

3-DIMENSIONAL MATHEMATICAL MODELLING OF TEMPERATURE DISTRIBUTION IN POROUS MEDIA

G. Owuor Lawi¹ §, W. Alfred Manyonge²

¹Department of Mathematics
Masinde Muliro University of Science and Technology
P.O. Box 190, Kakamega, 50100, KENYA

²Department of Mathematics and Applied Statistics
Maseno University
Private Bag Maseno, KENYA

Abstract: An alternative method of the numerical solution of the heat equation is presented. A three-dimensional modelling approach is used in this paper rather than one or two dimensional models in order to account for the lateral heterogeneous thermal conductivity coefficients or the heat sources or sinks commonly encountered in geothermal reservoir engineering studies.

AMS Subject Classification: 65C20

Key Words: geothermal reservoir, finite differences

1. Introduction

Geothermal reservoirs produce primarily from fractures. In the study of geothermal reservoir behaviour, on some occasion, it may be required to determine the temperature distribution in the field, particularly an analysis of the vertical thermal gradients. This calls for the use of a three dimensional heat model which account for lateral heat flow and the presence of sources or sinks for heat in the geothermal reservoir. Mathematical models for geothermal systems usually describe the three dimensional flow of water, steam or both and transport of heat in porous media. The basic governing equations may be expressed in terms of pairs of basic unknown thermodynamic quantities as independent variables, for example, fluid enthalpy and pressure or fluid density and internal

Received: June 4, 2011

© 2011 Academic Publications, Ltd.

§Correspondence author

energy, or pressure and temperature or pressure and saturation saturation. In this paper we present a model that can describe and predict the temperature distribution in a geothermal reservoir. A reasonable approximation of the heat transport mechanism in a geothermal field is the steady-state heat conduction model described by the following linear partial differential equation whose detailed derivation may be found in the works such as that of (see [1]).

$$\nabla \cdot [K(x, y, z) \nabla T(x, y, z)] + q(x, y, z) = 0 \quad (1)$$

in which x , y , and z are the orthogonal (Cartesian) curvilinear coordinates, T is the temperature of the reservoir, k the coefficient of thermal conductivity and q is the internal source/sink that is a function of heat production. Equation (1) may be written in space coordinates x , y , z as:

$$\frac{\partial}{\partial x} \left(K(x, y, z) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(K(x, y, z) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(K(x, y, z) \frac{\partial T}{\partial z} \right) + q(x, y, z) = 0. \quad (2)$$

2. The Heat Model

The media in which geothermal fluids flow are usually porous rocks and fractures. Our model in which heat flows will be represented by a cube, to account for both the aerial and vertical extent of the geotherm field. We align the orthogonal Cartesian coordinate system (x, y, z) with the origin 0 at one corner of the cube on the surface of the ground in such a manner that the x - and y -axes lie on the plane of the ground. The z -axis represents the vertical increase in the depth of the field with $z = 0$ on ground surface. We label the corners of the cube in some way with letters A through G . Let I , J and H represent the distances between the origin and the corner points of the cube along the x -, y - and z - axes respectively. We then solve equation (1) inside the heat model with the following boundary conditions:

(i) Surface temperature:

$$T(x, y, z) = T_0(x, y, z) \quad \text{at} \quad z = 0$$

(ii) Vertical boundaries:

$$\begin{aligned} \frac{T}{dx} &= 0 \quad \text{at} \quad x = 0, I, \\ \frac{T}{dy} &= 0 \quad \text{at} \quad y = 0, J \end{aligned}$$

indicating that the heat flow components are perpendicular to vertical boundaries, i.e. the temperature on the vertical surfaces are constants

(iii)

$$T_H(x, y, z) \quad \text{or} \quad q_H(x, y, z)$$

to be temperature or heat flow at the base of the model

Now, geothermal reservoirs usually have very high temperatures, in the order of 225°C (see[3]). Information about the temperature at the base of the model is usually not available. To determine the temperature at the base of the model, we use the average value of the observed heat flow on the surface of the geothermal field, i.e. $q_0(x, y)$. This heat flow is related to the temperature by the equation

$$K(x, y, z) \frac{dT}{dz} = q_0(x, y, 0). \quad (3)$$

Equation (2) together with its associated boundary conditions can now be solved inside and on the boundary of the heat model. Usually the assumptions that are made that are necessary in order to solve the mathematical model analytically are fairly restrictive. For example to obtain an analytic solution, we would require that the medium of the solution be homogeneous and isotropic, which is not true of most geothermal systems. For this reason we shall use a numerical method based on finite differences although there exist other numerical procedures or methods that can be used to solve the same heat equation. An example of these methods is the class of finite element methods.

3. Finite Difference Technique

The finite difference method entails two basic ideas, namely, that the domain of the solution of the partial differential equation is subdivided into a net with a finite number of mesh points and that the derivative of the function we are evaluating is replaced by a finite difference approximation. For the case of a three domain, the mesh points will form shapes such as those of cubes. Here, we shall illustrate the mathematical principle behind the finite difference representation. Consider a function $y = f(x)$. We perform a Taylor series expansion of this function at the point x_i and letting $x = x_{i+1}$, we have:

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f'(x_i) + \frac{(x_{i+1} - x_i)^2 f''(x_i)}{2!} + \dots \quad (4)$$

or

$$y_{i+1} = y_i + hy'_i + \frac{h^2 y''_i}{2!} + \frac{h^3 y'''_i}{3!} + \dots \tag{5}$$

and

$$y_{i-1} = y_i - hy'_i + \frac{h^2 y''_i}{2!} - \frac{h^3 y'''_i}{3!} + \dots \tag{6}$$

where $h = x_{i+1} - x_i$. From the above expressions, we may write

$$y'_i(x_i) \approx \frac{y_{i+1} - y_i}{h} \tag{7}$$

$$y'_i(x_i) \approx \frac{y_i - y_{i-1}}{h} \tag{8}$$

and

$$y'_i(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h} \tag{9}$$

The expressions in (7), (8) and (9) are known as the forward, backward and central quotient representations of the first derivative of the given function. Summing (5) and (6) and rearranging we find:

$$y''_i(x_i) = \frac{(y_{i+1} - 2y_i + y_{i-1}))}{h^2} + e_1 \tag{10}$$

in which $e = \frac{h^2}{12} y^{(iv)}$ tends to zero as h tends to zero. With this background in mind, we now turn to the heat equation in the form of equation (2). In this equation we can reasonably assume that the thermal conductivity K is a function of temperature, since the thermal conductivity varies considerably with the temperature of the solids. Let us define a subsidiary function ϕ such that:

$$K = \frac{d\phi}{dT} \tag{11}$$

this implies that

$$\phi = \int K dT \tag{12}$$

Substituting for K in equation (2) we find:

$$\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial T} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial T} \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial T} \frac{\partial T}{\partial z} \right) + q(x, y, z) = 0 \tag{13}$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + q(x, y, z) = 0 \tag{14}$$

or

$$\nabla^2\phi + q(x, y, z) = 0 \tag{15}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator. Let us introduce notation that will be useful in the few lines. Let i, j and k be counters in the x -, y - and z - directions respectively. Here $x = i\Delta x, y = j\Delta y$ and $z = k\Delta z$, where $\Delta x, \Delta y$ and Δz are spacing in the x, y and z directions of the Cartesian coordinate system respectively. Therefore

$$\phi(x, y, z) = \phi(i\Delta x, j\Delta y, k\Delta z) = \phi_{i,j,k} \tag{16}$$

Suppose that the interior mesh increments corresponding to x, y and z are h_1, h_2 and h_3 , i.e. $\Delta x = h_1, \Delta y = h_2$ and $\Delta z = h_3$. Then replacing each second derivative with its finite difference equivalent yields:

$$\begin{aligned} &\frac{(\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k})}{h_1^2} + \frac{(\phi_{i,j+1,k} - 2\phi_{i,j,k} + \phi_{i,j-1,k})}{h_2^2} \\ &+ \frac{(\phi_{i,j,k+1} - 2\phi_{i,j,k} + \phi_{i,j,k-1})}{h_3^2} + q_{i,j,k} = 0 \end{aligned} \tag{17}$$

which may be rearranged to give

$$\begin{aligned} \phi_{i,j,k} &= \alpha_1(\phi_{i+1,j,k} + \phi_{i-1,j,k}) + \alpha_2(\phi_{i,j+1,k} + \phi_{i,j-1,k}) \\ &+ \alpha_3(\phi_{i,j,k+1} + \phi_{i,j,k-1}) + \alpha q_{i,j,k} = 0, \end{aligned} \tag{18}$$

in which

$$\alpha_1 = \frac{(h_2h_3)^2}{2[(h_1h_2)^2 + (h_2h_3)^2 + (h_1h_3)^2]}, \tag{19}$$

$$\alpha_2 = \frac{(h_1h_3)^2}{2[(h_1h_2)^2 + (h_2h_3)^2 + (h_1h_3)^2]}, \tag{20}$$

$$\alpha_3 = \frac{(h_1h_2)^2}{2[(h_1h_2)^2 + (h_2h_3)^2 + (h_1h_3)^2]}, \tag{21}$$

$$\alpha = \frac{(h_1h_2h_3)^2}{2[(h_1h_2)^2 + (h_2h_3)^2 + (h_1h_3)^2]}. \tag{22}$$

From equation (18) we notice that one unknown is linked to six other unknowns. For simplicity, consider equal spacing in each of x, y and z directions, i.e. let

$h = h_1 = h_2 = h_3$ which would mean that $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{6}$ and $\alpha = \frac{h^2}{6}$ so that

$$\begin{aligned} \phi_{i,j,k} = \frac{1}{6} [& (\phi_{i+1,j,k} + \phi_{i-1,j,k}) + (\phi_{i,j+1,k} + \phi_{i,j-1,k}) \\ & + (\phi_{i,j,k+1} + \phi_{i,j,k-1}) + h^2 q_{i,j,k}] = 0. \end{aligned} \tag{23}$$

This is a finite difference equation. If we label the unknowns in positions

$$(i, j, k), (i + 1, j, k), (i - 1, j, k), (i, j + 1, k), (i, j - 1, k), (i, j, k + 1), (i, j, k - 1)$$

as

$$\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6,$$

respectively for ease of writing, then:

$$\frac{(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + h^2 q_0)}{6}$$

where ϕ_0 is calculated as the average of the six points around the centre. Using Taylor series expansion with equal spacing h on each ϕ at these positions and expanding to sufficiently higher order derivatives of ϕ at the point (i, j, k) and substituting into equation (23) and rearranging the terms, it can be shown that the local truncation error is of the order $O(h^2)$ indicating it is a second order method. We consider the heat model subdivided into 27 smaller cubes, 9 per layer for 3 layers. We examine the use of the model equation (23) vis-avis the nodes of the heat model. Equation (23) holds in the interior of the model and with Dirichlet boundary conditions namely:

$$\begin{aligned} b_1 = \phi(0, y, z), \quad b_2 = \phi(I, y, z), \quad b_3 = \phi(x, 0, z), \quad b_4 = \phi(x, J, z), \\ b_5 = \phi(x, y, 0), \quad b_6 = \phi(x, y, H). \end{aligned}$$

Here ϕ is specified on the boundary of the model. Equation (23) is solved by the following procedure: We order the points on the x and y planes then proceed plane by plane in the z direction to yield a set of linear algebraic equations. This procedure has been done as an example, for the heat model with the following features: Subdivide the interval I on the x - axis into three equal spacing, subdivide the interval J on the y - axis into three equation spacing and finally subdivide the interval H on the z - axis into three equal spacing. The size of the system of equations depends on the number of slices we make in the cube. To minimize the complexity for illustration purposes we use three slices, i.e. the mesh size $h = \frac{1}{3}$, on each axis. Label all the corners of the cubes of the heat model as $\phi_1, \phi_2, \dots, \phi_{64}$. This results in a total of 64 nodes out of

which 8 are interior. Applying equation (23) to each interior node of the model we find a system of 8 equations in 8 unknowns. On rearranging, this yields a set of linear algebraic equations of the form:

$$A\phi = \mathbf{b} \quad (24)$$

in which A is a block tri-diagonal and diagonally dominant matrix, ϕ is the vector of unknown quantities while \mathbf{b} is a vector of known quantities. This may be solved by known iterative methods such as the ADI method of Douglas and Rachford (see[2]). The solution of this system of equations yields the temperatures at the internal mesh points.

4. Model Verification

For purely illustrative purposes, we shall solve the heat equation inside the model when subjected to the following boundary conditions:

$$\phi(0, y, z) = 100^{\circ}C, \quad \phi(x, 0, z) = 160^{\circ}C, \quad \phi(x, y, 0) = 60^{\circ}C,$$

$$\phi(I, y, z) = 250^{\circ}C, \quad \phi(x, J, 0) = 280^{\circ}C, \quad \phi(x, y, H) = 280^{\circ}C.$$

For simplicity we shall in this example neglect heat source. Using the software MATLAB to compute the unknowns yields:

$$\phi_1 = 133.8^{\circ}C, \quad \phi_2 = 169.3^{\circ}C, \quad \phi_3 = 163.3^{\circ}C, \quad \phi_4 = 193.3^{\circ}C,$$

$$\phi_5 = 183.0^{\circ}C, \quad \phi_6 = 213.3^{\circ}C, \quad \phi_7 = 207.3^{\circ}C, \quad \phi_8 = 237.3^{\circ}C.$$

5. Conclusion

We note that if data for the surface heat flow is available, then this technique can be very useful in estimating the temperature distribution in geothermal reservoirs before exploitation of the reservoir.

References

- [1] A.V. Luikov, *Heat and Mass Transfer in Capillary-Porous Bodies*, First Edition, Pergamon Press, Oxford (1966).

- [2] G.F. Pinder, L. Lapidus, *Numerical Solution of Partial Differential Equations in Science and Engineering*, John Wiley and Sons (1982).
- [3] M.P. Hochstein, *Worldwide Occurrence of Geothermal Resources: Connective and Non – Connective Systems*, Geothermal Institute, University of Auckland, New Zealand, Private Bag, Auckland, New Zealand (1992).