisfy the following conditions:

$$\begin{aligned} &\frac{\triangle^2 f}{f},\ \left(\frac{\triangle f}{f}\right)^2 \in \ell^1,\ \ \frac{\triangle f}{f} \in \ell^2, \quad \triangle f = o(1), \\ &f = p_k, q_j, \quad k = 0, 1, 2, \quad j = 1, 2. \end{aligned} \tag{2.1}$$

For the highest order coefficient p_2 and the weight function w, we assume

$$p_2, \quad w > 0. \tag{2.2}$$

In order to define the discrete Hamiltonian system for (1.1), one introduces quasi-difference, see [7], [8], [28],

$$\begin{aligned} x_1(t) &= \hat{y}(t-1), \quad x_2 = \triangle \hat{y}(t-2), \\ u_1(t) &= p_1(t)\triangle \hat{y}(t-1) - \triangle\big(p_2(t)\triangle^2 \hat{y}(t-2)\big) \\ &\qquad + i\big\{\triangle\big(q_2(t)\triangle \hat{y}(t-1)\big) + q_2(t)\triangle^2 \hat{y}(t-2)\big\} - iq_1(t)\hat{y}(t), \\ u_2(t) &= p_2(t)\triangle^2 \hat{y}(t-2) - iq_2(t)\triangle \hat{y}(t-1). \end{aligned} \tag{2.3}$$

Now define the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$x(t) = (x_1(t), x_2(t))^{tr}, \quad u(t) = (u_1(t), u_2(t))^{tr}, \quad y(t) = (x(t), u(t))^{tr}$$

and the partial shift operator $Ry(t)$ by $Ry(t) = (x(t+1), u(t))^{tr}$, where tr denotes the vector transpose. Then (1.1) can be written in its discrete linear Hamiltonian form, see [28],

$$J\triangle y(t) = [zW(t) + P(t)]Ry(t), \tag{2.4}$$

where $W(t)$ and $P(t)$ are 4×4 complex Hermitian matrices, $W(t) = \operatorname{diag}(w(t), 0, 0, 0),\ x(t), u(t) \in \mathbb{C}^2,\ J$ is a canonical symplectic matrix, that is,

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}.$$

The nonzero elements of the 2×2 matrices A, B and C are given by

$$A_{1,2} = 1, \quad A_{2,2} = \frac{iq_2}{p_2}, \quad B_{2,2} = p_2^{-1},$$

$$C_{1,1} = p_0, \quad C_{1,2} = iq_1, \quad C_{2,1} = -iq_1, \quad C_{2,2} = p_1 - \frac{q_2^2}{p_2}.$$

Let $\ell_w^2(I)$ be a Hilbert space with weight function w and define this Hilbert space using the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$\ell_w^2(I) = \left\{ y : y = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C}^4 \ \text{ and } \sum_{t=0}^{\infty} (Ry^*)(t)W(t)(Ry)(t) < \infty \right\}.$$

Then the scalar product for the vector valued functions of the system is [28]

$$\sum_{t=0}^{\infty} \overline{y_1}(t+1)w(t)y(t+1) = \langle y_1, y \rangle_w, \quad y, y_1 \in \ell_w^2(I).$$

The system (2.4) will be assumed to satisfy some regularity conditions. There exists t_0 such that for all non-trivial solutions $y(., z)$ of (2.4) and for all $z \in \mathbb{C}$

$$\sum_{s \geq t_0} (Ry(s,z))^* W(t) Ry(s,z) > 0, \quad s \geq t_0, \tag{2.5}$$

$I_2 - A(t)$ is invertible and this ensures the existence and uniqueness of the solutions of any initial value problem for (1.1). In particular, this will only be required for the first half components of elements in $\ell_w^2(I)$.

It is well known that if definiteness conditions corresponding to

$$Jy'(t) = (P(t) + zW(t))y(t), \quad t \in [0, \infty),$$

is satisfied, then the minimal operator generated by this continuous system is densely defined and the maximal operator is well defined [2], [20], [21]. In this case the defect index of the minimal operator is equal to the number of linearly independent square integrable solutions [20], [21]. But if this corresponding definiteness condition is not satisfied, the minimal operator may not be densely defined and the maximal operator may be multivalued [2], [20], [21]. However, in the case of the discrete systems, the minimal operator may fail to be densely defined even if the condition (2.5) is satisfied and this is an important difference between differential and difference equations. Due to these technical difficulties, the spectral properties for difference equations have not been studied that much compared to differential equations.

Since the minimal operator generated by (1.1) may be neither densely defined nor single valued, its maximal operator may not be well defined and thus the selfadjoint extension operator for minimal operator cannot be discussed by application of von Neumann theory for densely defined Hermitian operators. This therefore requires the theory of Hermitian subspaces where the von Neumann theory has been extended in order to discuss the selfadjoint extension of the minimal Hermitian subspaces and for more details see [25], [26], [27], [29] and the references cited therein.

Let $\mathcal{M}$ be a linear subspace or a linear relation in $\ell^2_w(I) \times \ell^2_w(I)$, where the domain, range and kernel of $\mathcal{M}$ are defined by

$$D(\mathcal{M}) = \big\{y \in \ell^2_w(I) \ : \ (y, g) \in \mathcal{M} \ \text{ for some } \ g \in \ell^2_w(I)\big\},$$
$$R(\mathcal{M}) = \big\{g \in \ell^2_w(I) \ : \ (y, g) \in \mathcal{M} \ \text{ for some } \ y \in \ell^2_w(I)\big\},$$
$$K(\mathcal{M}) = \big\{y \in \ell^2_w(I) \ : \ (y, 0) \in \mathcal{M}\big\}$$

and finally define

$$\mathcal{M} - zI = \{(y, g - zy) \ : \ (y, g) \in \mathcal{M}\}$$

and

$$\mathcal{M}^* = \big\{(y, g) \in \ell^2_w(I) \times \ell^2_w(I) : \ \ \langle y, f\rangle = \langle g, x\rangle, \ \ \forall (x, f) \in \mathcal{M}\big\}$$

so that $\mathcal{M} \subset \ell^2_w(I) \times \ell^2_w(I)$ is called a Hermitian subspace if $\mathcal{M} \subset \mathcal{M}^*$. Now define $\dim(R(\mathcal{M} - zI))^\perp$ as the defect index of $\mathcal{M}$ and z. But since $R(\mathcal{M} - zI)^\perp = K(\mathcal{M}^* - \overline{z}I)$ the defect indices of $\mathcal{M}$ and its closure with respect to the same z are equal. We will denote

$$\operatorname{def} \mathcal{M} = \dim_\pm(\mathcal{M}) = \dim_{\pm i}(\mathcal{M}) = (\mathcal{N}_+, \mathcal{N}_-)$$

as positive and negative defect indices of $\mathcal{M}$.

Now define two semidefinite scalar product spaces

$$\ell(I) = \big\{y : \ y = \{y(t)\}_{t=a}^{\infty} \subset \mathbb{C}^4\big\}$$

and

$$\mathcal{L}^2_W(I) = \left\{ y \in \ell(I) : \ \sum_{t \in I} R^*(y)(t) W(t) R(y)(t) < \infty \right\}$$

with the semidefinite scalar product

$$\langle y_1, y_2\rangle = \sum_{t\in} R^*(y_2)(t) W(t) R(y_1)(t).$$

Then it follows that $\|y\| = (\langle y, y\rangle)^{\frac{1}{2}}$ for $y \in \mathcal{L}^2_W(I)$. Since $W(t)$ may be singular in I, $\|.\|$ is semi-norm. One thus defines a quotient space

$$L^2_W(I) = \mathcal{L}^2_W(I)/\big\{y \in \mathcal{L}^2_W(I) : \ \|y\| = 0\big\}.$$

It is therefore true that $L^2_W(I)$ is a Hilbert space with an inner product $\langle ., .\rangle$. For a function y which is a solution of (1.1) and is summable, denote by $\tilde{y}$ the corresponding class in $L^2_W(I)$ and for any $\tilde{y} \in L^2_W(I)$ by $y \in \mathcal{L}^2_W(I)$ denote a representative of $\tilde{y}$. It is evident that $\langle \tilde{y_1}, \tilde{y_2}\rangle = \langle y_1, y_2\rangle$ for any $\tilde{y_1}, \tilde{y_2} \in L^2_W(I)$. Now let π be a natural quotient map such that

$$\pi : \mathcal{L}^2_W(I) \longrightarrow L^2_W(I), \quad y \longrightarrow \tilde{y}.$$

Then π is surjective and not injective in general. One defines the natural difference operator corresponding to (1.1) by

$$\mathcal{L}(y)(t) = J\Delta y(t) - P(t)R(y)(t). \tag{2.6}$$

Further, we define

$$\mathcal{L}^2_{W\,0}(I) = \big\{y \in \mathcal{L}^2_W(I): \ \text{there exists two integers } \ s,k \in I \\ \text{with } \ s \le k \ \text{ such that } \ y(t) = 0 \ \text{ for } t \le s \text{ and } \ t \ge k+1\big\}$$

and

$$\tilde{H} = \big\{(\tilde{y},\tilde{g}) \in L^2_W(I) \times L^2_W(I): \ \text{there exists } y \in \mathcal{L}^2_W(I) \\ \text{such that } \ \mathcal{L}y(t) = W(t)R(g)(t), \ \ t \in I\big\},$$

$$H_{00} = \big\{(\tilde{y},\tilde{g}) \in \tilde{H}: \ \text{there exists } y \in \tilde{y} \ \text{ such that } \ y \in \mathcal{L}^2_{W\,0}(I) \\ \text{and } \ \mathcal{L}y(t) = W(t)R(g)(t), \ \ t \in I\big\}.$$

Then $\tilde{H}$ and H_{00} are both linear subspaces in $L^2_W(I) \times L^2_W(I)$. $\tilde{H}$ and H_{00} are called maximal and preminimal subspaces corresponding to $\mathcal{L}$, the natural difference operator generated by (1.1), respectively. $H_0 = \overline{H_{00}}$ is the minimal subspace corresponding to $\mathcal{L}$ in $L^2_W(I) \times L^2_W(I)$.

It follows that $H_{00} \subset \tilde{H} \subset H^*_{00}$ and consequently H_{00} is a Hermitian subspace in $L^2_W(I) \times L^2_W(I)$. It has been proved in [26] that the adjoint of preminimal subspace is the maximal subspace and thus $H^*_{00} = H^*_0 = \tilde{H}$.

In order to discuss the selfadjoint extension subspace for H_0, one needs that the system (2.4) satisfies definiteness condition. This will be stated as follows:

[A]: There exists a finite subinterval $I_1 \subset I$ such that for any $z \in \mathbb{C}$ and for any non-trivial solution $y(t)$ of (1.1) the following always holds:

$$\sum_{t \in I_1} R(y)^*(t)W(t)R(y)(t) > 0.$$

We need to point out here that this assumption has slight variation from that stated in (2.5). The definiteness condition A together with the fact that $I_2 - A(t)$ is invertible now guarantees the existence of a unique solution for (2.4). If $z \in \mathbb{C}\backslash\mathbb{R}$, G. Ren and Y. Shi [26] have shown that the dimension of the defect space of H_0 and also of H_{00} are equal to the number of linearly independent square summable solutions of (1.1) or (2.4). Assume that the defect index of H_0 are equal, that is, $\operatorname{def} H_0 = (n,n)$, where $2 \le n \le 4$, then H_0 has selfadjoint extension subspace in $L^2_W(I) \times L^2_W(I)$ denoted by H if there exists matrices $\alpha_1, \alpha_2 \in \mathbb{C}^{2\times 2}$ such that

$$\operatorname{rank}(\alpha_1,\alpha_2) = 2, \quad \alpha_1 \Phi^{tr}_a \alpha^*_1 = \alpha_2 \Phi^{tr}_\infty \alpha^*_2,$$

with $\Phi_a = ((y_1,y_2)(a-1))_{2\times 2}$ and Φ_∞ defined in a similar way though with the boundary condition fixed at limiting point, that is, as $t \to \infty$. These are called the selfadjoint boundary conditions. y_1 and y_2 are square summable solutions of (1.1). Then H is defined by

$$H = \left\{(\tilde{y},\tilde{g}) \in \tilde{H}: \ \alpha_1 \begin{pmatrix} (\tilde{y},y_1)(a-1) \\ (\tilde{y},y_2)(a-1) \end{pmatrix} + \alpha_2 \begin{pmatrix} (\tilde{y},y_1)(\infty) \\ (\tilde{y},y_2)(\infty) \end{pmatrix} = 0\right\}. \tag{2.7}$$

For more details see [29, Theorem 5.9] and Section 5 of the same reference in general.

It has been shown in [26] that $\operatorname{def} H_0$ is independent of the half-planes if z is nonreal. In that case, H_0 has a selfadjoint extension subspace H defined by (2.7). Moreover, if a closed Hermitian subspace has equal finite defect indices, then all its selfadjoint extension subspaces have the same essential spectrum [26]. For point spectrum of these subspaces, every isolated point of the spectrum of selfadjoint subspace is an eigenvalue of the subspace and therefore constitutes the point spectrum. Only those eigenfunctions that lose their square summability as $\operatorname{Im} z \to 0$ contribute to absolutely continuous spectrum.

The following theorem which is from [27, Thm. 4.1] gives a relationship between the spectral properties of a selfadjoint extension subspace to those of the corresponding selfadjoint extension operator if the minimal operator generated by (1.1) is densely defined and the maximal operator is single-valued. It is therefore considered bridging result between the spectral results of subspace theory to those of selfadjoint extension operators and its proof can be obtained from the same reference.

Theorem 2.1 *If H is a selfadjoint extension subspace in $L^2_W(I) \times L^2_W(I)$ and H_s is the selfadjoint operator defined in the subspace H, then*

$$\sigma_p(H) = \sigma_p(H_s), \quad \sigma_{ac}(H) = \sigma_{ac}(H_s), \quad \sigma_{ess}(H) = \sigma_{ess}(H_s).$$

Therefore to determine the number of solutions of (1.1) that are square summable, we use asymptotic summation and that requires (2.4) to be converted into its first order system. In this case, the first order system is of the form

$$\begin{aligned} \begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} &= S(t,z) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} E & EB \\ \hat{C}E & I - A^* + \hat{C}EB \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \end{aligned} \tag{2.8}$$

Here, $E = (I_2 - A)^{-1}$, $\hat{C} = C - z\mathrm{diag}(w, 0)$ and

$$S(t,z) = \begin{bmatrix} 1 & \frac{1}{1-\frac{iq_2}{p_2}} & 0 & \frac{1}{p_2 - iq_2} \\ 0 & \frac{1}{1-\frac{iq_2}{p_2}} & 0 & \frac{1}{p_2 - iq_2} \\ p_0 - zw & \frac{p_0 - zw + iq_1}{1-\frac{iq_2}{p_2}} & 1 & \frac{p_0 - zw + iq_1}{p_2 - iq_2} \\ -iq_1 & \frac{p_1 - iq_1 - \frac{q_2^2}{p_2}}{1-\frac{iq_2}{p_2}} & -1 & 1 + \frac{iq_2}{p_2} + \frac{p_1 - iq_1 - \frac{q_2^2}{p_2}}{p_2 - iq_2} \end{bmatrix}.$$

The aim of this paper is the analysis of the absolutely continuous spectrum of the subspace H, $\sigma_{ac}(H)$, and its multiplicity. For this, we will determine the absolutely continuous spectrum via the M-function. This in turn requires the asymptotics of the solutions of (1.1). Since it has been shown in [23] that the absolutely continuous spectrum and its spectral multiplicity are independent of the boundary conditions and the left regular endpoints, we will pay little attention to the left regular endpoints.

In order to do the analysis of the solutions of (1.1), we will need to solve the first order system (2.8). There are other ways of expressing (1.1) into its first order system and (2.8) is just but one of them. To determine the eigenfunctions of (1.1) respectively (2.8), we need to calculate the eigenvalues of $S(t,z)$. This demands that we calculate the associated characteristic polynomial of $S(t,z)$. From $\mathcal{P}(t,\lambda,z) = det(S(t,z) - \lambda.I_4) = 0$, multiplying both sides by $(p_2 - iq_2)\lambda^{-2}$ and factorising the resulting polynomial so that if λ is a root then $\overline{\lambda}^{-1}$ is also a root, one obtains

$$\begin{aligned} \frac{(p_2 - iq_2)}{\lambda^2} \mathcal{P}(t,\lambda,z) &= \sum_{k=0}^{2} p_k (1-\lambda)^k \left(1 - \lambda^{-1}\right)^k \\ &\quad + \sum_{j=1}^{2} q_j (1-\lambda)^{j-1} \left(1 - \lambda^{-1}\right)^{j-1} \left(i\lambda + (i\lambda)^{-1}\right) - zw. \end{aligned} \tag{2.9}$$

(2.9) is a polynomial of degree four. Though there exists a closed form for determining the roots of (2.9), because the coefficients p_k and q_j are permitted to be unbounded, a use of this closed form formula will lead to a lengthy computation with alot of approximation. We there opt to use some approximation techniques by imposing some conditions on the coefficients as done in [15, Sect. 3.3] in order to obtain the roots. The existence of such roots will be discussed later but they usually follow from an iteration procedure and the convergence of these iterates by simply applying Banach fix point theorem. In order to make the above polynomial a real polynomial, we apply a fractional linear approximation or Möbius transformation which transform the interior of a unit disc onto the left-hand plane while the unit circle on to the imaginary axis with the exterior of the unit disc onto the right hand plane. Therefore let

$$\lambda = \frac{is+1}{is-1} \tag{2.10}$$

and substitute this in (2.9) to obtain the Fourier transformed polynomial of the form

$$Q_4(s,z) = (p_0 - zw)\big(s^2+1\big)^2 + 4q_1\big(s^2+1\big)s + 4p_1\big(s^2+1\big) + 16q_2 s + 16p_2. \tag{2.11}$$

Thus (2.11) will be used to solve for the roots s respectively λ of Q_4.

Once the roots are determined, the system (2.8) needs to be transformed into Levinson-Benzaid-Lutz form or LBL-form [11]

$$y(t+1,z) = (\Lambda(t,z) + R(t,z))y(t,z), \tag{2.12}$$

where $\Lambda(t,z) = \operatorname{diag}(\lambda_i(t,z)),\ i = 1,\dots,4$, and $\lambda_i^{-1}(t,z)R_{ij}(t,z) \in \ell^1$. In order to achieve LBL-form and because of our assumptions in (2.1), we will need two diagonalisations and some $(I+Q)$ -transformations to reduce the remainder term $R(t,z)$ in (2.12) into summable terms. For more details on these transformations, see [4], [8]. In consideration of the resultant polynomial $Q_4(s,z)$ and its discriminant $\partial_s Q_4(s,z)$, one can show that there are only finitely many spectral values z for which $Q_4(s,z)$ has multiple roots, for more details, see [3]. Let $\omega_1 < \omega_2 < \dots < \omega_k$ denote all the real spectral values z leading to multiple roots. Following [3], the analysis will be restricted to small complex neighbourhoods of $z_0 \in (\omega_i, \omega_{i+1})$, $i = 0,\dots,k$, where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a given $z \in (\omega_i,\omega_{i+1})$, one can choose $\epsilon > 0$ and $a > 0$ such that $Q_4(s,z) = 0$ has no multiple or double roots for any

$$z \in \mathcal{K}_\epsilon(z_0) = \{z \mid |z - z_0| \le \epsilon,\ \ \operatorname{Im}(z) > 0\} = \mathcal{K},$$

and $t \ge 0$. This is possible because for any $z \in \mathcal{K} \cap \mathbb{R}$ the roots s of $Q_4(s,z)$ depend analytically on the coefficients p_k, q_j and the spectral parameter z. Throughout the study, it may be necessary to adjust a and ϵ repeatedly. This will be done without mentioning.

Since $Q_4(s,z)$ has 4 distinct s roots, $\mathcal{P}(t,\lambda,z)$ also has four distinct λ roots and therefore one can determine the corresponding eigenvectors. In the ideal situation, a time shift corresponds to a multiplication by λ. Somewhat, more generally, one has

$$\Delta \longrightarrow (\lambda - 1), \quad \hat{y}(t+k) \longrightarrow \lambda^k, \quad k \in \mathbb{Z}.$$

Here $\rightarrow$ means "replace by" or "corresponds to". With this and (2.3), one obtains

$$\begin{aligned} x_i &= (\lambda-1)^{i-1}\lambda^{-1}, \quad i = 1,2,\\ u_1 &= \sum_{l=1}^{2}(-1)^{l-1}(\lambda-1)^{2l-1}\big(p_l\lambda^{-l}\big)\\ &\quad - i\sum_{l=2}^{2}(-1)^{l-1}(\lambda-1)^{2l-2}\lambda^{-l}(q_l\lambda + q_l) - iq_1,\\ u_2 &= p_2(\lambda-1)^2\lambda^{-2} - iq_2(\lambda-1)\lambda^{-1}, \end{aligned} \tag{2.13}$$

as the components of the eigenvectors $v_\lambda = (x_1,x_2,u_1,u_2)^{tr}$, where tr means vector transpose. It is well known that the matrix $T(t)$ formed with the eigenvectors as columns will diagonalise $S(t,z)$. One thus transforms the system by

$$y(t,z) = T(t,z)v(t,z).$$

This results into a system of the form

$$\begin{aligned} v(t+1,z) &= T^{-1}(t+1,z)S(t,z)T(t,z)v(t,z)\\ &= (\Lambda + R)(t,z)v(t,z), \end{aligned} \tag{2.14}$$

where $R(t,z) = -T^{-1}(t+1,z)\Delta T(t,z)\Lambda(t,z)$.

The expression $R(t,z)$ consists of ℓ^2 terms and can be diagonalised again. Using the results of [4, Sect. 2.3], one diagonalises (2.14) using the matrix $[I + B(t)]$, $B(t) = o(1)$, formed with the eigenvectors of $[\Lambda + R](t)$. Here, one has

$$B_{ij} = (\lambda_j - \lambda_i)^{-1} R_{ij},\ i \neq j, \quad \text{and} \quad B_{ii} = 0.$$

Any remainder terms that are non-summable can be reduced to summable terms using $(I+Q)$-transformations [5], [7]. By application of Levinson-Benzaid-Lutz Theorem [11], the solutions of (1.1) respectively (2.4) are in the form

$$y_i(t,z) = (e_i + r_{ii}(t,z)) \prod_{l=t_0}^{t-1} \lambda_i(l,z), \quad r_{ii}(t,z) \in \ell^1, \quad i = 1\ldots,4,$$

where e_i is the normalised eigenvector. The term $r_{ii}(t,z)$ is the perturbing term from the remainder matrix $R(t,z)$.

Just like in the differential equations [6, Sect. 3], one can also employ the method of the M-matrix to obtain the result on the spectrum of H even in bounded and unbounded coefficients case. In [23], Remling has proposed two methods to determine Im $M(z)$. It is only the first one which is based on [23, (8)] that will be useful in our case and is given by

$$\langle F(.,z), F(.,z')\rangle(\overline{z} - z') = M_\alpha^*(z) - M_\alpha(z'). \tag{2.15}$$

Here $F(.,z)$ is the 2 by 4 system of square integrable solutions satisfying the boundary conditions (2.7) at 0 and the (i,j) matrix element is formed with the i-th and j-th vector. The method of proof shows that it is valid for discrete systems as well. Indeed, the discrete analogue is Theorem 6.3 in [28]. The second method is stated in [24] and has been used extensively in [6]. Nonetheless (2.15) is preferable, since it yields Im$M(z)$ for $z = z_0 \in \mathbb{R}$ directly as

$$\operatorname{Im} M(z_0) = \lim_{\eta \to 0^+} \operatorname{Im} M(z_0 + i\eta) = \lim_{\eta \to 0^+} \eta \langle F(., z_0 + i\eta), F(., z_0 + i\eta)\rangle. \tag{2.16}$$

If the boundary condition (2.7) does not give rise to a bound state, the limit on the right hand side exists boundedly and defines a continuous function of z_0.

3 Bounded coefficients

As in [3], define the coefficients p_k and q_j, $k = 0,1,2$, $j = 1,2$, to be almost constant if there exists constants c_k and d_j such that

$$p_k - c_k, \quad q_j - d_j \in \ell^1, \quad \text{as} \quad t \longrightarrow \infty.$$

One now has the following results:

Theorem 3.1 *Let p_k and q_j, $k = 0,1,2$, $j = 1,2$, be bounded and assume that* (2.1), (2.2) *as well as condition A are satisfied. Then*

(i) *If $q_1 = q_2 = 0$ and $p_1^2 < 4p_2(p_0 - zw)$, def $H_0 = (2,2)$ and $\sigma(H)$ is pure discrete.*

(ii) *If $w \to \infty$ as $t \to \infty$, then $\sigma(H)$ is pure discrete.*

(iii) *Assume all the coefficients p_k and q_j are almost constant and moreover, assume the limiting characteristic polynomial $\mathcal{P}_{4,0}(\lambda, z)$ of* (2.9) *has $2l$ zeroes of absolute value one $(0 \leq l \leq 2)$, then the selfadjoint extension subspace H has no singular continuous spectrum and the absolutely continuous spectrum $\sigma_{ac}(H)$ agrees with that of the constant coefficient limiting subspace with a spectral multiplicity of l.*

Proof.

(i) Assume that $q_1 = q_2 = 0$ then the polynomial (2.11) is a well-known biquadratic polynomial which can be solved explicitly. If $p_1^2 < 4p_2(p_0 - zw)$, then the discriminant $D(Q_4(s, z)) < 0$ implying that one has two pairs of roots which are in complex conjugate pairs. Assume the s roots are $\alpha_r \pm i\beta_r,\ \ r = 1, 2,$ then the roots $\alpha_r + i\beta_r$, $\beta > 0$, will lead to λ roots with absolute values less than 1 which eventually will lead to eigensolutions $y(t, z)$ which are square summable while those s-roots of the form $\alpha_r - i\beta_r$, $\beta > 0$, will eventually lead to non-square summable solutions. Thus the deficiency indices of the minimal subspace H_0 is $(2, 2)$, the minimal subspace is limit point at infinity and the eigenfunctions that are square summable are z-uniformly square summable. No eigenfunction in this case loses its square summability as $\operatorname{Im} z \to 0^+$ in order to contribute to absolutely continuous spectrum. It follows that absolutely continuous spectrum is absent and since by assumption singular continuous spectrum is absent, this leads only to discrete spectrum.

(ii) On the other hand, if all the coefficients are bounded and $w \to \infty$ as $t \to \infty$, then we have a compact resolvent. This can be shown by analysis of the corresponding Jacobi matrices since the Hamiltonian system will have a dominant diagonal.

(iii) Now consider $\mathcal{P}(t, \lambda, z)$ in (2.9) and assume that the coefficients are almost constant. If λ is a root, then $\overline{\lambda}^{-1}$ is also a root. By results of [8], those roots λ such that $|\lambda| > 1,\ \ \left|\overline{\lambda}^{-1}\right| < 1$ lead to non-square summable and square summable solutions regardless of the dichotomy condition. Thus only those roots such that $|\lambda| = \left|\overline{\lambda}^{-1}\right| = 1$ lead to eigenfunctions of which half of them lose their square summability as $\operatorname{Im} z \searrow 0$. It is these eigenfunctions that lose their square summability that contribute to absolutely continuous spectrum. The number of the eigenfunctions with such behaviour will correspond to the rank of the M-matrix and is therefore equal to the spectral multiplicity. For more details, see [8]. □

Corollary 3.2 *Assume that the minimal operator generated by* (1.1) *is densely defined and its corresponding maximal operator is not multivalued. Moreover, assume that all the conditions in Theorem* 3.1 *are satisfied, then the minimal difference operator L generated by* (1.1) *has selfadjoint extension H_s with spectral results as given in Theorem* 3.1.

The result of the above corollary are now similar to those in [8] for the case of $n = 2$. Its proof follow at once from Theorem 2.1 and Theorem 3.1.

4 Unbounded coefficients

In this section and the next section, Section 5, unless stated otherwise we will write p_0 to mean $p_0 - zw$. This is only to simplify the notations. As such we absorb zw into p_0. Now simplify (2.11) as follows

$$\begin{aligned} Q(s, z) = p_0 s^4 + 4q_1 s^3 + (2p_0 + 4p_1)s^2 \\ + (4q_1 + 16q_2)s + (p_0 + 4p_1 + 16p_2) = 0, \end{aligned} \tag{4.1}$$

and allow the coefficients to be unbounded. For us to approximate the s-roots of (4.1) we make the following assumptions on the coefficients

$$w = 1,\ \ p_0, p_2, q_1, q_2 = o(p_1) \quad \text{and} \quad |\, p_1 \,| \nearrow \infty, \quad \text{as} \quad t \nearrow \infty,$$

and

$$\left|\frac{2p_0 + 4p_1}{p_0}\right| \gg \left|\frac{p_0 + 4p_1 + 16p_2}{4p_1 + 2p_0}\right|. \tag{4.2}$$

Then apply the approach used in Eastham [15, Sect. 3.3] and [9]. For the regularity and smoothness condition, we will need that the coefficients are thrice differentiable with

$$\begin{aligned} &f^{-1}(\Delta(f)),\ f\Delta\big(p_1^{-1}\big) \in \ell^2, \quad f\Delta^2\big(p_1^{-1}\big),\ \big(f\Delta\big(p_1^{-1}\big)\big)^2 \in \ell^1, \\ &\Delta^2(f) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \quad f = w, p_0, p_1, p_2, q_1, q_2, . \end{aligned} \tag{4.3}$$

Because of the assumptions in (4.2), the s-roots can be put in different clusters or groups depending on their magnitudes. In this case, $\{p_0, 4q_1, 2p_0 + 4p_1\}$ and $\{2p_0 + 4p_1, 4q_1 + 16q_2, p_0 + 4p_1 + 16p_2\}$ will form the two cluster coefficients that will determine two clusters of eigenvalues with different magnitudes. For this kind of clustering to make sense, we will assume throughout this section that

$$p_0, \quad 2p_0 + 4p_1 \quad \text{and} \quad p_0 + 4p_1 + 16p_2$$
$$\text{are nowhere zero in the discretised interval } [0, \infty) \tag{4.4}$$

and hence form pivotal coefficients of the polynomial $Q_4(s, z)$ in (4.1).

Lemma 4.1 *Assume* (4.2) *and* (4.4)*, then the polynomial* (4.1) *can be written in terms of two polynomials*

$$Q_2(s, z) = s^2 + 4\frac{q_1}{p_0}s + \frac{(2p_0 + 4p_1)}{p_0} + R_2(s) \tag{4.5}$$

and

$$Q_1(s, z) = s^2 + \frac{(4q_1 + 16q_2)}{(2p_0 + 4p_1)}s + \frac{(p_0 + 4p_1 + 16p_2)}{(2p_0 + 4p_1)} + R_1(s), \tag{4.6}$$

where

$$R_2(s) = p_0^{-1}(4q_1 + 16q_2)s^{-1} + p_0^{-1}(p_0 + 4p_1 + 16p_2)s^{-2}$$

and

$$R_1(s) = p_0 s^4 (2p_0 + 4p_1)^{-1} + 4q_1 s^3 (2p_0 + 4p_1)^{-1}.$$

$R_i(s) = o\big(s_{i\pm}^2\big)$, $i = 1, 2$, *and are very small in the absolute value sense.*

P r o o f. In order to prove this result, it suffices to show that $R_i(s) = o\big(s_{i\pm}^2\big)$, $i = 1, 2$. Thus by assuming (4.2) to hold, write

$$\begin{aligned} R_1(s) &= \big|p_0 s_{1\pm}^4 (2p_0 + 4p_1)^{-1} + 4q_1 s_{1\pm}^3 (2p_0 + 4p_1)^{-1}\big| \\ &\le |p_0||2p_0 + 4p_1|^{-1}|s_{1\pm}|^4 + |4q_1||2p_0 + 4p_1|^{-1}|s_{1\pm}|^3. \end{aligned}$$

Considering the first term on the right-hand side, one has $|p_0||2p_0 + 4p_1|^{-1}|s_{1\pm}|^4 \approx |s_{2\pm}|^{-2}|s_{1\pm}|^4 \approx o\big(s_{1\pm}^2\big)$. For the second term on the right-hand side, one has

$$\begin{aligned} |4q_1||2p_0 + 4p_1|^{-1}|s_{1\pm}|^3 &\approx \big|2p_0^2 + 4p_1 p_0\big|^{\frac{1}{2}} |2p_0 + 4p_1|^{-1}|s_{1\pm}|^3 \\ &\approx |p_0|^{\frac{1}{2}} |2p_0 + 4p_1|^{-\frac{1}{2}} |2p_0 + 4p_1||2p_0 + 4p_1|^{-1}|s_{1\pm}|^3 \\ &\approx |s_{2\pm}|^{-1}|s_{1\pm}|^3 \\ &\approx o\big(s_{1\pm}^2\big). \end{aligned}$$

To show that $R_2(s)$ is small in the absolute value sense, consider

$$\begin{aligned} |R_2(s)| &= \big|(p_0 + 4p_1 + 16p_2)p_0^{-1}s_{2\pm}^{-2} + (4q_1 + 16q_2)p_0^{-1}s_{2\pm}^{-1}\big| \\ &\le |p_0 + 4p_1 + 16p_2|\big|p_0^{-1}\big|\big|s_{2\pm}^{-2}\big| + |4q_1 + 16q_2|\big|p_0^{-1}\big|\big|s_{2\pm}^{-1}\big|. \end{aligned}$$

Simplify the first term to the right-hand side as follows

$$\begin{aligned} &|p_0 + 4p_1 + 16p_2|\big|p_0^{-1}\big| \mid p_0 \mid |2p_0 + 4p_1|^{-1} \\ &\quad = |p_0 + 4p_1 + 16p_2||2p_0 + 4p_1|^{-1}|2p_0 + 4p_1||p_0|^{-1} \cdot |p_0||2p_0 + 4p_1|^{-1} \\ &\quad \approx |s_{1\pm}|^2.|s_{2\pm}|^{-2}.|s_{2\pm}|^2 \approx o\big(s_{2\pm}^2\big) \end{aligned}$$

and rewrite the second term as

$$
\begin{aligned}
&|4q_1 + 16q_2||p_0|^{-1}|p_0|^{\frac{1}{2}}|2p_0 + 4p_1|^{-\frac{1}{2}} \\
&\quad \approx |2p_0 + 4p_1|^{\frac{1}{2}}|p_0|^{-\frac{1}{2}}|p_0 + 4p_1 + 16p_2|^{\frac{1}{2}} \cdot |2p_0 + 4p_1|^{-\frac{1}{2}}|p_0|^{\frac{1}{2}}|2p_0 + 4p_1|^{-\frac{1}{2}} \\
&\qquad \cdot |2p_0 + 4p_1|^{\frac{1}{2}}|p_0|^{-\frac{1}{2}} \\
&\quad \approx |s_{2\pm}|.|s_{1\pm}|.|s_{2\pm}|^{-1}|s_{2\pm}|.
\end{aligned}
$$

It follows that $R_i(s) = o\big(s_{i\pm}^2\big)$ as claimed. □

Theorem 4.2

(a) *To each root s of $Q(s,z)$, there exists a unique root s_j of $Q_j(s,z)$, $j = 1,2$, such that* $|\ s - s_j\ | \le o(1)$.

(b) *s is real if and only if s_j is real. Similarly, s is non-real if and only if s_j is non-real.*

Here, a is choosen so large so that all the eigenvalues of $Q(s,z)$ are distinct and satisfy $|\ s - s_j\ | \le o(1)$.

The proof of this theorem follows immediately from [10, Lemma 3.3] and [22, Thm. 2.2.4]. One can also use Kantorovic theorem [1] to show the existence of the roots of $Q(s,z)$ near the roots of $Q_j(s,z)$, $j = 1,2$.

Lemma 4.1 and Theorem 4.2 now implies that the s-roots of $Q(s,z)$ are approximately given by

(i) $s_{1\pm} \approx \pm 2i\big(\frac{p_1}{p_0}\big)^{\frac{1}{2}}$ and

(ii) $s_{2\pm} \approx \big(-\big(\frac{16p_2+4p_1+p_0}{4p_1+2p_0}\big)\big)^{\frac{1}{2}} \approx \pm i(1+b)^{\frac{1}{2}}$ where $b = \frac{16p_2}{4p_1+2p_0}$.

which results from an iteration procedure applied on $Q_1(s,z)$ and $Q_2(s,z)$. In this case, one picks a root s_j of these polynomials which are close enough to an s-root of the polynomial $Q(s,z)$. In that way, one obtains infinitely many iterates that will eventually converge to the desired root. The uniqueness of this limit point will follow at once from Banach fix point theorem.

In addition to (4.2) and (4.3), assume that the roots are non-degenerate. Using the relation (2.10) one thus obtains the four λ-roots of the polynomial $\mathcal{P}(t,\lambda,z)$ which can be approximated as

$$
\begin{aligned}
\lambda_1 &\approx 1 + \left(\frac{p_0}{p_1}\right)^{\frac{1}{2}} + \frac{p_0}{2p_1} + \frac{p_0^{\frac{3}{2}}}{4p_1^{\frac{3}{3}}} + O\big(p_1^{-2}\big), \\
\lambda_2 &\approx 1 - \left(\frac{p_0}{p_1}\right)^{\frac{1}{2}} + \frac{p_0}{2p_1} - \frac{p_0^{\frac{3}{2}}}{4p_1^{\frac{3}{3}}} + O\big(p_1^{-2}\big), \\
\lambda_3 &\approx \frac{p_1}{p_2} + \frac{p_0}{4p_2} + 1, \quad \lambda_4 \approx \frac{4p_2}{4p_1 + 2p_0}.
\end{aligned}
\tag{4.7}
$$

Once the approximate values of the roots of (4.1) are known, one uses Levinson-Benzaid-Lutz Theorem [11] to obtain the eigenfunctions of (2.4) respectively (1.1) by asymptotic summation. In order to do this, one has to establish that the roots of (4.1) or the eigenvalues of the matrix $S(t,z)$, satisfy the z-uniform dichotomy condition. The Levinson-Benzaid-Lutz Theorem states that the solutions of a system

$$
y(t+1) = [\Lambda(t) + R(t)]y(t), \tag{4.8}
$$

where $\Lambda(t)$ is diagonal and invertible, look like the solutions of the unperturbed system $y(t+1) = \Lambda(t)y(t)$, if $R(t)$ is sufficiently small and $\Lambda(t) = \operatorname{diag}(\lambda_i(t))$ satisfy a dichotomy condition. In the Levinson-Benzaid-Lutz Theorem [11], small means absolutely summable, that is, $\lambda_i^{-1}R(t) \in \ell^1$, forall i. The dichotomy in this case amounts to: for any pair of indices i and j such that $i \ne j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t)| \ge \delta$ for all $t \ge t_0$, then either $\big|\frac{\lambda_i(t)}{\lambda_j(t)}\big| \ge 1$ or $\big|\frac{\lambda_i(t)}{\lambda_j(t)}\big| \le 1$ for large t. As in the continuous case, in the spectral theory of the difference operators, the matrix elements and $\lambda_i(t)$ will generally depend also on the spectral parameter z. Thus one writes $\lambda_i = \lambda_i(t,z)$ for this. In our case, we need Levinson-Benzaid-Lutz Theorem which is proved uniformly in z. For this version, see [22, Thm. 1.2.2] for more details.

To simplify the analysis of the uniform dichotomy condition, Behncke [5, Thm. 5.1] has shown that those roots λ_k, such that $|\lambda_k| > 1$ and $\left|\lambda_k^{-1}\right| < 1$ lead to non-square summable and square summable eigenfunctions respectively regardless of the uniform dichotomy condition. It thus suffices to check the uniform dichotomy condition only for the λ- roots of $\mathcal{P}(t,\lambda,z)$ with $|\lambda| = 1$.

Lemma 4.3 *Let $z \in \mathcal{K}$ such that $\operatorname{Im} z > 0$ and assume that the λ-roots of $\mathcal{P}(t,\lambda,z)$ are distinct. Then the λ-roots of $\mathcal{P}(t,\lambda,z)$ satisfy the uniform dichotomy condition. Moreover, if the coefficients are almost constant, then assume that $\frac{ds_{j+}}{dz} \neq \frac{ds_{j-}}{dz}$ and again the uniform dichotomy condition is satisfied.*

Proof. Assume the coefficients are unbounded. Then by [5, Thm. 5.1], we only need to check the uniform dichotomy condition between λ_1 and λ_2 since λ_3 and λ_4 will lead to non-square summable and square summable eigenfunctions respectively regardless of the uniform dichotomy condition. But as $t \to \infty$ $|\lambda_1| \approx |\lambda_2| \approx 1$. In this case we need to check on the effect of the spectral parameter z. Now let $z = z_0 + i\eta$ where $z_0 \in \mathbb{R}$ and $\eta > 0$ is small. Then by absorbing $-z_0$ into p_0, we can write

$$\lambda_1 \approx 1 + \frac{p_0^{\frac{1}{2}}}{p_1^{\frac{1}{2}}} - \frac{i\eta}{(p_0p_1)^{\frac{1}{2}}} + O\big(p_1^{-1}\big) \quad \text{and}$$

$$\lambda_2 \approx 1 - \frac{p_0^{\frac{1}{2}}}{p_1^{\frac{1}{2}}} + \frac{i\eta}{(p_0p_1)^{\frac{1}{2}}} + O\big(p_1^{-1}\big). \tag{4.9}$$

Here the perturbation term as a result of the spectral parameter has different signs and hence as $\eta \to 0$, $|\lambda_1| > 1$ while $|\lambda_2| < 1$ which is the desired dichotomy condition. If the coefficients are almost constant, then the dichotomy condition follows at once from the results of [8, Sect. 4]. □

One can now compute the eigenvectors of the transformation matrix $T(t)$ using (2.13). In order to work with bounded transformation matrix $T(t)$, we normalise the eigenvectors as follows. For the eigenvectors of λ_1, λ_2, λ_3, and λ_4 multiply vector elements by $\lambda_1(p_0p_1)^{-\frac{1}{2}}$, $-\lambda_2(p_0p_1)^{-\frac{1}{2}}$, $\lambda_3 p_1^{-1}$, and $\lambda_4 p_1^{-1}$, respectively so that $\|T(t)\| = O\big((p_0p_1)^{-\frac{1}{2}}\big)$. Then a lengthy calculation now gives

$$\det T(t) = -2(p_0p_1)^{-\frac{1}{2}}(p_2 - iq_2) + O\Big(p_1^{-\frac{3}{2}}\Big).$$

Thus for asymptotic summation, we need the following smoothness and decay conditions:

$$p_1\Delta(f) = o(1),\ f^{-1}\Delta(f), p_1 f^{-1}\Delta(f), \Delta(f), \left(\frac{p_0}{p_1}\right)^{\frac{1}{2}}\Delta(f), p_1 f\Delta(f) \in \ell^2. \tag{4.10}$$

Here, $f = p_0, p_1, p_2, q_1, q_2$. The reason for rather strict conditions given above is because two eigenvalues λ_1 and λ_2 are close together leading to unbounded terms in the remainder matrix R. In particular, R_{13} and R_{23} elements are unbounded eventhough all other elements will be bounded. This problem is solved by second diagonalisation using a transformation matrix of the form $I + B(t)$ as explained in Section 2. One can easily observe that an extra factor of $\left(\frac{p_1}{p_0}\right)^{\frac{1}{2}}$ is needed for smoothness conditions as a result of computing matrix $B(t)$. At this point, we make an assumption that the difference $\Delta(f)$ will beat $\left(\frac{p_1}{p_0}\right)^{\frac{1}{2}}$. In order to achieve LBL form after the second diagonalisation, we need the following conditions:

$$p_1^{\frac{3}{2}} p_0^{-\frac{1}{2}} f^{-1}\Delta^2(f),\ p_1^{\frac{3}{2}} p_0^{-\frac{1}{2}} f\Delta^2(f),\ \big(f^{-1}\Delta(f)\big)^2 \in \ell^1. \tag{4.11}$$

One therefore has the following result which is an extension of Theorems 3.2 and 3.6 of [4] to the discrete system and is therefore the main result of this section.

Theorem 4.4 *Consider the fourth order difference equation* (1.1) *such that the assumption A is satisfied as well as* (2.1)–(2.2), (4.2)–(4.4), (4.10) *and* (4.11). *Then one has the following*:

(a) *Assume* $p_0^{-\frac{1}{2}} p_1^{-\frac{1}{2}}$ *is summable, then def* $H_0 = (3,3)$ *and* $\sigma(H)$ *is discrete.*

(b) *If* $p_0^{-\frac{1}{2}} p_1^{-\frac{1}{2}}$ *is not summable and the coefficients are unbounded, then def* $H_0 = (2,2)$ *and* $\sigma_{ac}(H,1) = \mathbb{R}$.

(c) *If* $p_0^{-\frac{1}{2}} p_1^{-\frac{1}{2}}$ *is not summable and* p_0 *is bounded then def* $H_0 = (2,2)$ *and* $[\bar{c},\infty) \subset \sigma_{ac}(H,1)$ *if* $p_1 > 0$ *and* $\sigma_{ac}(H,1) \supset (-\infty,\bar{c}]$ *if* $p_1 < 0$.

Here $\bar{c} = \limsup p_0$ *and* $\underline{c} = \liminf p_0$.

Proof. Here, λ_3 and λ_4 lead to z-uniformly non-square and square summable eigenfunctions. Thus, they only contribute to discrete spectrum and deficiency index $(1,1)$. On the other hand, if one has to do the analysis for the square summability of the eigenfunctions associated to λ_1 and λ_2, we need the correction terms to the diagonals after the first diagonalisation since the second diagonalisation will only contribute summable terms to the diagonal and thus can be neglected. These terms are given by

$$R_{11} = \frac{\Delta(p_0p_1)^{\frac{1}{2}}}{2(p_0p_1)^{\frac{1}{2}}}\lambda_1, \quad R_{22} = \frac{\Delta(p_0p_1)^{\frac{1}{2}}}{2(p_0p_1)^{\frac{1}{2}}}\lambda_2.$$

Thus the eigenfunctions associated to these eigenvalues are of the form

$$y_{1/2}(t,z) = \pm(p_0p_1)^{-\frac{1}{2}}(t)\lambda_{1/2}(e_{1/2} + r_{11/22})\prod_{s=t_0}^{t-1}\left(1 + \frac{\Delta(p_0p_1)^{\frac{1}{2}}(s)}{2(p_0p_1(s))^{\frac{1}{2}}}\right)\lambda_{1/2}(s). \tag{4.12}$$

Now application of natural logarithms together with Euler summation formula with the fact that $\mid \lambda_{1/2}(t) \mid \approx 1$ gives

$$\ln\prod_{s=t_0}^{t-1}\frac{\Delta(p_0p_1)^{\frac{1}{2}}(s)}{2(p_0p_1)^{\frac{1}{2}}(s)} \approx \frac{1}{2}\sum_{s=t_0}^{t-1}\frac{\Delta(p_0p_1)^{\frac{1}{2}}(s)}{(p_0p_1)^{\frac{1}{2}}(s)} \approx \frac{1}{2}\int_{t_0}^{t-1}\frac{d(p_0p_1)^{\frac{1}{2}}}{(p_0p_1)^{\frac{1}{2}}} \approx \frac{1}{4}\ln(p_0p_1)(s).$$

Now substituting this result in (4.12) and evaluating

$$\|y_{1/2}(t)\| \approx \|p_0(t)p_1(t)\|^{-\frac{1}{4}}$$

shows that the solutions decay slowly for a real spectral parameter z as $t \to \infty$. But off the real axis, that is, if $z = z_0 + i\eta$, $z_0 \in \mathbb{R}$, $\eta > 0$ then the z-uniform square summability of these two eigenfunctions will depend on the summability of $(p_0p_1)^{-\frac{1}{2}}$. Thus if $p_0^{-\frac{1}{2}} p_1^{-\frac{1}{2}}$ is summable, then both the eigenfunctions of λ_1 and λ_2 will be z-uniformly square summable and hence will contribute $(2,2)$ to the deficiency index and discrete spectrum at most. If $p_0^{-\frac{1}{2}} p_1^{-\frac{1}{2}}$ is not summable, then the eigenfunction associated with λ_1 will lose its square summability as $\eta \to 0$ and hence will contribute to absolutely continuous spectrum of multiplicity one. It remains now to show that $\operatorname{Im} M(z)$ is bounded. To see this, it suffices to show that $\lim_{\eta\to 0}\eta\|y_2(t,z)\|^2 \to 0$ as $t \to \infty$ but this follow from

$$y_2(t,z) \approx \int_{t_0}^{\infty}\left(-(p_0p_1)^{-\frac{1}{2}}(t)\right)\left(\int_{t_0}^{t}\frac{-\eta}{(p_0p_1)^{\frac{1}{2}}(s)}\,ds\right)dt$$

if one applies natural logarithm and Euler summation fomula in (4.12) and thus shows that the eigenfunction decays to zero as $t \to \infty$.

If p_0 is unbounded, then the absolutely continuous spectrum is the whole of real line. But if p_0 is bounded, expressing the eigenvalue as in (4.9) now gives the desired range of the absolutely continuous spectrum. □

Remark 4.5 It should be noted here, however, that one cannot obtain a deficiency index of $(4,4)$ like in the differential case because by construction if λ is a root then $\overline{\lambda}^{-1}$ is also a root and $\mid \lambda_+\lambda_- \mid = 1$ so that if $\mid \lambda_- \mid < 1$ then $\mid \lambda_+ \mid > 1$. However, this result confirms the fact that if p_1 is unbounded and the other coefficients are bounded such that they are very small in the absolute value sense compared to p_1, then they are just bounded perturbation terms and hence the deficiency index and spectral results are similar to the case where the odd coefficients are assumed to be zero.

Remark 4.6 If we assume that the minimal operator is densed and well defined as well as the maximal operator not multivalued, then one obtains similar results in Theorem 4.4 for difference operators.

5 $q_1 \to +\infty$ as $t \to \infty$

In this section, we will continue with our spectral analysis of the fourth order difference equation (1.1) with the assumption that one of the odd coefficients q_1 is unbounded. Besides, we will absorb $-zw$ into p_0 as mentioned in Section 4. For simplicity, we will assume that

$$q_1 \nearrow \infty, \quad \text{as} \quad t \to \infty, \quad \text{and also} \quad p_2, w = 1, \quad p_0, p_1, q_2 = o(q_1). \tag{5.1}$$

For regularity and smoothness conditions on the coefficients, we will assume that

$$\begin{aligned} &\Delta^2(p_1),\ \Delta^2(p_0),\ \Delta^2(q_2),\ \longrightarrow 0,\\ &\frac{f}{q_1} = o(1),\ \Delta\left(\frac{f}{q_1}\right),\ q_1\Delta\left(q_1^{-1}\right) \in \ell^2,\ \Delta^k\left(\frac{f}{q_1}\right) \in \ell^1, \quad k \geq 2,\\ &\ f = p_0, p_1, q_2. \end{aligned} \tag{5.2}$$

These conditions are necessary for asymptotic summation. This will enable the first order system to be in Levinson-Benzaid-Lutz form after two diagonalisations and the elements of the remainder matrix $R(t)$ will be summable a condition necessary for application of Levinson-Benzaid-Lutz Theorem.

Theorem 5.1 *Consider the difference equation* (1.1) *with A satisfied as well as* (2.1)–(2.2), (5.1) *and* (5.2). *Then*:

(i) *If q_1^{-1} is summable then def $H_0 = (3,3)$, $\sigma(H)$ is discrete and $\sigma_{ac}(H)$ is empty.*
(ii) *If q_1^{-1} is not summable then def $H_0 = (2,2)$ and $\sigma_{ac}(H,1) = \mathbb{R}$.*

Proof. By application of quasi-differences, one transforms (1.1) into its Hamiltonian system and then to its corresponding first order system. An approximation of the eigenvalues of the transfer matrix $S(t,z)$ leads to

$$\begin{aligned} Q(s,z) = p_0 s^4 + 4q_1 s^3 + (2p_0 + 4p_1)s^2 \\ + (4q_1 + 16q_2)s + (p_0 + 4p_1 + 16) = 0. \end{aligned} \tag{5.3}$$

The s-roots of this polynomial can then be approxiamted as

$$\begin{aligned} s_1 &\approx -\frac{16 + 4p_1 + p_0}{16q_2 + 4q_1} + O\left(q_1^{-3}\right),\\ s_2 &\approx -\frac{4q_1}{p_0} + \frac{(p_0 + 2p_1)}{2q_1} - \frac{(4q_2 + q_1)p_0}{4q_1^2} + O\left(q_1^{-3}\right). \end{aligned}$$

The corresponding eigenvalues can then be determined with (2.10). One gets

$$\begin{aligned} &\lambda_1 = -\exp 2i\beta(s) \quad \text{with} \quad \beta(s) = \arctan s_1 \approx s_1, \quad \text{and}\\ &\lambda_2 \approx \exp(2i\gamma) \quad \text{with} \quad \gamma = \arctan\left(\frac{p_0}{4q_1}\right). \end{aligned} \tag{5.4}$$

This allows us to determine the dependence on Imz. With $z = z_0 + i\eta$ we get

$$\begin{aligned} &\lambda_1(t, z_0 + i\eta) \approx \lambda_1(t, z_0)\exp\left(-\frac{2\eta}{(16q_2 + 4q_1)}\right) \quad \text{and}\\ &\lambda_2(t, z_0 + i\eta) \approx \lambda_2(t, z_0)\exp\left(\frac{2\eta}{4q_1}\right). \end{aligned} \tag{5.5}$$

Meanwhile the other two s-roots are given by

$$s_\pm \approx -\frac{(2p_1 + (p_0 - z))}{4q_1} \pm \left\{\frac{(2p_1 + (p_0 - z))^2}{16q_1^2} - \frac{(4q_2 + q_1)}{q_1}\right\}^{\frac{1}{2}}. \tag{5.6}$$

In this case, the corresponding λ-roots are given by

$$\lambda_{3/4} = \frac{(is_{3/4}+1)}{(is_{3/4}-1)} \approx \frac{\left(\mp\left(1+\frac{4q_2}{q_1}\right)^{\frac{1}{2}}+1\right)}{\left(\mp\left(1+\frac{4q_2}{q_1}\right)^{\frac{1}{2}}-1\right)}.$$

In this case s_3 and s_4 are almost purely imaginary so that $\mid \lambda_3 \mid < 1$ and $\mid \lambda_4 \mid > 1$ uniformly in t. Thus it suffices to check for the uniform dichotomy condition for λ_1 versus λ_2. But this will follow at once from Lemma 4.3.

From (2.13), one then computes the corresponding eigenvectors. In this case we would want to work with a bounded diagonalising matrix $T(t)$. Thus one normalises the eigenvectors by multiplying each component of the eigenvector by $q_1^{-1}\lambda$. A lengthy calculation then gives

$$\det T(t) = \frac{-2i\left(-\left(4\frac{q_2}{q_1}+1\right)^{\frac{3}{2}}\right)}{q_1 q_2}.$$

It thus follows that $T^{-1}(t)$ is unbounded and hence the remainder matrix $R(t) = -T^{-1}(t+1)T(t)\Lambda(t)$ can even be approximated by $-T^{-1}(t)T(t)\Lambda(t)$ since at the limit point, $T^{-1}(t+1)$ and $T^{-1}(t)$ differe only by summable terms.

The most important contribution to the diagonals are those from R_{11} and R_{22} since λ_1 and λ_2 are almost of absolute value 1. In this case we get $-R_{11} = R_{22} \approx \frac{\Delta(q_1)}{2q_1}$. The off diagonal terms will be of the form $O\big(f^{-1}\Delta(f)\big)$ where $f = p_k, q_j$, $k = 0,1$ and $j = 1,2$. Thus after the first diagonalisation, the system will not yet be in its LBL form and therefore a second diagonalisation will be necessary to convert it into LBL form. The eigenfunctions will therefore be given by

$$y_k(t) = q_1^{-1}\lambda_k(e_k + R_{kk})\prod_{s=t_0}^{t-1}(\lambda_k(s,z)). \tag{5.7}$$

Since λ_3 and λ_4 lead to z-uniformly square and non-square summable eigenfunctions respectively, they contribute $(1,1)$ to the deficiency index and discrete spectrum at most.

Just like in Theorem 4.5 above, the eigenfunctions associated to λ_1 and λ_2 are slowly decaying and to see this one again uses natural logarithms and Euler summation formula. Here, we will assume that $\tilde{\lambda}_1 = (1 + \frac{\Delta q_1}{2q_1})\lambda_1$ where $\frac{\Delta q_1}{2q_1}$ is the correction term to the diagonal after the first diagonalisation. We obtain the following approximations:

$$\ln \prod_{s=t_0}^{t-1}\tilde{\lambda}_1(s) \approx \frac{1}{2}\sum_{s=t_0}^{t-1}\frac{\Delta(q_1(s))}{q_1(s)} \approx \frac{1}{2}\int_{t_0}^{t-1}\frac{dq_1(s)}{q_1(s)} \approx \frac{1}{2}\ln q_1(t-1).$$

In the limiting sense, it follows that

$$\|y_1(t)\| = \left| q_1^{-1}(t)\lambda_1(t)\prod_{s=t_0}^{t-1}\tilde{\lambda}_1(s)\right| \approx |q_1(t)|^{-\frac{1}{2}}.$$

The same analysis applies for $y_2(t)$. As $t \to 0$ the solutions decays slowly to zero. In the case of a spectral parameter with non-zero imaginary part, the z-uniform square summability of these solutions will depend on the summability of q_1^{-1}. Thus if q_1^{-1} is summable then the eigenfunctions associated to λ_1 and λ_2 will be z-uniformly square summable contributing $(2,2)$ to the deficiency index and discrete spectrum only. It follows that def $H_0 = (3,3)$ and $\sigma_{ac}(H)$ is discrete.

If q_1^{-1} is not summable, then the eigenfunction associated to λ_1 loses its square summability as $\eta \to 0$ and hence contributes to absolutely continuous spectrum. In this case $\operatorname{def} H_0 = (2,2)$. The bounded property of $\operatorname{Im} M(z)$ can be shown in the same way like in Theorem 4.5. □

Remark 5.2 The results for $q_1 \searrow -\infty$ as $t \to \infty$ can be analysed the same way. Moreover, if the minimal difference operator generated by (1.1) is densely defined and the maximal difference operator is not multivalued, then under the same conditions, the results of Theorem 5.1 can be extended to the selfadjoint extension operator of the minimal operator.

The following example shows how absolutely continuous spectrum can easily change in the case of a two term difference equation depending on the p_0 term. The example shows that if $p_0 \to \infty$, as $t \to \infty$, that is, dominant p_0, then the spectrum is discrete but if $p_0 \to 0$ as $t \to \infty$, then absolutely continuous spectrum is obtained of multiplicity 1.

Example 5.3 Consider a 4th order difference equation of the form

$$\Delta^4 \hat{y}(t-2) + t^\alpha \hat{y}(t) = zw(t)\hat{y}(t),$$

where $t \in \mathbb{N}$, $w(t) = 1$, α is a nonzero real-valued constant and z the spectral parameter. Thus, one obtains a ν-charateristic polynomial from this equation of the form

$$\mathcal{F}(t, \nu, z) = (2-\nu)^2 + t^\alpha - z,$$

where $\nu = \lambda + \lambda^{-1}$. By adjusting a and ϵ as mentioned in Section 2 whenever necessary, and choosing $z \in \mathcal{K}_\epsilon(z_0)$ appropriately, one can obtain two distinct ν-roots of the above polynomial. Note that the absolutely continuous spectrum can only be obtained for the values of $\nu \in [-2, 2]$.

Thus, by determining the values of z that satisfy the inequality $\mid \nu \mid = \left|2 - (z - t^\alpha)^{\frac{1}{2}}\right| \le 2$, one obtains $t^\alpha \le z \le 4^2 + t^\alpha$. Now assume that $\alpha > 0$, then as $t \to \infty$, $\operatorname{def} H_0 = (3, 3)$, the spectrum is discrete and $\sigma_{ac}(H) = \emptyset$. On the other hand, if $\alpha < 0$, $\operatorname{def} H_0 = (2, 2)$ and one obtains absolutely continuous spectrum of multiplicity 1, that is, $(0, 16] \subset \sigma_{ac}(H, 1)$. Thus, a dominant p_0 term, $p_0 \to \infty$ as $t \to \infty$, leads to discrete spectrum.

The following example shows how the sign of the coefficient can easily change the composition of the spectrum.

Example 5.4 Consider a fourth order difference equation (1.1) with

$$p_2 = 1, \;\; p_1 = bt^\beta, \;\; p_0 = 0 = q_j, \;\; w = 1, \;\; 0 < \beta < 2, \;\; j = 1, 2.$$

Then if $b > 0$, $\operatorname{def} H_0 = (2, 2)$ and $\sigma(H)$ is discrete, that is, $\sigma_{ac}(H)$ is empty. On the other hand if $b < 0$, then $\operatorname{def} H_0 = (2, 2)$ and $\sigma_{ac}(H, 1) = (-\infty, 0]$.

References

[1] R. Ansorge, T. Meis, and W. Törnig, Iterative Solutions of Non-linear Systems of Equations: Proceedings, Oberwolfach, Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1982).
[2] F. V. Atkinson, Discrete and Continuous Boundary Problems (Academic Press, Inc., New York, 1964).
[3] H. Behncke, A spectral theory of higher order differential operators, Proc. Lond. Math. Soc. (3) **92**, 139–160 (2006).
[4] H. Behncke, Spectral analysis of fourth order differential operators III, Math. Nachr **283**(11), 1558–1574 (2010).
[5] H. Behncke, Asymptotically constant linear systems, Proc. Am. Math. Soc. (4) **138**, 1387–1393 (2010).
[6] H. Behncke, D. B. Hinton, and C. Remling, The Spectrum of differential operators of order $2n$ with almost constant coefficients, J. Difference Equ. Appl. **175**, 130–162 (2001).
[7] H. Behncke and F. O. Nyamwala, Spectral theory of difference operators with almost constant coefficients, J. Difference Equ. Appl. **17**(5), 677–695 (2011).
[8] H. Behncke and F. O. Nyamwala, Spectral theory of difference operators with almost constant coefficients II, J. Difference Equ. Appl. **17**(5), 821–829 (2011).
[9] H. Behncke and F. O. Nyamwala, Spectral theory of differential operators with unbounded coefficients, Math. Nachr. **285**(1), 56–73 (2012).
[10] H. Behncke and F. O. Nyamwala, Spectral theory of differential operators with unbounded coefficients II, Math. Nachr. **285**(2–3), 181–201 (2012).
[11] Z. Benzaid and D. A. Lutz, Asymptotic representation of solutions of perturbed systems of linear difference equations, Stud. Appl. Math. **77**, 195–221 (1987).
[12] S. L. Clark, A spectral analysis of selfadjoint operators generated by a class of second order difference equations, J. Math. Anal. **197**, 267–285 (1996).
[13] S. L. Clark and F. Gesztesy, On Weyl-Titchmarsh theory for singular finite hamiltonian systems, J. Comput. Appl. Math. **171**, 151–184 (2004).
[14] P. A. Cojuhari and J. Janas, Discreteness of the spectrum for some unbounded jacobi matrices, Acta Sci. Math. (Szeged) **73**, 649–667 (2007).

[15] M. S. P. Eastham, The Asymptotic Solution of Linear Differential Sytems, London Math. Soc. Monographs New Series (Oxford University Press, Oxford, 1989).
[16] S. Elaydi, An Introduction to Difference Equations, third edition, Springer Science + Business Media (New York, 2005).
[17] A. Fischer and C. Remling, The absolutely continuous spectrum of discrete canonical systems, Trans. Am. Math. Soc. **361**, 793–818 (2008).
[18] D. B. Hinton and A. Schneider, On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems I, Math. Nachr. **163**, 323–342 (1993).
[19] D. B. Hinton and J. K. Shaw, On the Titchmarsh-Weyl $M(\lambda)$-Functions for linear hamiltonian systems, J. Difference Equ. Appl. **40**, 316–342 (1981).
[20] A. M. Krall, $M(\lambda)$ theory for singular hamiltonian systems with one singular point, SIAM J. Math. Anal **20**, 664–700 (1989).
[21] A. M. Krall, $M(\lambda)$ theory for singular hamiltonian systems with two singular points, SIAM J. Math. Anal **20**, 701–715 (1989).
[22] F. O. Nyamwala, Spectral Theory of Differential and Difference Operators in Hilbert Spaces, Doctoral Dissertation (Universität Osnabrück, Germany, 2010).
[23] C. Remling, Spectral analysis of higher order differential operator I, general properties of the M-Functions, J. Lond. Math. Soc (2) **58**, 367–380 (1998).
[24] C. Remling, Spectral analysis of higher order differential operator II, fourth order equations, J. Lond. Math. Soc (2) **59**, 188–206 (1999).
[25] G. Ren and Y. Shi, The defect index for singular symmetric linear difference equations with real coefficients, Proc. Am. Math. Soc. **138**(7), 2463–2475 (2010).
[26] G. Ren and Y. Shi, Defect indices and definiteness conditions for a class of discrete linear hamiltonian systems, Appl. Math. Comput. **218**, 3414–3429 (2011).
[27] G. Ren and Y. Shi, Self-adjoint extensions for discrete linear hamiltonian systems, Manuscript.
[28] Y. Shi, Weyl-Titchmarsh theory for a class of discrete linear hamiltonian systems, Linear Algebra Appl. **416**, 452–519 (2006).
[29] Y. Shi and H. Sun, Self-adjoint extensions for second-order symmetric linear difference equations, Linear Algebra Appl. **434**, 903–930 (2011).
[30] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Mathematics 1258 (Springer, Berlin, 1987).