COMPUTATION OF EFFICIENT NASH EQUILIBRIA FOR EXPERIMENTAL ECONOMIC GAMES

by

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A thesis submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Applied Mathematics

School Of Mathematics, Statistics And Actuarial Science

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Acknowledgement

My sincere gratitude goes to my supervisors Dr. Edgar Ouko Otumba and Prof. Alfred W. Manyonge. They availed the necessary resources and sacrificed their time and through their guidance I was able to complete this programme successfully.

I thank all my lecturers including Prof. Rao, Prof. Omollo Ongati and Dr. Oduor Michael and friends at the School of Mathematics and Actuarial Sciences for their support. I will always feel indebted to Dr. David Stern who introduced me to Biomathematics Modeling and enabled me develop more passion in this area of study. To all my colleagues; Purity, Ahomo, Denis, Ondiek, Okewa and Peter thank you for your encouragement.

I appreciate my family and in particular my husband, Ronald Ngala Ongutte, for his understanding, encouragement, comfort and love throughout this period. Special thanks also go to my sisters and brothers whose love for education saw me through up to this stage. Last but not least, I acknowledge all other persons who made this work a success.

I thank God for the gift of life and provision of good health during this period.

Dedication

To my beloved husband Ronny and sons Ian-Shawn and Gordon-Griffins

Abstract

Game theory has been used to study a wide variety of human and animal behaviours. It looks for states of equilibrium, sometimes called solutions. Nash equilibrium is the central solution concept with diverse applications for most games in game theory. However some games have no Nash equilibrium, others have only one Nash equilibrium and the rest have multiple Nash equilibria. For games with multiple equilibria, different equilibria can have different rewards for the players thus causing a challenge on their choice of strategies. In this study, to solve the problems associated with existence of multiple equilibria in games,we identified and computed the most efficient Nash equilibrium in such experimental economic games. To achieve this we described and carried out an experiment on a game that was modelled as a three-player experimental economic game. The results were recorded and by the best response sets method we identified all the Pure Nash equilibria and computed the most efficient Nash equilibrium for our experimental economic game. Using the Brauwer's fixed point theorem we verified the existence of mixed Nash equilibrium in the experimental economic game. The findings were that the most efficient equilibrium varied from one player to the other. An individual whose aim was to minimize risks played the risk dominant strategies whereas for those aiming to maximize their profits, the payoff dominant strategies were played in cooperation to achieve the most efficient Nash Equilibrium for the experimental economic game. The computation of most efficient Nash Equilibrium in games can be applied to most situations in competitive Economic environment that are faced with multiple choices on which strategy is optimal.

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List of Notations

Lower case

n:	The number of players in a game
s:	Strategies chosen by all n players
s^*:	Pure strategy profile
s_i:	Strategy chosen by player i
s_j:	The $(n-1)$ dimensional vector strategies played by all other players
s'_i:	Alternate strategy available to player i
s'_j:	Alternate strategies played by all other players
$u_i(s)$:	Utility (pay-offs in game theoretic usage) incurred by player i

Upper case

N:	The number of players in a game
S:	The set of possible strategies for all n players
S_i:	Set of possible strategies for player i

Greek case

Γ:	Represents a game
β:	Strategy profile for the players in the modelled game
ρ_i:	Mixed strategy for player i

Glossary of Terms and Acronyms

Actions Choices available to a player.

Common knowledge A fact is common knowledge if all players know it, and know that they all know it and so on.

Efficient Nash equilibrium An equilibrium is efficient if there is no other equilibrium in which someone is better off and no one is worse off.

Equilibrium A stable result. They are not necessarily good outcomes.

ESS Evolutionary Stable Strategy.

Experimental economics The application of experimental methods to study economic questions.

Game A formal description of a strategic situation.

Game theory A mathematical methodology for analyzing calculated circumstances, such as in games, where a person's success is based upon the choices of the others.

LP Linear Program.

NE Nash Equilibrium.

Payoff A number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason.

Player An agent who makes decisions in a game.

PPAD Polynomial Parity Arguments on Directed Graphs.

Rationality A player is said to be rational if he seeks to play in a manner which maximizes his own payoff. It is often assumed that rationality of all players is common knowledge.

Strategies Rules that tell a player what actions to take at each point of the game.

z-Tree Zurich Toolbox for Ready-made Economic Experiments.

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CHAPTER 1

INTRODUCTION

1.1 Background to the Problem

Game theory is the formal study of conflict and cooperation. It deals with strategic interactions among multiple decision makers, called players (and in some context agents), with each player's preference ordering among multiple alternatives captured in an objective function for that player, which he or she tries to maximize (in which case the objective function is a utility, payoff or benefit function) or minimize (in which case we refer to the objective function as a cost or loss function). The concept of game theory provides a language to formulate, structure, analyze and understand strategic scenarios. The games studied in game theory are well defined mathematical objects with a set of players, a set of moves (strategies) available to those players and a specification of payoffs or costs for each combination of strategies [1].

There are two branches of game theory, namely cooperative and non-cooperative game theory [15]. Cooperative game theory studies friction-less negotiation among rational players who can make binding agreements about how to play a game. The emphasis is on the groups or coalitions of players. Non-cooperative game theory is mainly concerned with individual behaviour: what decision should each rational player use, or how will rational players actually choose their actions and what is the most likely outcome of the game.

Game theory has further been broadly classified into four main subcategories: Classical game theory, combinatorial game theory, dynamic game theory and other topics such as evolutionary game theory, experimental game theory and economic game theory [17].

A strong solution concept, which is applicable to all games in game theory, is the Nash equilibrium which captures the notion of a stable solution. As much as some experimental economic games have a unique Nash equilibrium, others have none whereas the rest have multiple equilibria. On the other hand, most experiments that have been conducted involve two players yet in real life application we normally have more than two agents interacting. For this study, we considered a three player experimental economic game with multiple equilibria and computed the most efficient Nash equilibria for the game. Using the Brouwer's fixed point theorem we verified that mixed Nash Equilibrium existed in the experimental economic game.

1.2 Basic Concepts

1.2.1 Game Theory

Game theory is a mathematical methodology that studies mathematical models of conflict and cooperation between intelligent rational decision makers [16]. It was initially developed in economics to understand a large collection of economic behaviours, including behaviour of firms, markets and consumers. To represent a game, we will use the notation

$$\Gamma = \langle N, (S_i), (u_i) \rangle \tag{1.1}$$

where N is the number of players, S_i the available strategies, u_i the payoff to the player i and $i = 1, 2, 3$.

A game can either be static or dynamic. A static game is only played once while a dynamic game is played multiple times. There are two types of dynamical games:

(a) Sequential games in which players play one after another.

(b) Stage games which are static games played for a finite number of repetitions.

In evolutionary game theory, the concept of rationality is not meaningful but the idea that evolutionary forces like natural selection and mutation are the driving forces of change is very important. Some representative games of evolutionary game theory include hawk and dove, war of attrition, stag hunt, tragedy of commons, prisoners' dilemma among others. On the other hand, classical game theory is based upon a number of severe assumptions about the structure of the game. Classical game theory describes the behaviour of rational players and also attempts to mathematically capture behaviour of strategic situations, in which an individual's success in making choices depends on the choices of the others. The mission of evolutionary game theory is to remedy some key deficiencies of the classical game theory: the lack of dynamics and equilibrium selection in the case of multiple Nash equilibrium [15].

In experimental economics cash is used to motivate subjects, in order to mimic real world incentives [1]. Experiments are used to help understand how and why markets and other exchange system function as they do. Experiments may be conducted in the field or in laboratory settings, whether of individual or group behaviour.

1.2.2 Solution Methods

A solution concept is a formal rule for predicting how a game will be played. These predictions are called "solutions" and describe which strategies will be adopted by players and, therefore, the result of the game. The most commonly used solution concepts are the equilibrium concepts. When a solution is required for a game, it is necessary to first specify the form and type of game under investigation. Cooperative solutions are designed to capture a stable outcome of a bargaining problem whereas Non-cooperative solutions involve solving a non-cooperative point of the game, for example, Nash equilibrium [3] . Some solution concepts have been discussed below.

Dominant Strategy

A dominant strategy is a basic solution concept in game theory. It implies that each player has a unique best strategy, independent of the strategies played by the other players [25]. We will use s to denote the strategies chosen by all n players, S as the set of possible strategies for all n players, s_i as the stategy chosen by player i and S_i as the set of possible strategies for player i. $u_i(s)$ is the utility (pay-offs in game theoritic usage) incurred by player i.

More formally, a strategy vector $\mathbf{s} \in \mathbf{S}$ is a dominant strategy solution, if for each player i, and each alternate strategy vector $\mathbf{s}' \in \mathbf{S}$, we have that

$$u_i(\mathbf{s}_i, \mathbf{s}'_j) \geq u_i(\mathbf{s}'_i, \mathbf{s}'_j) \tag{1.2}$$

where $\mathbf{s}_j$ denotes the $(n-1)$ dimensional vector strategies played by all other players. Note that:

$\Gamma = \langle N, (\mathbf{S}_i), (u_i) \rangle$

$N = \{1, 2, \cdots, n\}$

$\mathbf{S} = S_1 \times S_2 \times \cdots \times S_n$

$\mathbf{s} = (s_1, s_2, \cdots, s_n)^T \in \mathbf{S}$

$\mathbf{s}_j = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)^T$

$u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$

Dominant strategy equilibria (strongly dominant or weakly dominant), if they exist, are very desirable, but rarely do they exist because the conditions to be satisfied are too demanding [17]. Dominant strategy solution may not give optimal payoff to any of the players. One limitation of this method is that most games rarely posses dominant strategy solutions [25].

Nash Equilibrium

The notion of an equilibrium is the basic ingredient in game theory. Every finite game has an equilibrium point [24]. Nash (1951) proved that every game with a finite number of players, each having a finite set of strategies, has a Nash Equilibrium of mixed strategies [18]. Nash equilibrium is the central solution concept in game theory with extremely diverse applications. It captures the notion of stable solution in which no single player can individually improve his or her welfare by deviating.

More formally, a strategy vector $\mathbf{s} \in \mathbf{S}$ is said to be a Nash equilibrium if for all players i

and each alternate strategy $\mathbf{s}'_i \in \mathbf{S}$, we have that

$$u_i(\mathbf{s}_i, \mathbf{s}_j) \geq u_i(\mathbf{s}'_i, \mathbf{s}_j). \tag{1.3}$$

In other words, no player i can change his chosen strategy from $\mathbf{s}_i$ to $\mathbf{s}'_i$ and thereby improve his payoff, assuming that all other players stick to the strategies they have chosen in $\mathbf{s}$. Such a solution is self enforcing in the sense that once the players are playing such a solution, it is in every player's best interest to stick to his or her strategy. Clearly a dominant strategy solution is a Nash equilibrium [25].

Moreover if the solution is strictly dominating (switching to it always strictly improves the income), it is also a unique Nash equilibrium. However Nash equilibrium may not always be unique. For example some games may have multiple equilibria. For games with multiple Nash equilibria, different equilibria can have (widely) different payoffs for the players. A game can have either a pure strategy and/or a mixed strategy Nash equilibria [11].

Given a game (1.1) with pure strategies, the strategy profile

$$\mathbf{s}^* = (s_1^*, \cdots, s_n^*)^T$$

is said to be a pure strategy Nash equilibrium of (1.1) if

$$u_i(\mathbf{s}_i^*, \mathbf{s}_j^*) \geq u_i(\mathbf{s}_i, \mathbf{s}_j^*) \tag{1.4}$$

$\forall \mathbf{s}_i \in \mathbf{S}_i \ \forall \quad i = 1, 2 \cdots n.$

That is each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players. Therefore for the game (1.1), the strategy profile $(s_1^*, \cdots, s_n^*)^T$ is a Nash equilibrium if and only if

$$\mathbf{s}_i^* \in B_i(\mathbf{s}_j^*) \quad \forall i = 1, \cdots, n.$$

Definition 1.2.1. A strategy profile $(\mathbf{s}_i^*, \mathbf{s}_j^*)$ is a strict Nash Equilibrium if for every player i,

$$u_i(\mathbf{s}_i^*, \mathbf{s}_j^*) > u_i(\mathbf{s}_i, \mathbf{s}_j^*),$$

for every $\mathbf{s}_i^* \neq \mathbf{s}_j^*$.

The difference from the original definition of Nash Equilibrium is only in the strict inequality sign.

Consider a pure strategy game: (1.1). A pure strategy or a deterministic strategy for player i specifies the deterministic choice $s_i(I)$ at each information set I. Let $\mathbf{S}_i$ be finite for each $i = 1, 2 \cdots n$. If player i randomly chooses one element of the set S_i, we have a mixed strategy or a randomized strategy.

More formally, given a player i with $\mathbf{S}_i$ as the set of pure strategies, a mixed strategy ρ_i for player i is a probability density function over $\mathbf{S}_i$. That is, $\rho_i : \mathbf{S}_i \mapsto [0, 1]$ assigns to each pure strategy $\mathbf{s}_i \in \mathbf{S}_i$, a probability $\rho_i(\mathbf{s}_i)$ such that

$$\sum_{\mathbf{s}_i \in \mathbf{S}_i} \rho_i(\mathbf{s}_i) = 1. \tag{1.5}$$

A mixed strategy profile is a Nash equilibrium if the mixed strategy for each player is a best response to the mixed strategies of the rest; that is, it attains the maximum possible utility among all possible mixed strategies of this player. The support of a mixed strategy is the set of all pure strategies that have non-zero probability in it. A mixed strategy is a best response if and only if all pure strategies in its support are best responses [6]. If each player in n-player game has a finite number of pure strategies, then there exists at least one equilibrium in mixed strategy[18]. If there are no pure strategy equilibria, there must be a unique mixed strategy equilibrium. However, it is possible for pure strategy and mixed strategy Nash equilibria to co-exist in games.

Brouwer's Fixed Point Theorem

Classical game theory and economics rely heavily on fixed point theorems to prove the existence of various solution concepts. Brouwer's fixed point theorem plays a central role in the proof of existence of general equilibrium in the market economies. Nash's original proof relies on Brouwer's fixed point theorem stating that

Theorem 1.2.2. *every continuous function f from the n-dimensional unit ball to itself has a fixed point: a point x such that $f(x) = x$ [6].*

Given any game, Nash constructed a Brouwer map whose fixed points are precisely equilibria of the game. As a result, the existence of equilibria followed as a result of the existence of fixed points [18].

The contribution of Nash was to define a mixed strategy Nash Equilibrium for any game with a finite set of actions and prove that at least one (mixed strategy) Nash must exist in such a game. He was able to use Brouwer's fixed point theorem tp prove that there had to exist at least one set of mixed strategies that mapped back to themselves for non-zero sum games, namely, a set of strategies that did not call for a shift in strategies that could improve payoffs.

Nash considered a game

$$G = (N, A, u) \tag{1.6}$$

where N is the original number of players, u is the payoff from the chosen actions and $A = A_1 \times \cdots \times A_N$ is the action set of players. All action sets A_i are finite.

He let $\Delta = \Delta_1 \times \cdots \times \Delta_N$ denote the set of mixed strategies for the players. The finiteness of A_i ensures the compactness of Δ. He then defined the gain function for player i, G_i. For a mixed strategy $\sigma \in \Delta$, we let the gain for player i on action $a \in A_i$ be

$$G_i(\sigma, a) = max\{0, U_i(a, \sigma_j) - U_i(\sigma_i, \sigma_j)\},$$

where σ_i is the mixed strategy for player i and σ_j is the mixed strategy for all other players in the game (1.6). The gain function represents the benefit a player gets by unilaterally changing his strategy.

He defines $g = (g_1, \cdots, g_N)$ where $g_i(\sigma, a) = \sigma_i(a) + G_i(\sigma, a)$ for $\sigma \in \Delta$, $a \in A_i$.

$$\sum_{a \in A_i} g_i(\sigma, a) = \sum_{a \in A_i} \sigma_i(a) + G_i(\sigma, a) = 1 + \sum_{a \in A_i} G_i(\sigma, a) > 0.$$

He used g to define $f : \Delta \mapsto \Delta$ as follows:
Let

$$f_i(\sigma, a) = \frac{g_i(\sigma, a)}{\sum_{a \in A_i} g_i(\sigma, a)}$$

for $a \in A_i$.

f_i is a valid mixed strategy in Δ_i. Also each f_i is a continuous function of σ, and hence f is a continuous function [18]. Now Δ is the cross product of a finite number of compact convex sets, and so Δ is also compact and convex. Therefore he applied the Brauwer fixed point theorem to f. So f had a fixed point in Δ, call it σ^*.

He claimed that σ^* is a Nash equilibrium in the game (1.6). For this purpose it sufficed to show that

$$\forall \quad 1 \leq i \leq N, \forall \quad a \in A_i, G_i(\sigma^*, a) = 0.$$

This simply states that each player gains nothing by unilaterally changing his strategy, which is exactly the necessary condition for Nash equilibrium.

He assumes that the gains are not zero. $\exists \quad i, 1 \leq i \leq N$ and $a \in A_i$ such that

$$G_i(\sigma^*, a) > 0.$$

Note then that

$$\sum_{a \in A_i} g_i(\sigma^*, a) = 1 + \sum_{a \in A_i} G_i(\sigma^*, a) > 1.$$

So let

$$C = \sum_{a \in A_i} g_i(\sigma^*, a).$$

He denotes $G(i, \cdot)$ as the gain vector indexed by actions in A_i. Since

$$f(\sigma^*) = \sigma^*,$$

we clearly have that

$$f_i(\sigma^*) = \sigma_i^*.$$

Therefore

$$\sigma_i^* = \frac{g_i(\sigma^*)}{\sum_{a \in A_i} g_i(\sigma^*, a)}.$$

$$\Rightarrow \sigma_i^* = \frac{\sigma_i^* + G_i(\sigma^*, \cdot)}{C}$$

$$C\sigma_i^* = \sigma_i^* + G_i(\sigma^*, \cdot)$$

$$(C-1)\sigma_i^* = G_i(\sigma^*, \cdot)$$

$$\sigma_i^* = (\frac{1}{C-1})G_i(\sigma^*, \cdot).$$

Since $C > 1$, he had that σ_i^* is some positive scaling of the vector $G_i(\sigma^*, \cdot)$.

He claims that

$$\sigma_i^*, a)(u_i(a_i, \sigma_j^*) - u_i(\sigma_i^*, \sigma_j^*)) = \sigma_i^*(a)G_i(\sigma^*, a)\forall a \in A_i.$$

To see this, he first notes that if

$$G_i(\sigma^*, a) > 0,$$

then this is true by the definition of the gain function.

He assumes that

$$G_i(\sigma^*, a) = 0.$$

By our previous statements we have that

$$\sigma_i^*, a = \frac{1}{C-1} G_i(\sigma^*, a) = 0,$$

and so the left term is zero, giving the entire expression as 0 as needed.

So finally he had that

$$0 = (U_i(a_i, \sigma_j^*) - U_i(\sigma_i^*, \sigma_j^*)$$

$$= \sum_{a \in A_i} (\sigma_i^*(a) U_i(a_i, \sigma_j^*) - U_i(\sigma_i^*, \sigma_j^*))$$

$$= \sum_{a \in A_i} (\sigma_i^*(a)(U_i(a_i, \sigma_j^*) - U_i(\sigma_i^*, \sigma_j^*))$$

$$= \sum_{a \in A_i} \sigma_i^*(a) G_i(\sigma^*, a)$$

by the previous statements.

$$= \sum_{a \in A_i} (C - 1)\sigma_i^*(a)^2 > 0$$

where the last inequality follows since σ_i^* is a non-zero vector.

But this is a clear contradiction, so all the gain must indeed be zero.
Therefore σ^* is a mixed Nash equilibrium for the game (1.6) as needed [18].

1.3 Statement of the Problem

Multiple Nash Equilibria is one of the fundamental problems in game theory. For games with multiple equilibria, different equilibria can have (widely) different payoffs or costs for the players and this poses a challenge on the choice of strategies to be played. Therefore there was need to identify and compute the most efficient Nash equilibrium in such games.

1.4 Objectives of the Study

The main objective of this study was to compute efficient Nash Equilibria for experimental economic games. The specific objectives of the study were to:

(i) define and describe a three-player experimental economic game;

(ii) identify all the Nash equilibria in the three-player experimental economic game;

(iii) compute the most efficient Nash equilibrium in the three-player experimental economic game.

1.5 Research Methodology

For this study, a game modelled as an experimental economic game with three players was defined and described. It was played by high school students, three in each group, and all the possible outcomes recorded. At the end of the experiment, all the outcomes were analyzed and the payoffs for every outcome cell calculated and all pure Nash equilibria identified from the best response sets of the players. Using the Brouwer's fixed point theorem we verified the existence of mixed Nash equilibrium as a solution concept in the experimental economic game and using some refinements of the Nash Equilibrium we identified and computed the most efficient Nash equilibrium for the three-player experimental economic game.

1.6 Significance of the Study

The study of game theory will enable us to model competing behaviours of interacting agents in mathematical economics and business. Using game theory we will be able to look at states of equilibria, sometimes called solutions. The computation of efficient Nash equilibria will be applicable in a competitive and interactive economic environment because it will provide a way of advising and predicting what will happen if several people or several institutions are making decisions at the same time, and if the outcomes depend on the decisions of the others.

1.7 Outline of the Thesis

The thesis is structured as follows:
In chapter 2 the review of literature related to this study was done. In section 2.1 some experimental economic games that have already been done were defined and described. In section 2.2 an explanation on how Nash Equilibrium has been computed for both zero-

sum and non-zero sum games was covered.

Chapter 3 contains the results and discussion on the subject of study. Areas covered in this chapter include description of the experimental economic game in section 3.1 where the stages of the game were outlined and all possible outcomes of the game recorded . In section 3.2 computation and identification of efficient Nash equilibria in the experimental economic game was done. In this section, the Brouwer fixed point theorem has been used to verify that a mixed Nash Equilibrium existed in the experimental economic game.
The summary, conclusions and the recommendations have been summarized at the end of the last chapter.

CHAPTER 2

LITERATURE REVIEW

In this chapter we review the development of some experimental economic games. We also briefly discuss how Nash Equilibria have been computed for some of these games and the shortcomings that justify our study.

2.1 Experimental Economic Games

The first experimental economic games were published in 1960s. The earliest experiments in economics journals focused almost exclusively on markets with particular emphasis on finding the conditions under which a market would converge to the competitive price and quantity [9].

Meanwhile on parallel tracks, psychologists and game theorists began to investigate simple games such as the prisoner's dilemma which was initially conducted as an experiment by Melvin Dresher and Merill Flood in January 1950. Prisoner's dilemma game is an example of early experiments concerning interactive behaviour [1]. This game involves two prisoners who are separately given the choice between testifying against the other (non-cooperation) and keeping silent (cooperation). The payoffs are such that each of them is better off testifying against the other but if they both pursue this strategy they are both worse off than by remaining silent. Prisoner's dilemma helps us understand what governs the balance between cooperation and competition in business.

In 1972 Peter Bohm conducted the first multi-person continuous prisoner's dilemma game referred to as public goods game [9]. A public good is a resource from which all may benefit regardless of whether they have contributed to the good. Free riders enjoy the good without making any contribution. Altruists contribute heavily to the public pot and continue to do so even when others ride free. Conditional consenters start by contributing some of their wealth but when they realize others free ride they no longer cooperate. In public good the best strategy is to free ride, but when all do so the payoff is worse off than if all contributed.

In 1982, three German economists, (Guth, Schmittberger and Schwarze) conducted the first ultimatum game experiment. Their purpose was to strip bargaining down to its essentials by creating the simplest possible bargaining situation [10]. The game has two players namely the Proposer and the Responder. An amount of money is made available to the pair by the experimenter. The Proposer's task is to determine the division of money and the Responder's task is to either accept or reject the offer. If accepted the money is divided as proposed; but if rejected both players receive zero earnings and the money reverts to the experimenter [1]. Behaviour in the ultimatum game has implications for a broad set of important economic problems involving worker motivation, contracting and the notion of fair price [26].

In 1994 an improvement on the ultimatum game was made and the game was referred to as the dictator game. The first player, "the Proposer" determines the allocation (split) of some endowment (such as cash prize). The second player "the Responder" simply receives the remainder of the endowment left by the Proposer. The Responder's role is entirely passive (he has no strategic input into the outcome of the game).

Trust game, which is an investment game was first presented by Berg, Dickhaut and McCabe in 1995 [4]. Like in public goods game, there is no gain in cooperation, but to achieve those gains, the first mover must first trust by putting his payoffs in the hands of the second mover, with no promise of return. The amount sent by the first mover is "trust" and the amount returned is "reciprocity". This game has been adapted to measure trust at individual levels [9].

2.2 Computation of Nash Equilibrium

The idea of Nash equilibria can be traced back to the work of Antoine [2]. He developed a model called “duopoly”; a model of competitive markets and mathematically derived an equilibrium solution.

In 1940’s Neumann and Morgenstern [27] studied two-player zero-sum games (e.g. the rock-paper and scissors game) where one player’s gain is another player’s loss under the same competitive equilibrium concept. A proof of the existence of the equilibria for two-player zero-sum game was established via Von Neumann’s minimax theorem. A finite two person zero sum game in strategic form is denoted by

$$(X, Y, A), \tag{2.1}$$

where $X = \{x_1, \cdots x_m\}$, $Y = \{y_1, \cdots y_n\}$ and $A = a_{ij} = A(x_i, y_j)$. X and Y are the strategies for the row player and the column player respectively for the game (2.1). The von Neumann’s minimax theorem states:

Theorem 2.2.1. *For every $(m \times n)$ matrix A, there is a stochastic row vector $\boldsymbol{x}^* = (x_1^*, \cdots, x_m^*)$ and a stochastic column vector $\boldsymbol{y}^* = (y_1^*, \cdots y_n^*)$ such that*

$$min_{\boldsymbol{y} \in \Delta(S_2)} \boldsymbol{x}^* A \boldsymbol{y} = max_{\boldsymbol{x} \in \Delta(S_1)} \boldsymbol{x} A \boldsymbol{y}^*$$

The most accessible proof is through linear programming duality theorem, a special case of Von Neumann’s minimax theorem. According to Von Neumann’s minimax theorem, every finite game has a value, and both players have minimax strategies. The key implication of the minimax theorem is the existence of a mixed strategy Nash equilibrium in any matrix game.

Thus, if a game is zero-sum, Nash equilibrium has been formulated in terms of linear programming and linear programs have been solved efficiently. Oskar Morgenstern and John Von Neumann showed that the mixed strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is in fact a mixed strategy NE of the matrix A in the game (2.1). For this they considered

$$\mathbf{x}^* A \mathbf{y}^* \geq min_{\mathbf{y} \in \Delta(S_2)} \mathbf{x}^* A \mathbf{y}$$

$$= max_{\mathbf{x}\in\Delta(S_1)}\mathbf{x}A\mathbf{y}^*$$

$$\geq \mathbf{x}A\mathbf{y}^*$$

$\forall \quad \mathbf{x} \in \Delta(S_1)$. This implies

$$u_1(\mathbf{x}^*, \mathbf{y}^*) \geq u_1(\mathbf{x}, \mathbf{y}^*)$$

$\forall \quad \mathbf{x} \in \Delta(S_1)$.

Further

$$\mathbf{x}^*A\mathbf{y}^* \geq min_{\mathbf{y}\in\Delta(S_2)}\mathbf{x}^*A\mathbf{y}$$

$$= max_{\mathbf{x}\in\Delta(S_1)}\mathbf{x}A\mathbf{y}^*$$

$$\geq \mathbf{x}^*A\mathbf{y}$$

$\forall \quad \mathbf{y} \in \Delta(S_2)$. This implies

$$u_2(\mathbf{x}^*, \mathbf{y}^*) \geq u_2(\mathbf{x}^*, \mathbf{y})$$

$\forall \quad \mathbf{y} \in \Delta(S_2)$. Thus $(\mathbf{x}^*, \mathbf{y}^*)$ is a mixed strategy NE or a randomized saddle point. This means that the minimax theorem guarantees the existence of the mixed strategy NE for any matrix game.

Nash John was the first to study the more general non-zero-sum games (e.g the prisoner's dilemma) with two or more players along a competitive equilibrium approach and he proved in [18] the existence of a competitive equilibrium approach. It was based on fixed-point theorem of Brauwer :

Theorem 2.2.2. *Every continuous map f from $[0, 1]$ to itself must have a fixed point $X^* \in [0, 1]$ such that $f(X^*) = X^*$.*

This line of techniques of proving the existence of equilibria has proven to be a powerful tool in the subsequent development of mathematical economics. For non-zero sum games, it can be formulated as a linear complementary problem [7].

A few years after Nash's work, Debreu and Arrow derived a first rigorous proof for the existence of a market clearing equilibrium, under quite mild assumptions on the utility functions of market participants. Again their proof was based on fixed point theorems [8]. Debreu and Arrow result has been regarded by many as one of the most beautiful application of mathematical theories developed in the last century.

For games that are not zero-sum, several algorithms have been proposed over the past century, but all of them are either of unknown complexity, or known to require, in the worst case, exponential time [6]. During the same decades that these concepts were being explored by the game theorists, computer science theorists were busy developing independently a theory of algorithm addressing precisely the kind of problems raised above.

In 1964 Mangasarian represented all equilibria of a bi-matrix game as convex combinations of the vertices of certain polyhedra defined by the payoff matrix [14]. During the same year, Lemke and Howson developed a path-following, simplex-like algorithm for general two player games [13]. The Lemke-Howson algorithm for bi-matrix games provides both an elementary proof of the existence of equilibrium point and an efficient computation method for finding atleast one equilibrium point. It is based upon a simple pivoting strategy which corresponds to following a path whose endpoint is a Nash equilibrium [17].

Rosenmuller extended Lemke-Howson algorithm to n-person finite games [21]. Later inspired by the path-following approach of Lemke and Howson, Herbert Scarf (1973) developed the first converging algorithm to compute fixed points and to compute equilibrium prices of competitive markets [22]. His algorithm was also for n-person games. Today the path-following method has been recognized as one of the most important algorithm paradigms in optimization.

In 1979, Khachiyan designed an ellipsoid algorithm that can solve a linear program in polynomial time. The existence of such algorithm implies that Nash equilibrium in two-player zero-sum game can be found in polynomial time [12].

Most experimental economic games have been modelled as prisoner's dilemma and an easy numerical way that has been used to identify the Nash equilibria is by use of a payoff matrix. The matrix has been helpful especially in two-person games where players have more than two strategies. To find the Nash equilibria, one way has been to check the pairs of strategies, and ask for each one of them whether the individual strategies are best responses to each other. The other way has been to compute each player's best response(s) to each strategy of the other player, and then find strategies that are mutual best responses.

Despite the considerable effort that has been devoted to the search of efficient algorithm for Nash's problem, no polynomial time algorithm has been found. The first step towards understanding complexity of Nash equilibria began in 1994 when Christos [6] defined the complexity class, Polynomial Parity Arguments on Directed Graphs (PPAD), to characterize that mathematical proofs rely on parity arguments. Such an argument appears in proofs of many important theorems, including Sperner's lemma, which led to one of the most elegant proofs of Brouwer's fixed-point theorem. Therefore both a discreet version of the fixed point problem and the Nash equilibrium problem are members of PPAD.

Nash equilibrium is not always unique. Some games have no Nash equilibrium whereas others have multiple Nash equilibria. For the games with multiple Nash equilibria, it becomes difficult to predict what strategies will be chosen by the players and there is need for players to make the best choices so as to optimize from the outcomes of the game. Two-person games do not take us very far because many of the games that are most important in real world involve considerably more than two players, for example, economic competition, highway congestion, over-exploitation of the environment and monetary exchange. Therefore there was need to put more emphasis on games with more than two players. In this study, the identification and computation of efficient Nash equilibria in a three-player experimental economic game with multiple equilibria was done.

CHAPTER 3

RESULTS AND DISCUSSION

In this chapter, a social cooperation circumstance in a school has been described and modelled as a stag hunt (assurance) game and played by high school students. The possible outcomes of the three player experimental economic game have also been displayed and the payoff for all the players in every outcome cell calculated. In the last section, identification of pure Nash Equilibria in the game (3.1) was done. The mixed strategy Nash equilibrium was also computed. The fixed point theorem of Brauwer was used to verify the existence of mixed Nash equilibrium in the game (3.1). Finally the computation and identification of most efficient Nash Equilibrium in the game (3.1) was done.

3.1 Description of the Game

3.1.1 Stag Hunt Game

Many circumstances that have been described as prisoner's dilemma, which is an experimental economic game, might be interpreted as stag hunt, depending on how fitness is calculated. The original stag hunt game was described by the philosopher Jean-Jacques Rousseau in the year 1755 . This game is a well known coordination game in which two players go out to hunt together. If they cooperate they have a chance of capturing a stag, constituting a high reward. On their own, the hunters can only hope to capture a hare yielding a lower payoff. Should one player try to cooperate, while the other chooses to hunt alone (defects), the cooperator will fail and get nothing, whereas the defector can still get a hare. In order to make the stag hunt game to be more applicable in real world, it was generalized into an N-player Stag hunt game [20].

For this study, the following social cooperation situation was modelled as three- player stag hunt game: In a certain High School, students were given two assignments to attempt. The first assignment (stag) was quite challenging and for an individual to succeed he must have the cooperation of one of his partners. The second assignment (hare) is simpler and can be done by any one student without a problem.

The students were required to make a choice between attempting the first assignment or attempting the second assignment. Attempting the first assignment together and obtaining a correct solution was more rewarding than individually finding a solution to the second assignment. Any student who cooperated with any other to correctly complete the first assignment was given a payoff of 10 and whoever cooperated with any other student to complete the second assignment correctly was given a payoff of 7 points. Attempting the first assignment individually was doomed to failure and had a payoff of zero.

The following assumptions were made:

(i) All the players (students) were rational as they made their choice.

(ii) All the players had the same ability in making choices.

(iii) All the players had the same strategy profile.

We denote the above game as

$$\Gamma_1 = \langle N, (S_i), (u_i) \rangle \tag{3.1}$$

where N is the number of players, S_i the available strategies, u_i the payoffs to the players and $i = 1, 2, 3$. The game (3.1) is a pure strategy game with the strategy profile $\beta = (\beta_1, \beta_2)$ where:
β_1 represents the first pure strategy (choosing the first assignment - stag), β_2 represents the second pure strategy (choosing the second assignment - hare. Note that $\beta \in S_i$.

3.1.2 Stages of the Game (3.1)

The stag hunt game and the game modelled as the stag hunt game, (3.1), was explained to students so that they had the full knowledge of the game and they made their choices independently. It was a dynamical stage game in that the students were allowed to play it for a finite number of repetitions as they varied their strategies as well. The students had complete information about the game since all the parameters and the rules of the game were well known by all of them.

The game modelled in (3.1) had three players (three students) in each group. The two strategies available to the players were:
(β_1) and (β_2). The following steps were followed:

(i) Students were asked to choose their strategies, β_1 or β_2. Note that players chose their strategies independently.

(ii) The outcomes were recorded before they were allowed to repeat the same game.

(iii) The payoffs for all the possible outcome cells from the game were calculated.

3.1.3 Outcomes of the Game (3.1)

In the three person stag hunt game (3.1) modelled above, each player had two choices, attempting the first assignment or attempting the second assignment. This resulted to eight possible outcomes (cells) for the three players, (Player 1, Player 2, Player 3) respectively as: $(\beta_1, \beta_1, \beta_1)$; $(\beta_1, \beta_1, \beta_2)$; $(\beta_1, \beta_2, \beta_1)$; $(\beta_1, \beta_2, \beta_2)$; $(\beta_2, \beta_1, \beta_1)$; $(\beta_2, \beta_1, \beta_2)$; $(\beta_2, \beta_2, \beta_1)$ and $(\beta_2, \beta_2, \beta_2)$.
Therefore to find the number of possible outcomes we use the expression S^N where S represents the number of strategies available to the players and N represents the number of players. Thus $2^3 = 8$.

Payoffs were calculated by examining each pair-wise payoff set among players, and the payoffs for three players were calculated by considering the type of interaction they had. For example, the payoffs for three players for $(\beta_1, \beta_2, \beta_1)$ was as follows: Player 1 received 0 points for the interaction with player 2 and 10 points for cooperating with player 3. Player 2 received 7 points for not cooperating with player 1 and 7 points for not cooperating with player 3. Player 3 received 10 points for cooperating with player 1 and 0 points for not cooperating with player 2. Therefore $(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$. This implies that

$$u_1(\beta_1, \beta_2, \beta_1) = 10,$$

$$u_2(\beta_1, \beta_2, \beta_1) = 14$$

$$u_3(\beta_1, \beta_2, \beta_1) = 10$$

where u_1, u_2 and u_3 are the payoffs of player 1, player 2 and player 3 respectively. Applying the same rules, we have the summary for the payoffs to the three players as per the eight possible outcomes as shown below:

(Player 1, Player 2, Player 3)

$(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$

$(\beta_1, \beta_1, \beta_2) = (10, 10, 14)$

$(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$

$(\beta_1, \beta_2, \beta_2) = (0, 14, 14)$

$(\beta_2, \beta_1, \beta_1) = (14, 10, 10)$

$(\beta_2, \beta_1, \beta_2) = (14, 0, 14)$

$(\beta_2, \beta_2, \beta_1) = (14, 14, 0)$

$(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$

(see Figure 3.1).

The payoffs for all the three players, $u_i(\beta)$ will be as shown below:

$u_1(\beta_1, \beta_1, \beta_1) = 20$, $u_2(\beta_1, \beta_1, \beta_1) = 20$ and $u_3(\beta_1, \beta_1, \beta_1) = 20$

$u_1(\beta_1, \beta_1, \beta_2) = 10$, $u_2(\beta_1, \beta_1, \beta_2) = 10$ and $u_3(\beta_1, \beta_1, \beta_2) = 14$

$u_1(\beta_1, \beta_2, \beta_1) = 10$, $u_2(\beta_1, \beta_2, \beta_1) = 14$ and $u_3(\beta_1, \beta_2, \beta_1) = 10$

$u_1(\beta_1, \beta_2, \beta_2) = 0$, $u_2(\beta_1, \beta_2, \beta_2) = 14$ and $u_3(\beta_1, \beta_2, \beta_2) = 14$

$u_1(\beta_2, \beta_1, \beta_1) = 14$, $u_2(\beta_2, \beta_1, \beta_1) = 10$ and $u_3(\beta_2, \beta_1, \beta_1) = 10$

$u_1(\beta_2, \beta_1, \beta_2) = 14$, $u_2(\beta_2, \beta_1, \beta_2) = 0$ and $u_3(\beta_2, \beta_1, \beta_2) = 14$

$u_1(\beta_2, \beta_2, \beta_1) = 14$, $u_2(\beta_2, \beta_2, \beta_1) = 14$ and $u_3(\beta_2, \beta_2, \beta_1) = 0$

$u_1(\beta_2, \beta_2, \beta_2) = 14$, $u_2(\beta_2, \beta_2, \beta_2) = 14$ and $u_3(\beta_2, \beta_2, \beta_2) = 14$

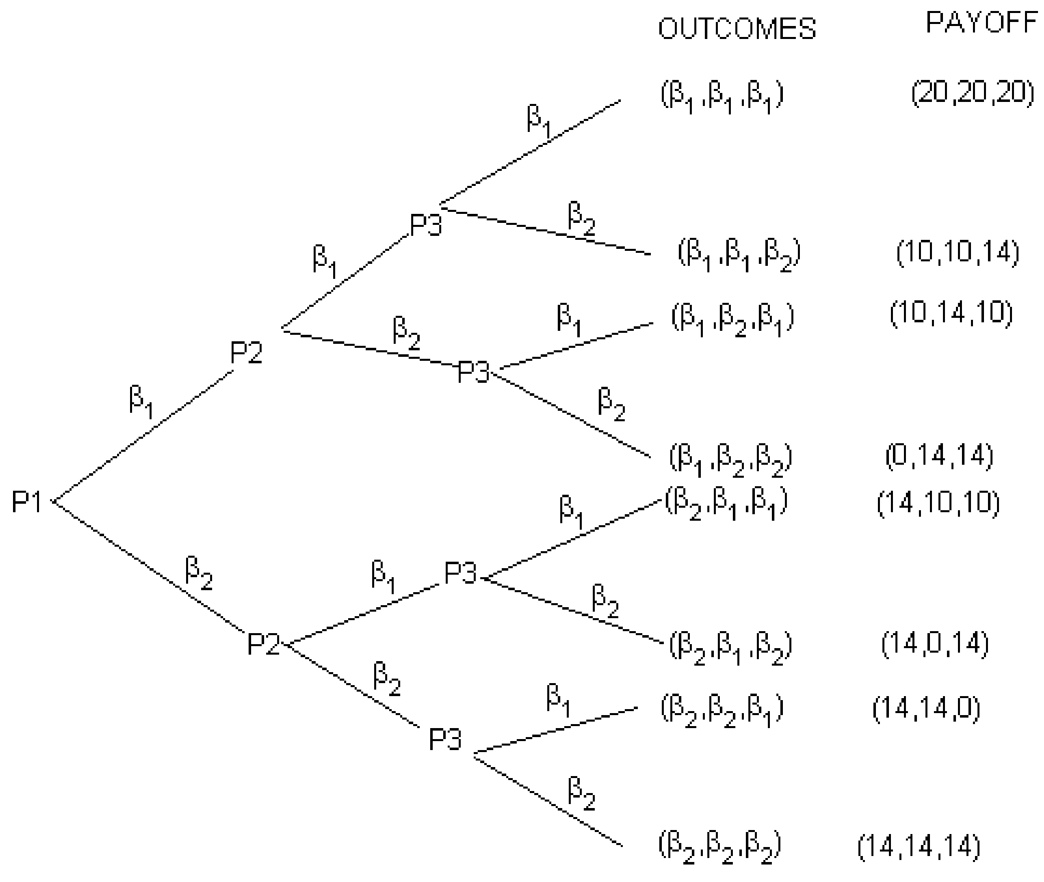

Figure 3.1: Tree Diagram on outcomes and payoffs

3.2 Computation and Identification of Efficient Nash Equilibria

3.2.1 Identification of Pure NE in the Game (3.1)

The game (3.1) is an example of non-cooperative coordination game. Nash equilibrium is based on the premises that

(i) each individual acts rationally given their beliefs about the other players' actions and that

(ii) these beliefs are correct.

It is the second element which makes this an equilibrium concept. Nash equilibrium is an action profile with the property that no player can do better by changing their actions, given the other players actions. We can alternatively define a Nash equilibrium as an action profile for which every player's action is the best response to the other players' actions.

Definition 3.2.1. A pure strategy game (1.1), the strategy profile $(s_1^*, \cdots, s_n^*)^T$ is a Nash equilibrium if and only if $\mathbf{s}_i^* \in B_i(\mathbf{s}_j^*), \forall \quad i = 1, \cdots, n$ [17].

This means that the pure strategy chosen by player i will lead to Nash Equilibrium if it is a best response for player i to the strategies chosen by all other players, where $\mathbf{s}_i^*$ is the pure strategy profile for player i, B_i is the best response for player i and $\mathbf{s}_j^*$ is the pure strategy profile for all other players.

The game (3.1) has eight different action profiles:
$(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$, $(\beta_1, \beta_1, \beta_2) = (10, 10, 14)$, $(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$, $(\beta_1, \beta_2, \beta_2) = (0, 14, 14)$, $(\beta_2, \beta_1, \beta_1) = (14, 10, 10)$, $(\beta_2, \beta_1, \beta_2) = (14, 0, 14)$, $(\beta_2, \beta_2, \beta_1) = (14, 14, 0)$ and $(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$ for the three players, (Player 1, Player 2, Player 3), respectively. Since the game has only a few actions, we found Nash equilibria for the game by examining each action profile in turn to determine if it satisfied the conditions for equilibrium.

The best response sets for the game (3.1) were:

(i) $B_i(\beta_1; i = 1, 2, 3) = \beta_1$, that is, the best response for player i when he or she plays β_1 is β_1 and

(ii) $B_i(\beta_2; i = 1, 2, 3) = \beta_2$ which means that the best response for player i when β_2 is played is β_2.

Therefore
$B_1(\beta_1) = \beta_1$; $B_1(\beta_2) = \beta_2$,
$B_2(\beta_1) = \beta_1$; $B_2(\beta_2) = \beta_2$ and
$B_3(\beta_1) = \beta_1$; $B_3(\beta_2) = \beta_2$.
B_1, B_2 and B_3 are the best response for player 1, 2 and 3 respectively.
Since $\beta_1 \subset B_1(\beta_1)$, $\beta_1 \subset B_2(\beta_1)$ and $\beta_1 \subset B_3(\beta_1)$, then $(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$ is a pure Nash Equilibrium.

Similarly, since $\beta_2 \subset B_1(\beta_2)$,$\beta_2 \subset B_2(\beta_2)$ and $\beta_2 \subset B_3(\beta_2)$, then $(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$ is a pure Nash equilibrim.

The other profiles: $(\beta_1, \beta_1, \beta_2) = (10, 10, 14)$, $(\beta_1, \beta_2, \beta_1) = (10, 14, 10)$, $(\beta_1, \beta_2, \beta_2) = (0, 14, 14)$, $(\beta_2, \beta_1, \beta_1) = (14, 10, 10)$, $(\beta_2, \beta_1, \beta_2) = (14, 0, 14)$ and $(\beta_2, \beta_2, \beta_1) = (14, 14, 0)$ are not pure Nash equilibria since:
$\beta_1 \subsetneq B_i(\beta_2)$and
$\beta_2 \subsetneq B_i(\beta_1)$
In summary, the results of the game (3.1) are as shown below:

(i) $(\beta_1, \beta_1, \beta_1)$ is a pure Nash equilibrium because each player prefers this profile to that in which she chooses β_2 alone. A player is better off remaining attentive to the pursuit of a stag, β_1, than running after a hare, β_2, if all other players remain attentive).

(ii) $(\beta_2, \beta_2, \beta_2)$ is a pure Nash equilibrium because each player prefers this profile to that in which she pursues a stag, (β_1), alone. A player is better off catching a hare, (β_2), than pursuing a stag, (β_1), if no one else pursues a stag.

(iii) No other profile is a pure Nash equilibrium because in any other profile at least one player chooses a stag, (β_1), and at least one player chooses a hare, (β_2), so that any player choosing (β_1) is better off switching to (β_2)

Since $(\beta_1, \beta_1, \beta_1)$ is a strict NE, then β_1 is evolutionary stable. $(\beta_2, \beta_2, \beta_2)$ is also a strict NE, therefore β_2 is also evolutionary stable.

3.2.2 Mixed Nash Equilibrium in the Game (3.1)

Since this game had multiple equilibrium points, the optimal choice is a mixed strategy. Thus randomization of the pure strategies was done as shown below:

Suppose (ρ_1, ρ_2, ρ_3) is a mixed strategy profile. This means that ρ_1 is a probability density function on $S_1 = \{\beta_1, \beta_2\}$, ρ_2 is a probability density function on $S_2 = \{\beta_1, \beta_2\}$ and ρ_3 is a probability density function on $S_3 = \{\beta_1, \beta_2\}$

Let us represent:

$$\rho_1 = (\rho_1(\beta_1)\rho_1(s_2)),$$

$$\rho_2 = (\rho_2(\beta_1)\rho_2(s_2))$$

and

$$\rho_3 = (\rho_3(\beta_1)\rho_3(s_2)).$$

We have

$$S = S_1 \times S_2 \times S_3$$

$$= \{(\beta_1, \beta_1, \beta_1)(\beta_1, \beta_1, \beta_2)(\beta_1, \beta_2, \beta_1)(\beta_1, \beta_2, \beta_2)(\beta_2, \beta_1, \beta_1)(\beta_2, \beta_1, \beta_2)(\beta_2, \beta_2, \beta_1)(\beta_2, \beta_2, \beta_2).\}$$

We computed the payoff functions u_1, u_2 and u_3. Note that

$$u_i(\rho_1, \rho_2, \rho_3) = \sum_{S_1, S_2, S_3 \in S} \rho(S_1, S_2, S_3) u_i(S_1, S_2, S_3)$$

and $i = 1, 2, 3$. That is:

$$\begin{aligned} u_1(\rho_1, \rho_2, \rho_3) = &\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1)u_1(\beta_1, \beta_1, \beta_1) + \rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_2)u_1(\beta_1, \beta_1, \beta_2) + \\ &\rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_1)u_1(\beta_1, \beta_2, \beta_1) + \rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_2)u_1(\beta_1, \beta_2, \beta_2) + \\ &\rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_1)u_1(\beta_2, \beta_1, \beta_1) + \rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_2)u_1(\beta_2, \beta_1, \beta_2) + \\ &\rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_1)u_1(\beta_2, \beta_2, \beta_1) + \rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_2)u_1(\beta_2, \beta_2, \beta_2). \end{aligned} \tag{3.2}$$

$$\begin{aligned} u_1(\rho_1, \rho_2, \rho_3) = \; &20\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_2) + \\ &10\rho_1(\beta_1)\rho_2(\beta_2)\rho_3(\beta_1) + 14\rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_1) + \\ &14\rho_1(\beta_2)\rho_2(\beta_1)\rho_3(\beta_2) + 14\rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_1) + \\ &14\rho_1(\beta_2)\rho_2(\beta_2)\rho_3(\beta_2). \end{aligned} \tag{3.3}$$

$$\begin{aligned} u_1(\rho_1, \rho_2, \rho_3) = \; &20\rho_1(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1)(1-\rho_3)(\beta_1) + \\ &10\rho_1(\beta_1)(1-\rho_2)(\beta_1)\rho_3(\beta_1) + 14(1-\rho_1)(\beta_1)\rho_2(\beta_1)\rho_3(\beta_1) + \\ &14(1-\rho_1)(\beta_1)\rho_2(\beta_1)(1-\rho_3)(\beta_1) + 14(1-\rho_1)(\beta_1)(1-\rho_2)(\beta_1)\rho_3(\beta_1) + \\ &14(1-\rho_1)(\beta_1)(1-\rho_2)(\beta_1)(1-\rho_3)(\beta_1). \end{aligned} \tag{3.4}$$

$$u_1(\rho_1, \rho_2, \rho_3) = 14 - 14\rho_1(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1) + 10\rho_1(\beta_1)\rho_3(\beta_1). \tag{3.5}$$

Similarly,

$$u_2(\rho_1, \rho_2, \rho_3) = 14 - 14\rho_2(\beta_1) + 10\rho_1(\beta_1)\rho_2(\beta_1) + 10\rho_2(\beta_1)\rho_3(\beta_1). \tag{3.6}$$

and finally we had

$$u_3(\rho_1, \rho_2, \rho_3) = 14 - 14\rho_3(\beta_1) + 10\rho_1(\beta_1)\rho_3(\beta_1) + 10\rho_2(\beta_1)\rho_3(\beta_1). \tag{3.7}$$

Basing on the assumption that all the students were rational, they had the same strategy profile to choose from and their ability in making choices were the same, we let

$$\rho_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \rho_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \rho_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Then

$$u_1(\rho_1,\rho_2,\rho_3)=\frac{104}{9},u_2(\rho_1,\rho_2,\rho_3)=\frac{104}{9},u_3(\rho_1,\rho_2,\rho_3)=\frac{104}{9}.$$

Suppose (ρ_1,ρ_2,ρ_3) is a mixed strategy profile. It can be seen that

$$u_1(\rho_1,\rho_2,\rho_3)=14-14\rho_1(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1),$$

$$u_2(\rho_1,\rho_2,\rho_3)=14-14\rho_2(\beta_1)+10\rho_1(\beta_1)\rho_2(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1)$$

and

$$u_3(\rho_1,\rho_2,\rho_3)=14-14\rho_3(\beta_1)+10\rho_1(\beta_1)\rho_3(\beta_1)+10\rho_2(\beta_1)\rho_3(\beta_1).$$

Let $(\rho_1^*,\rho_2^*,\rho_3^*)$ be a mixed strategy equilibrium. Then

$$\begin{aligned} u_1(\rho_1^*,\rho_2^*,\rho_3^*) &\geq u_1(\rho_1,\rho_2^*,\rho_3^*) \quad \forall \quad \rho_1\in\Delta(S_1);\\ u_2(\rho_1^*,\rho_2^*,\rho_3^*) &\geq u_1(\rho_1^*,\rho_2,\rho_3^*) \quad \forall \quad \rho_2\in\Delta(S_2);\\ u_3(\rho_1^*,\rho_2^*,\rho_3^*) &\geq u_1(\rho_1^*,\rho_2^*,\rho_3) \quad \forall \quad \rho_3\in\Delta(S_3). \end{aligned} \tag{3.8}$$

The inequalities (3.8) are equivalent to:

$$\begin{aligned} &14-14\rho_1^*(\beta_1)+10\rho_1^*(\beta_1)\rho_2^*(\beta_1)+10\rho_1^*(\beta_1)\rho_3^*(\beta_1)\\ &\geq 14-14\rho_1(\beta_1)+10\rho_1(\beta_1)\rho_2^*(\beta_1)+10\rho_1(\beta_1)\rho_3^*(\beta_1)\\ &\forall \rho_1\in\Delta(S_1);\\ \\ &14-14\rho_2^*(\beta_1)+10\rho_1^*(\beta_1)\rho_2^*(\beta_1)+10\rho_2^*(\beta_1)\rho_3^*(\beta_1)\\ &\geq 14-14\rho_2(\beta_1)+10\rho_1^*(\beta_1)\rho_2(\beta_1)+10\rho_2(\beta_1)\rho_3^*(\beta_1)\\ &\forall \rho_2\in\Delta(S_2);\\ \\ &14-14\rho_3^*(\beta_1)+10\rho_1^*(\beta_1)\rho_3^*(\beta_1)+10\rho_2^*(\beta_1)\rho_3^*(\beta_1)\\ &\geq 14-14\rho_3(\beta_1)+10\rho_1^*(\beta_1)\rho_3(\beta_1)+10\rho_2^*(\beta_1)\rho_3(\beta_1)\\ &\forall \rho_3\in\Delta(S_3). \end{aligned} \tag{3.9}$$

These inequalities (3.9) are equivalent to:

$$10\rho_1^*(\beta_1)\rho_2^*(\beta_1) + 10\rho_1^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_1^*(\beta_1) \geq 10\rho_1(\beta_1)\rho_2^*(\beta_1) + 10\rho_1(\beta_1)\rho_3^*(\beta_1) - 14\rho_1(\beta_1)$$
$$\forall \rho_1 \in \Delta(S_1);$$

$$10\rho_1^*(\beta_1)\rho_2^*(\beta_1) + 10\rho_2^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_2^*(\beta_1) \geq 10\rho_1^*(\beta_1)\rho_2(\beta_1) + 10\rho_2(\beta_1)\rho_3^*(\beta_1) - 14\rho_2(\beta_1)$$
$$\forall \rho_2 \in \Delta(S_2);$$

$$10\rho_1^*(\beta_1)\rho_3^*(\beta_1) + 10\rho_2^*(\beta_1)\rho_3^*(\beta_1) - 14\rho_3^*(\beta_1) \geq 10\rho_1^*(\beta_1)\rho_3(\beta_1) + 10\rho_2^*(\beta_1)\rho_3(\beta_1) - 14\rho_3(\beta_1)$$
$$\forall \rho_3 \in \Delta(S_3). \tag{3.10}$$

In turn the inequalities (3.10) are equivalent to:

$$\begin{aligned}
&\rho_1^*(\beta_1)\{10\rho_2^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \geq \rho_1(\beta_1)\{10\rho_2^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \\
&\forall \rho_1 \in \Delta(S_1); \\
&\rho_2^*(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \geq \rho_2(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_3^*(\beta_1) - 14\} \\
&\forall \rho_2 \in \Delta(S_2); \\
&\rho_3^*(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_2^*(\beta_1) - 14\} \geq \rho_3(\beta_1)\{10\rho_1^*(\beta_1) + 10\rho_2^*(\beta_1) - 14\} \\
&\forall \rho_3 \in \Delta(S_3).
\end{aligned} \tag{3.11}$$

Some of the possible cases are:

(i) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} > 1$ which leads to the pure strategy profile $\beta_1, \beta_1, \beta_1$ that is a NE.

(ii) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} < 1$ which leads to the pure strategy profile $\beta_2, \beta_2, \beta_2$ that is a NE.

(iii) $\frac{5}{7}\{\rho_2^*(\beta_1) + \rho_3^*(\beta_1)\} = 1$ which leads to a mixed strategy profile that we indeed showed that it was also a NE.

Verification of Existence of Equilibria in the Game (3.1)

Considering the game (3.1) analyzed above, the two multiple equilibria (pure Nash equilibria) were $(\beta_1, \beta_1, \beta_1) = (20, 20, 20)$ and $(\beta_2, \beta_2, \beta_2) = (14, 14, 14)$. We proved the existence of mixed Nash equilibrium using the Brauwer's fixed point theorem as shown below:
We had the game (3.1),

$$\Gamma_1 = \langle N, (S_i), (u_i) \rangle,$$

where N is the number of players and $S_i = S_1 \times S_2 \times S_3$ is the action set for the players. All the action sets S_i are finite.

Using the method used to obtain the results for game (1.6), we let $\Delta = \Delta_1 \times \cdots \times \Delta_N$ denote the set of mixed strategies for the players in the game (3.1). The finiteness of S_i ensures the compactness of Δ.

We then defined the gain function for player i, G_i. For a mixed strategy $\rho \in \Delta$, we let the gain for player i on action $\beta \in S_i$ be

$$G_i(\rho, \beta) = max\{0, u_i(\beta, \rho_j) - u_i(\rho_i, \rho_j)\},$$

where ρ_i is the mixed strategy for player i and ρ_j is the mixed strategy for all other players in the game (3.1). The gain function represents the benefit a player gets by unilaterally changing his strategy.

We now define $g = (g_1, \cdots, g_N)$ where $g_i(\rho, \beta) = \rho_i(\beta) + G_i(\rho, \beta)$ for $\rho \in \Delta$, $\beta \in S_i$.
We see that

$$\sum_{\beta \in S_i} g_i(\rho, \beta) = \sum_{\beta \in S_i} \rho_i(\beta) + G_i(\rho, \beta) = 1 + \sum_{\beta \in S_i} G_i(\rho, \beta) > 0.$$

We now use g to define $f : \Delta \mapsto \Delta$ as follows:

Let

$$f_i(\rho, \beta) = \frac{g_i(\rho, \beta)}{\sum_{\beta \in S_i} g_i(\rho, \beta)}$$

for $\beta \in S_i$.

It is easy to see that f_i is a valid mixed strategy in Δ_i. It is also easy to check that each f_i is a continuous function of ρ, and hence f is a continuous function. Now Δ is the cross product of a finite number of compact convex sets, and so we get that Δ is also compact and convex. Therefore we may apply the Brouwer fixed point theorem to f. So f has a fixed point in Δ, call it ρ^*.

We claim that ρ^* is a Nash equilibrium in the game (3.1). For this purpose it suffices to show that

$$\forall \quad 1 \leq i \leq N, \forall \quad \beta \in S_i, G_i(\rho^*, \beta) = 0.$$

This simply states that each player gains nothing by unilaterally changing his strategy, which is exactly the necessary condition for Nash equilibrium.

Now assume that the gains are not zero. $\exists \quad i, 1 \leq i \leq N$ and $\beta \in S_i$ such that

$$G_i(\rho^*, \beta) > 0.$$

Note then that

$$\sum_{\beta \in S_i} g_i(\rho^*, \beta) = 1 + \sum_{\beta \in S_i} G_i(\rho^*, \beta) > 1.$$

So let

$$C = \sum_{\beta \in S_i} g_i(\rho^*, \beta).$$

We denote $G(i, \cdot)$ as the gain vector indexed by actions in S_i. Since

$$f(\rho^*) = \rho^*,$$

we clearly have that

$$f_i(\rho^*) = \rho_i^*.$$

Therefore we see that

$$\rho_i^* = \frac{g_i(\rho^*)}{\sum_{\beta \in S_i} g_i(\rho^*, \beta)}.$$

$$\Rightarrow \rho_i^* = \frac{\rho_i^* + G_i(\rho^*, \cdot)}{C}$$

$$C\rho_i^* = \rho_i^* + G_i(\rho^*, \cdot)$$

$$(C-1)\rho_i^* = G_i(\rho^*, \cdot)$$

$$\rho_i^* = (\frac{1}{C-1})G_i(\rho^*, \cdot).$$

Since $C > 1$, we have that ρ_i^* is some positive scaling of the vector $G_i(\rho^*, \cdot)$.

Now we claim that

$$\rho_i^*(\beta)(U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*)) = \rho_i^*(\beta)G_i(\rho^*, \beta)\forall\beta \in S_i.$$

To see this, we first note that if

$$G_i(\rho^*, \beta) > 0,$$

then this is true by the definition of the gain function.

We assume that

$$G_i(\rho^*, \beta) = 0.$$

By our previous statements we have that

$$\rho_i^*(\beta) = \frac{1}{C-1}G_i(\rho^*, \beta) = 0,$$

and so the left term is zero, giving the entire expression as 0 as needed.

So finally we have that

$$0 = (U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*)$$

$$= \sum_{\beta \in S_i}(\rho_i^*(\beta)U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*))$$

$$= \sum_{\beta \in S_i}(\rho_i^*(\beta)(U_i(\beta_i, \rho_j^*) - U_i(\rho_i^*, \rho_j^*))$$

$$= \sum_{\beta \in S_i} \rho_i^*(\beta) G_i(\rho^*, \beta)$$

by the previous statements.

$$= \sum_{\beta \in S_i} (C-1) \rho_i^*(\beta)^2 > 0$$

where the last inequality follows since ρ_i^* is a non-zero vector.

But this is a clear contradiction, so all the gain must indeed be zero.
Therefore ρ^* is a mixed Nash equilibrium for the game (3.1) as needed.

More often, most situations involve population of players and to study multi-player games effectively we need to deviate from classical game theory to Evolutionary Game Theory. Edgar (2012)[19] presented an approach that deviates from classical game theory in regard to rationality of players, belief about the behaviour of other players and the alignment of such beliefs across players. This is important because in a multi-player game, some players may make their choices irrationally. Evolutionary Game Theory will effectively enable us determine equilibria of games played by a population of players, where the fitness (payoff) of the players is derived from the success each player has in playing the game.

Together with Evolutionary Game Theory, new concepts were developed such as the Evolutionary Stable Strategy which is applied to study the stability of populations [23]. ESS is an equilibrium refinement of NE. It is a NE that is evolutionary stable in the sense that if adopted by a population of players in a given environment, it cannot be invaded by any alternative strategy that is initially rare. It is known that any ESS is an asymptotically stable strategy [5]. In particular, in games with multiple ESS, we resolve the problem of equilibrium selection by choosing the one that is stochastically stable.

Suppose in the game (3.1), a third pure strategy (attempting the third assignment, β_3) is introduced such that attempting the first assignment (β_1) together is still more rewarding than individually attempting either the second assignment (β_2) or the third assignment

(β_3.) We denote the new game as

$$\Gamma_2 = \langle N, (S_i), (u_i) \rangle \tag{3.12}$$

where N is the number of players, S_i the available strategies, u_i the payoffs to the players and $i = 1, 2, 3$.

The game (3.12) is a pure strategy game with the strategy profile $\beta = (\beta_1, \beta_2, \beta_3)$, where
β_1 represents the first pure strategy (choosing the first assignment),
β_2 represents the second pure strategy (choosing the second assignment),
β_3 represents the third pure strategy (choosing the third assignment) and N is the number of players.

The rewards for β_1 and β_2 are maintained as in the game (3.1). However the third assignment (β_3) has the lowest reward of 5. We considered two cases: where the third assignment (β_3) could be completed successfully on its own and where the reward for β_3 depends on cooperation among the students. The result was twenty seven possible outcome cells and their respective payoffs were calculated by examining each pair-wise payoff set among players, and the payoffs for three players were calculated by considering the type of interaction they had as was done in the game (3.1). The first eight possible outcomes of this game and their respective payoffs for the three players were the same as the outcomes in the game (3.1). However the other 19 possible outcome cells and their respective payoffs for the three players were calculated and the results were as displayed in Table (3.1).

In the game (3.12), we had multiple equilibria. Assuming that all players acted rationally, the two pure Nash Equilibria are $(\beta_1, \beta_1, \beta_1)$ and $(\beta_2, \beta_2, \beta_2)$ profiles since no player had an incentive to deviate from either the first or second equilibria. Since the two pure NE are strict, then the two pure strategies of the game (3.12) are evolutionary stable strategies. The mixed strategy that resulted from the two pure Nash Equilibria is also evolutionary stable.

In some cases, some students decided to behave irrationally by attempting the third as-

Table 3.1: Possible outcomes and their respective payoffs for the game (3.12)

OUTCOMES	FIRST PAYOFF, $U_i(\beta)$	SECOND PAYOFF,$U_i(\beta)$
$\beta_3, \beta_3, \beta_3$	$(10, 10, 10)$	$(10, 10, 10)$
$\beta_3, \beta_3, \beta_1$	$(10, 10, 0)$	$(5, 5, 0)$
$\beta_3, \beta_1, \beta_1$	$(10, 10, 10)$	$(0, 10, 10)$
$\beta_3, \beta_1, \beta_3$	$(10, 0, 10)$	$(5, 0, 5)$
$\beta_3, \beta_3, \beta_2$	$(10, 10, 14)$	$(5, 5, 14)$
$\beta_3, \beta_2, \beta_2$	$(10, 14, 14)$	$(0, 14, 14)$
$\beta_3, \beta_2, \beta_3$	$(10, 14, 10)$	$(5, 14, 5)$
$\beta_1, \beta_2, \beta_3$	$(0, 14, 10)$	$(0, 14, 0)$
$\beta_3, \beta_2, \beta_1$	$(10, 14, 0)$	$(0, 14, 0)$
$\beta_2, \beta_3, \beta_1$	$(14, 10, 0)$	$(14, 0, 0)$
$\beta_2, \beta_3, \beta_3$	$(14, 10, 10)$	$(14, 5, 5)$
$\beta_1, \beta_3, \beta_3$	$(0, 10, 10)$	$(0, 5, 5)$
$\beta_1, \beta_3, \beta_1$	$(10, 10, 10)$	$(10, 0, 10)$
$\beta_1, \beta_1, \beta_3$	$(10, 10, 10)$	$(10, 10, 0)$
$\beta_1, \beta_3, \beta_2$	$(0, 10, 14)$	$(0, 0, 14)$
$\beta_2, \beta_1, \beta_3$	$(14, 0, 10)$	$(14, 0, 0)$
$\beta_3, \beta_1, \beta_2$	$(10, 0, 14)$	$(0, 0, 14)$
$\beta_2, \beta_2, \beta_3$	$(14, 14, 10)$	$(14, 14, 0)$
$\beta_2, \beta_3, \beta_2$	$(14, 10, 14)$	$(14, 0, 14)$

signment, β_3 which was less rewarding than the first two assignments. Since β_1 and β_2 were evolutionary stable strategies, any student with mutant behaviour who decided to adopt the third strategy, β_3, could not successfully invade this population of players.

More precisely, β_1 is an ESS if either:

(i) the payoff for playing β_1 against other players playing β_1 is greater than that of playing any other strategy β_3 against players playing β_1, for example,

$$U_i(\beta_1, \beta_1, \beta_1) > U_i(\beta_3, \beta_1, \beta_1),$$

(ii) the payoff of playing β_1 against itself is equal to that of playing β_3 against β_1 but the payoff of playing β_3 against β_3 is less than that of playing β_1 against β_3, for example

$$U_i(\beta_1, \beta_1, \beta_1) = U_i(\beta_3, \beta_1, \beta_1)$$

and

$$U_i(\beta_1, \beta_1, \beta_3) > U_i(\beta_3, \beta_3, \beta_3).$$

Alternatively, β_2 is an ESS if either:

(i) the payoff for playing β_2 against other players playing β_2 is greater than that of playing any other strategy β_3 against players playing β_2, for example,

$$U_i(\beta_2, \beta_2, \beta_2) > U_i(\beta_3, \beta_2, \beta_2),$$

(ii) the payoff of playing β_2 against itself is equal to that of playing β_3 against β_2 but the payoff of playing β_3 against β_3 is less than that of playing β_2 against β_3, for example

$$U_i(\beta_2, \beta_2, \beta_2) = U_i(\beta_3, \beta_2, \beta_2)$$

and

$$U_i(\beta_2, \beta_2, \beta_3) > U_i(\beta_3, \beta_3, \beta_3).$$

Note that for both evolutionary stable strategies, either (i) or (ii) will do and that the former is a stronger condition than the latter. It is most likely that players will always adopt the evolutionary stable strategies since no mutant strategy can successfully invade this game.

3.2.3 Identification of Efficient Nash Equilibria in the Game (3.1)

The game (3.1) modelled in this study is an example of coordination game with multiple Nash equilibria. Some equilibria may give higher payoffs, some may be naturally more salient, others may be safer and/or fairer. When there are several NE, how will a rational agent decide on which of the several equilibria is the right one to settle upon? Attempts to resolve this problem have produced a number of refinements to the concept of NE. This necessitated the need to identify which equilibria is efficient in the case of multiple equilibria.

Risk dominance and payoff dominance are two related refinements of NE solution concept in game theory. A NE is considered payoff dominant if it is Pareto superior to all other NE in the game. When faced with a choice among equilibria, all players would agree on the payoff dominant equilibrium since it offers each player at least as much payoff as the other NE. This implies that

$$u_i(\beta_1, \beta_1, \beta_1) > u_i(\beta_2, \beta_2, \beta_2).$$

In the game modelled in (3.1), $(\beta_1, \beta_1, \beta_1)$ is a payoff dominant equilibrium because each player prefers this profile to that in which she chooses β_2 alone. A player is better off remaining attentive in attempting the first assignment, β_1, than attempting the second assignment, β_2, if all other players remain attentive since this will give them a higher reward.

Conversely, a NE equilibrium is considered risk dominant if it has the largest basin of attraction. This implies that the more uncertainty players have about the actions of the other player(s), the more likely they will choose a strategy corresponding to it. In the game (3.1), $(\beta_2, \beta_2, \beta_2)$ is a risk dominant equilibrium because each player prefers this profile to that in which she attempts the first assignment, (β_1), alone. A player is better off attempting the second assignment, (β_2), than the first assignment, (β_1), if no one else attempts the first assignment because this option is less risky.

In conclusion, the Pareto dominant and the risk dominant strategies are both ESS. A player who wishes to have a higher gain will always play the cooperative equilibrium (attempting the first assignment with a higher payoff). The player who wishes to minimize risks will play the defective equilibrium(attempting the second assignment because it is safer).

Summary, Conclusions and Recommendations

This study was set up with an objective of computing efficient Nash Equilibria for experimental economic games. We refer back to the introductory section of this study where we set up our objectives, and seek to answer the question of whether we met the study objectives. We effectively managed to :

(i) define and describe a three-player experimental economic game (3.1)

(ii) identify all the Nash Equilibia in our experimental economic game (3.1)

(iii) compute and identify the most efficient Nash equilibrium in the game (3.1).

In conclusion, the experimental economic game (3.1) had two Pure Nash equilibria that were evolutionary stable and the Mixed Nash equilibrium that resulted from randomization of pure strategies was also evolutionary stable. Since all the Nash Equilibria in the game (3.1) were evolutionary stable, they were all considered as most efficient equilibria because any student with mutant behaviour could not successfully invade the population of students playing the evolutionary stable strategy.

Alternatively, any student whose main objective was to maximize the points earned played the payoff dominant strategy, β_1, and this implied that $(\beta_1, \beta_1, \beta_1)$ was the most efficient Nash Equilibrium in the game (3.1). On the other hand, students who were uncertain about the strategies chosen by the other players and wanted to avoid the risk of completely loosing in the game (3.1) played the risk dominant strategy, β_2, and thus $(\beta_2, \beta_2, \beta_2)$ being the risk dominant Nash Equilibrium was most efficient in that case.

A major contribution that this study has made is that since most situations in economics such as cooperative projects and security dilemma are usually faced with multiple choices which challenge players in this field, and if Economics strives to be a predictive Science, then multiplicity of equilibria is a problem that needs to be dealt with. More often, in selecting from multiple equilibria, economists make use of efficiency considerations and that not only equilibria that are payoff dominant should be chosen, but also risk dominance should be considered as well.

In any Economic environment, we may have more than three firms interacting and this has also necessitated further study on multi-player games not only in the case of multiplicity of equilibria, but also in future, finding a more appropriate polynomial time algorithm for efficient Nash equilibrium.

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Proof of Minimax Theorem

We refer to the minimax theorem (2.2.1) and the game (2.1). Given a matrix A, linear programs LP_1 and LP_2 were derived. LP_1 represented the optimal strategy of row player while LP_2 represented the optimal strategy for the column player. First an observation was made that the linear program LP_2 was the dual to the linear program LP_1. If an LP has an optimal solution, then its dual also has an optimal solution: moreover the optimal value of the dual is the same as the optimal value of the original(primal) LP(strong duality theorem)[17].

We considered the row player's optimization problem, P_1(maxminimization) and the column player's optimization problem, P_2(minmaximization).

Maximize

$$min_j \sum_{i=1}^{m} a_{ij} x_i$$

subject to

$$\sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, \cdots, m.$$

Therefore

$$P_1 = max_{x \in \Delta(S_1)} min_{y \in \Delta(S_2)} xAy.$$

Also minimize

$$max_i \sum_{j=1}^{n} a_{ij} y_j$$

subject to

$$\sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, \cdots, n.$$

Therefore

$$P_2 = min_{y \in \Delta(S_2)} max_{x \in \Delta(S_1)} xAy.$$

Applying the strong duality theorem, it was observed that the problem P_1 had an optimal solution by the very nature of the problem. Since LP_1 was equivalent to the problem P_1, the immediate implication was that LP_1 had an optimal solution. Thus we had two LP_s, LP_1 and LP_2 which are duals of each other. Then by strong duality theorem, LP_2 also had an optimal solution and the optimal value for LP_2 was the same as the optimal value of LP_1.

Let $z^*, x_1^*, \cdots, x_m^*$ be an optimal solution of LP_1. Then we had

$$z^* = \sum_{i=1}^{m} a_{ij^*} x_i^*$$

for some $j^* \in \{1, \cdots, n\}$. By the feasibility of the optimal solution of LP_1, we had

$$\sum_{i=1}^{m} a_{ij^*} x_i^* \leq \sum_{i=1}^{m} a_{ij} x_i^*$$

for $j = 1, \cdots, n$. This implied that

$$\sum_{i=1}^{m} a_{ij^*} x_i^* = min_j \sum_{i=1}^{m} a_{ij} x_i^*$$
$$= min_{y \in \Delta(S_2)} x^* Ay.$$

Thus

$$z^* = min_{y \in \Delta(S_2)} x^* Ay.$$

Similarly, we let $w^*, y_1^*, \cdots, y_n^*$ be an optimal solution of LP_2. Then

$$w^* = \sum_{j=1}^{n} a_{i^*j} y_j^*$$

for some $i^* \in \{1, \cdots, m\}$. By the feasibility of the optimal solution of LP_2, we had

$$\sum_{j=1}^{n} a_{i^*j} y_j^* \geq \sum_{j=1}^{n} a_{ij} y_j^*$$

for $j = 1, \cdots, n$. This implied that

$$\sum_{j=1}^{n} a_{i^*j} y_j^* = max_i \sum_{j=1}^{n} a_{ij} y_j^*$$

$$= max_{x \in \Delta(S_1)} xAy^*.$$

Thus

$$w^* = max_{x \in \Delta(S_1)} xAy^*.$$

By strong duality theorem, the optimal values of the primal and the dual were the same and therefore $z^* = w^*$. This meant that

$$min_{y \in \Delta(S_2)} x^* Ay = max_{x \in \Delta(S_1)} xAy^*.$$

This proved the minimax theorem.